Eva Gedeonová

Jordan-Hölder Theorem for Lines

Matematický časopis, Vol. 22 (1972), No. 3, 177--198

Persistent URL: http://dml.cz/dmlcz/126520

MATEMATICKÝ ČASOPIS
ROČNÍK 22 1972 ČÍSLO 3

JORDAN—HÖLDER THEOREM FOR LINES

EVA GEDEONOVÁ, Bratislava

The aim of this paper is to find such nonmodular lattices in which the Jordan—Hölder theorem for lines is true. The notion of a line is a natural generalization of the notion of a chain in a lattice. M. Kolibiar in his paper [2] has shown that two neighbouring elements of a connected line in a modular lattice are comparable and form a priminterval. He has also shown that the Jordan—Hölder theorem for lines is true in modular lattices. We shall prove that if every two comparable neighbouring elements of any connected line in a finite lattice form a priminterval, then this lattice is modular (see Theorem 1). Hence two neighbouring elements of a connected line in a semimodular lattice need not form a priminterval. But the Jordan—Hölder theorem for lines holds for some semimodular lattices by considering the correspondence of simple pairs of lines. It can be shown that if a lattice has a connected line which has two uncomparable neighbouring elements, then this lattice contains lines with different lengths. If a lattice is p-modular (i. e. it does not contain a sublattice with diagram in Figure 1) then any two neighbouring elements of any its connected line are comparable. In this paper it is proved that the Jordan—Hölder theorem for lines is valid in a p-modular and semimodular lattice. An example of a p-modular and semimodular lattice which is not modular is given.

Basic concepts and properties

Throughout the paper S denotes a lattice. Let a, b, $x \in S$. We say that x is *between* a and b and write axb if $(a \cap x) \cup (x \cap b) = x = (a \cup x) \cap (x \cup b)$. When the lattice S is a chain then axb iff $a \leqq x \leqq b$ or $b \leqq x \leqq a$. The relation "between" in S possesses the following properties:

(α_1) xyz implies zyx

(α_2) xyz and xzy imply $y = z$

(t_1) xyz and xzu imply yzu.

Four different elements $a, b, c, d \in S$ form a *pseudolinear quadruple* when they satisfy abc, bcd, cda, dab. If axb, then $a \cap b \leqq x \leqq a \cup b$. Clearly, axb and $a \leqq b$ implies $a \leqq x \leqq b$.

If A, B are subsets of some lattices and a bijection φ from A onto B is given, so that abc if and only if $\varphi(a)\varphi(b)\varphi(c)$, we say that A, B are *b-equivalent*. A subset A of S is called a *line* if there exists a b-equivalent chain to A. An element a is an endelement of a line A, if $a \in A$ and for any two elements of the line A is ayx or ayx. Evidently, a chain in S is a line in S. The relation "between" in a line has the following property:

(t_2) $\quad xyz$, yzu and $y \neq z$ imply xyu.

Let A be a line in S with an endelement a. For $x, y \in A$ set $x \prec y$ iff axy. Evidently, $(A, \prec)$ is a chain and xyz, $x, y, z \in A$, if and only if $x \prec y \prec z$ or $z \prec y \prec x$. A line $A \subset S$ is called connected when it has the following property: If $x \in S$ and if there exist elements $a, b \in A$, such that axb and $A \cup \{x\}$ is a line in S, then $x \in A$.

In paper [2] the following equivalent definition of a line is given: A subset of a lattice is a line if and only if it satisfies the following two conditions:

(i) for all three elements $x, y, z \in A$ one (at least) of the relations xyz, yzx, zxy, holds.

(ii) A does not contain a pseudolinear quadruple.

In the paper [4] there is the following statement: If a subset A of a lattice has more then four elements and satisfies the condition (i) of the preceding definition then A is a line.

Let A be a line in S. Two elements $a, b \in A$, $a \neq b$, are called *neighbouring* if $\{x \mid x \in A, axb\} = \{a, b\}$.

An interval $[a, b] \,(= \{x \in S \mid a \leqq x \leqq b\})$, $a \neq b$, is called *priminterval* if $[a, b] = \{a, b\}$. If $[a, b]$ is a priminterval we say that b covers a, and denote $a \lhd b$. Two elements $a, b \in S$ are incomparable, if neither $a \leqq b$ nor $b \leqq a$ holds, we write $a \parallel b$.

We say that the lattice S satisfies the *upper priminterval condition*, if for every two elements $a, b \in S$, $a \cap b \lhd b$ implies $a \lhd a \cup b$. Dually, we say that the lattice S satisfies the *lower priminterval condition*, if for every two elements $a, b \in S$, $a \lhd a \cup b$ implies $a \cap b \lhd b$.

Neighbouring elements in a line

Definition 1. *A line A in S has the property (α) if every two neighbouring comparable elements of A form a priminterval.*

Theorem 1. *If every connected line in a lattice S has the property (α), then the lattice S satisfies the lower and the upper priminterval conditions.*

Proof. Let $u, v \in S$, $u \parallel v$, $u \cap v \lhd v$. The elements u, $u \cup v$, v form a line. Let K be a connected line which contains the elements u, $u \cup v$, v. Let K contain an element x such that $x \neq u$, $x \neq u \cup v$, $ux(u \cup v)$. Consequently,

$$u \prec x \prec u \cup v. \tag{1}$$

Since $x \in K$, uxv. Then

$$x = (u \cap x) \cup (x \cap v) = u \cup (x \cap v). \tag{2}$$

From (1) it follows $u \cap v \leqq x \cap v \leqq v$. In view of $u \cap v \lhd v$ either $u \cap v = x \cap v$ or $x \cap v = v$. Assuming $u \cap v = x \cap v$ we get from (2) $x = u \cup (u \cap v) = u$, which cannot be by (1). If $x \cap v = v$, then by (2) $x = u \cup v$, which is impossible by (1).

Consequently, in the line K there does not exist an element x such that $ux(u \cup v)$, $u \neq x \neq u \cup v$. It means that the elements u, $u \cup v$ are neighbouring elements of the line K. Considering the fact that the line K has the property (α), we have $u \lhd u \cup v$.

We have proved the upper priminterval condition. The lower priminterval condition follows by duality.

Lemma 1. *Let A be a subset of a lattice S having the following properties:*

(i) *There exist two elements $a, b \in A$ such that $a \cap b \in A$ and $A' = [a \cap b, a] \cap A$, $A'' = [a \cap b, b] \cap A$ are chains.*

(ii) $A' \cup A'' = A$

(iii) *If $x, y \in A$ (A''), $x \geqq y$, $z \in A''(A\ )$, then xyz*
Then A is a line with endelements a, b.

Proof. The set $\breve{A}'' \oplus A'$ is a chain ($\breve{A}''$ is a dual chain to A'', $\oplus$ means the ordinal sum). We shall prove that $\breve{A}'' \oplus A'$ and A are b-equivalent. Let $\varphi\colon \breve{A}'' \oplus A' \to A$ be an identical morphism. We shall denote the relation "between,, in the chain $\breve{A}'' \oplus A'$ as $(x, y, z)\beta$. If $x, y, z \in \breve{A}''(A')$, then $(x, y, z)\beta \Leftrightarrow \varphi(x)\varphi(y)\varphi(z)$. If $x \in \breve{A}''$, $y, z \in A'$, then $(x, y, z)\beta$ implies $y \leqq z$, from which it follows $\varphi(x)\varphi(y)\varphi(z)$ by (iii). If $x, y \in \breve{A}''$, $z \in A'$, then $(x, y, z)\beta$ implies $x \geqq y$, hence $\varphi(x)\varphi(y)\varphi(z)$ by (iii). Clearly, $\varphi(x)\varphi(y)\varphi(z)$ implies $(x, y, z)\beta$.

Lemma 2. *If A is a line with endelements a, b, $a \parallel b$, $a \cap b \in A$, then $A = A' \cup A''$, where $A' = A \cap [a \cap b, a]$, $A'' = A \cap [a \cap b, b]$, A', A'' are chains.*

Proof. The line A is b-equivalent with some chain B hence there exists a bijection φ from A onto B. Let $A_1 = \{x \in A \mid \varphi(x) \leqq \varphi(a \cap b)\}$ and $A_2 = \{x \in A \mid \varphi(x) \geqq \varphi(a \cap b)\}$. Then $A = A_1 \cup A_2$. If $a \in A_1$, then $A_1 = A'$, $A_2 = A''$. If $x, y \in A'$, then $x \leqq a$, $y \leqq a$. From $axy(ayx)$ it follows $y \leqq x \leqq a$ $(x \leqq y \leqq a)$. Therefore A' is a chain.

Remark. If A is a line with endelements $a, b, a \parallel b, a \cap b \in A$, then we shall denote the set $A \cap [a \cap b, a]$ by A' and the set $A \cap [a \cap b, b]$ by A''.

Lemma 3. *Let A be a line in the lattice S with endelements $a, b, a \parallel b, a \cap b \in A$. Let an element $u \in [a \cap b, a] \cup [a \cap b, b]$ satisfy the following conditions:*
(i) *aub*
(ii) *$A' \cup \{u\}$ or $A'' \cup \{u\}$ is a chain.*
Then the set $A \cup \{u\}$ is a line.

Proof. The conditions (i), (ii) of Lemma 1 are fulfilled. Thus it remains to prove the condition (iii). Let $u \in [a \cap b, a]$. We shall consider three possibilities, the others are symmetrical.

a) If $x, y \in A''$, $x \leqq y$, then

$$x = x \cup (a \cap b) = (x \cap y) \cup (a \cap b) = (x \cap y) \cup (x \cap u). \tag{1}$$

Since A is a line and a is an endelement, then axy, hence

$$x = (x \cup a) \cap (y \cup x) \geqq (x \cup u) \cap (y \cup x) \geqq x. \tag{2}$$

uxy holds by (1) and (2).

b) Let $x \in A'$, $y \in A''$, $x \geqq u$. Considering the fact that aub, we get $u = (u \cup a) \cap (u \cup b) \geqq (u \cup x) \cap (u \cup y) \geqq u$. Since the second identity holds, trivially xuy follows.

c) Let $x \in A'$, $y \in A''$, $x \leqq u$. Since axy,

$$x = (a \cup x) \cap (x \cup y) \geqq (u \cup x) \cap (x \cup y) \geqq x.$$

The second identity holds trivially, hence uxy.

Lemma 4. *The relation xab implies $x \cap a \geqq x \cap b$.*

Proof. From xab we get $a = (x \cup a) \cap (a \cup b) \geqq x \cap b$. Hence $x \cap a \geqq x \cap b$.

Theorem 2. *Let K be a line in the lattice S with endelements a, b $a \parallel b$. Then*

$$K^{\cap} = \{a \cap x \mid x \in K\} \cup \{b \cap x \mid x \in K\}$$

is a line in S with endelements a, b.

Proof. For every element $x \in K$, axb. Hence by Lemma 4 $a \cap x \geqq a \cap b$ and $b \cap x \geqq a \cap b$. Therefore

$$K^{\cap} = (K^{\cap} \cap [a \cap b, a]) \cup (K^{\cap} \cap [a \cap b, b]),$$

which means that the condition (ii) of Lemma 1 is fulfilled. We show that the condition (i) holds too. Let $x, y \in K^{\cap} \cap [a \cap b, a]$. Hence, there exist elements $x_1, y_1 \in K$ such that

$$x = a \cap x_1, \quad y = a \cap y_1.$$

Either ax_1y_1 or ay_1x_1 holds, therefore either $a \cap x_1 \geqq a \cap y_1$ or $a \cap y_1 \geqq \geqq a \cap x_1$ by Lemma 4, hence $x \geqq y$ or $y \geqq x$. We see that the condition (i) is fulfilled.

It remains to prove the validity of the condition (iii) of Lemma 1. Let $x, y \in K^{\cap} \cap [a \cap b, a]$ and

$$x > y \tag{1}$$

and let $z \in K^{\cap} \cap [a \cap b, b]$. Hence, there exist elements $x_1, y_1, z_1 \in K$ such that

$$x = x_1 \cap a, \quad y = y_1 \cap a, \quad z = z_1 \cap b.$$

Either ax_1y_1 or ay_1x_1 holds. From the relation ay_1x_1 there follows $a \cap y_1 \geqq \geqq a \cap x_1$ by Lemma 4, hence $y \geqq x$, which is impossible by (1). Since $x \neq y$, we get $x_1 \neq y_1$. Therefore

$$ax_1y_1, \quad x_1 \neq y_1. \tag{2}$$

The elements x_1, y_1, z_1 satisfy one of the relations: a) $z_1x_1y_1$, b) $x_1z_1y_1$, c) $x_1y_1z_1$.

a) Let $z_1x_1y_1$. Since $y_1 \in K$, ay_1b. This and the relation (2) imply the relation x_1y_1b, by (t_1). From this and from the relation $z_1x_1y_1$ there follows the relation z_1y_1b, by (t_2), which implies by Lemma 4

$$b \cap y_1 \geqq b \cap z_1. \tag{3}$$

The relation (3) and ay_1b imply $y = a \cap y_1 = a \cap ((a \cap y_1) \cup (b \cap y_1)) \geqq \geqq a \cap ((a \cap y_1) \cup (b \cap z_1)) = a \cap (y \cup z) \geqq x \cap (y \cup z) = (x \cup y) \cap (y \cup z) \geqq \geqq y$, hence

$$y = (x \cup y) \cap (y \cup z).$$

Since the second identity holds trivially, we get xyz.

b) Let $x_1z_1y_1$. The relation ay_1b and (2) imply the relation x_1y_1b, by (t_1). From this and from $x_1z_1y_1$ it follows that by (t_1) z_1y_1b. From this xyz follows exactly as in the case a).

c) Let $x_1y_1z_1$. This relation and (1) imply $y \leqq (x \cup y) \cap (y \cup z) = x \cap (y \cup z) = (a \cap x_1) \cap ((a \cap y_1) \cup (b \cap z_1)) \leqq (a \cap x_1) \cap (y_1 \cup z_1) = a \cap (x_1 \cap (y_1 \cup z_1)) \leqq a \cap ((x_1 \cup y_1) \cap (y_1 \cup z_1) = a \cap y_1 = y$. The second identity is easy to prove, hence xyz.

Definition 2. *Let K be a line with endelements a, b. The pair of elements $x, y \in K$ is called a simple pair $\langle x, y\rangle$ with respect to a if axy and the elements x, y are neighbouring in the line K.*

Remark. If we shall consider a line with endelements a, b, we shall call a simple pair $\langle x, y\rangle$ with respect to a shortly a simple pair $\langle x, y\rangle$.

Lemma 5. *Let K be a line with endelements $a, b, a \parallel b$. Let $\langle x, y\rangle$ be a simple pair of the line K and $x \ngeqq y$ ($x \nleqq y$). Then $\langle x \cap b, y \cap b\rangle$ ($\langle x \cap a, y \cap a$) is a simple pair of the line $K^\cap$.*

Proof. We suppose $x \ngeqq y$. Evidently, axy and ayb. From these two relations there follows by (t_1)

$$xyb. \tag{1}$$

This implies by Lemma 4

$$y \cap b \geqq x \cap b, \quad x \cap y \geqq x \cap b. \tag{2}$$

If $y \cap b = x \cap b$, then (1), (2) imply $y = (x \cap y) \cup (y \cap b) = (x \cap y) \cup \cup (x \cap b) = (x \cap y)$. Hence $y \leqq x$, which is impossible (we have supposed $x \ngeqq y$). Hence $x \cap b < y \cap b$. The line $K^\cap$ fulfils the conditions (i), (ii), (iii) of Lemma 1 (see the proof of Theorem 2). From the condition (iii) it follows

$$a(x \cap b)\,(y \cap b).$$

It remains to show that the elements $x \cap b$, $y \cap b$ are neighbouring in the line $K^\cap$. If $c \in K^\cap$ exists such that

$$y \cap b > c > x \cap b \tag{3}$$

then, since $c \in K^{\cap\prime\prime}$, there exists an element $c_1 \in K$ such that $c = c_1 \cap b$. Since x, y are neighbouring elements of the line K, either c_1xy or xyc_1. The relation c_1xy cannot hold, because the relations c_1xy and xyb imply c_1xb by (t_2), which implies $b \cap x \geqq b \cap c_1 = c$, contrary to (3). On the other hand the relations xyc_1 and axy imply ayc_1 by (t_2). From this and from ac_1b we have yc_1b by (t_1). By Lemma 4 $c = b \cap c_1 \geqq b \cap y$, which is also impossible. We see that the elements $x \cap b$, $y \cap b$ are neighbouring in the line $K^\cap$. The assertion in the brackets can be proved analogously.

Lemma 6. *Let K be a line with endelements $a, b, a \parallel b$. Let $\langle x_i, y_i\rangle$, $i = 1, 2$, be two simple pairs of the line K different from each other.* If $x_i \ngeqq y_i$, $i = 1, 2$ ($x_i \nleqq y_i$, $i = 1, 2$), *then $b \cap x_1 \neq b \cap x_2$ ($a \cap y_1 \neq a \cap y_2$).*

Proof. By assumption, ax_iy_i, $i = 1, 2$. Since ay_ib, $i = 1, 2$, we get by (t_1)

$$x_iy_ib, \quad i = 1, 2. \tag{1}$$

Since the pairs $\langle x_1, y_1\rangle$, $\langle x_2, y_2\rangle$ are different from each other, $x_1 \neq x_2$. Let

$$(2) \qquad b \cap x_1 = b \cap x_2 .$$

We can consider x_1x_2b, because the case x_2x_1b is symmetrical. The relations x_1x_2b and (2) imply $x_2 = (x_2 \cap x_1) \cup (x_2 \cap b) = (x_2 \cap x_1) \cup (x_1 \cap b) \leqq x_1$, hence

$$(3) \qquad x_2 \leqq x_1 .$$

Considering the fact, that x_1x_2b and x_1, y_1 are neighbouring elements, we get $x_1y_1x_2$. This and (3) gives

$$y_1 = (x_1 \cap y_1) \cup (y_1 \cap x_2) = (x_1 \cap y_1) .$$

Hence $y_1 \leqq x_1$, which contradicts the assumption. The assertion in the brackets can be proved analogously.

Definition 3. *Let K be a finite line. The length* $\mathrm{d}K$ *of the line K is the number of its simple pairs.*

Definition 4. *The line K has the property (β), if any two neighbouring elements of the line K are comparable.*

Theorem 3. *Let K be a finite line of the lattice S with endelements $a, b, a \parallel b$. Then*

a) *if the line K has the property (β), then* $\mathrm{d}K = \mathrm{d}K^{\cap}$.

b) *if the line K has not the property (β), then* $\mathrm{d}K < \mathrm{d}K^{\cap}$.

Proof. Let us denote the set of all simple pairs of the line $K(K^{\cap})$ by $K^*(K^{\cap *})$. Let us define a map φ from the set K^* into the set $K^{\cap *}$ as follows. Let $\langle x, y\rangle \in K^*$. If $x > y$, then $\varphi(\langle x, y\rangle) = \langle a \cap x, a \cap y\rangle$ and if $x \ngeqq y$, then $\varphi(\langle x, y\rangle) = \langle b \cap x, b \cap y\rangle$. By Lemma 5 $\varphi(\langle x, y\rangle)$ are simple pairs of the line $K^{\cap}$. We show that the map φ is $1 - 1$. Let $\langle x_1, y_1\rangle \neq \langle x_2, y_2\rangle$. If $x_1 \not> y_1$, $x_2 \not> y_2$ or $x_1 > y_1$, $x_2 > y_2$, then $\varphi(\langle x_1, y_1\rangle) \neq \varphi(\langle x_2, y_2\rangle)$ by Lemma 6. If $x_1 \ngeqq y_1, x_2 > y_2$, then $\varphi(\langle x_1, y_1\rangle) = \langle b \cap x_1, b \cap y_1\rangle \in K''^*$ and $\varphi(\langle x_2, y_2\rangle = = \langle a \cap x_2, a \cap y_2\rangle \in K'$. The case $x_1 > y_1$, $x_2 \ngeqq y_2$ is similar to the preceding case. We have already proved that $\langle x_1, y_1\rangle \neq \langle x_2, y_2\rangle$ implies $\varphi(\langle x_1, y_1\rangle) \neq \varphi(\langle x_2, y_2\rangle)$.

We first assume that the line K has the property (β). We show that φ is a map from the set K^* onto the set $K^{\cap *}$. Let $\langle c, d\rangle \in K^{\cap *}$. By Lemma 2 either $c \in K^{\cap\prime}$ or $c \in K^{\cap\prime\prime}$. Let, for example, $c \in K^{\cap\prime\prime}$. Then

$$(1) \qquad a \cap b \leqq c < d .$$

Since $c, d \in K^{\cap\prime\prime}$, there exist $x, y \in K$ such that

$$(2) \qquad c = x \cap b, \quad d = y \cap b .$$

If there were yxb, then by Lemma 4 there would be $x \cap b \geqq y \cap b$, hence $c \geqq d$, which contradicts (1). Consequently xyb and since axb, hence by (t_1)

$$axy. \tag{3}$$

The relation $x \geqq y$ implies $x \cap b \geqq y \cap b$, hence $c \geqq d$, which contradicts (1). Hence $x \ngeqq y$. Since the line K is finite, there exist elements x_i ($i = 1, 2, \ldots, n$) such that

$$x = x_1 \prec x_2 \prec \ldots \prec x_{n-1} \prec x_n = y$$

($x \prec y \Leftrightarrow axy, x, y \in K$) and $\langle x_i, x_{i+1} \rangle$, $i = 1, 2, \ldots, n - 1$, are simple pairs. From $x \ngeqq y$ and from the property (β) it follows that there exist an i, $1 \leqq \leqq i \leqq n - 1$, such that

$$x_i < x_{i+1}. \tag{4}$$

In view of $x \prec x_i \prec x_{i+1} \prec y \prec b$ there holds xx_ib and $x_{i+1}yb$. Hence by Lemma 4 it follows

$$c = x \cap b \leqq x_i \cap b \leqq x_{i+1} \cap b \leqq y \cap b = d.$$

By Lemma 5 $\varphi(\langle x_i, x_{i+1} \rangle) = \langle x_i \cap b, x_{i+1} \cap b \rangle$, hence $x_i \cap b \neq x_{i+1} \cap b$. Considering the fact that $\langle c, d \rangle$ is a simple pair, we see that

$$\langle c, d \rangle = \langle x_i \cap b, x_{i+1} \cap b \rangle = \varphi(\langle x_i, x_{i+1} \rangle).$$

Let us assume that the line K has not the property (β). Then there exist two neighbouring elements c, d of the line K, which are incomparable. Let acd. Since $c \ngeqq d$, $\varphi(\langle c, d \rangle) = \langle b \cap c, b \cap d \rangle$. By Lemma 5 the elements $a \cap c$, $a \cap d$ form the simple pair $\langle a \cap c, a \cap d \rangle$. Let $\langle x, y \rangle \in K^*$ such that $\varphi(\langle x.y \rangle) = = \langle a \cap x, a \cap y \rangle = \langle a \cap c, a \cap d \rangle$. Then $x > y$. Since $c \nleqq d$, $x \nleqq y$ and $c \parallel d$, it follows by Lemma 6 that $a \cap d \neq a \cap y$. But this contradicts the fact that $\varphi(\langle x, y \rangle) = < a \cap x, a \cap y \rangle$. Hence no simple pairs are mapped on the simple pair $\langle a \cap c, a \cap d \rangle$. This gives $\mathrm{d}K < \mathrm{d}K^{\cap}$.

Remark. The last theorem shows that if a finite lattice contained a connected line K with endelements a, b, which has not the property (β), then in this lattice the Jordan—Hölder Theorem for lines would not hold. Let us find a sufficient condition that every line of the lattice S have the property (β).

Definition 5. *A lattice S is partly modular (p-modular), iff for every $a, b, a_1, b_1 \in S$,* which satisfy the condition

$$(a_1 \cup b) \cap a = a_1, \quad (a \cup b_1) \cap b = b_1, \tag{1}$$

we have $a_1 \cup b_1 = (a_1 \cup b) \cap (a \cup b_1)$.

Theorem 4. *A lattice S is p-modular of and only if it does not contain a sublattice with the diagram of Figure* 1.

Proof. If a lattice contains a sublattice of Figure 1, then by Definition 5 it is not p-modular.

Now we assume that the lattice S does not contain a sublattice with the diagram of Figure 1. Let $a, b, a_1, b_1 \in S$ and let (1) of the Definition 5 hold. If $a \leqq b$, then by (1), $a_1 = a$ and $(a \cup b_1) \cap b \leqq (b \cup b_1) \cap b = b$, hence $b_1 \leqq b$. Then $a_1 \cup b_1 = a \cup b_1 = (a \cup b) \cap (a \cup b_1) = (a_1 \cup b) \cap (a \cup b_1)$. The case $a \geqq b$ is symmetrical. Let $a \parallel b$. Let us denote $a_2 = a \cup b_1$, $b_2 = a_1 \cup b$. Then from (1) it follows

$$(2) \qquad a \cap b \leqq a_1 \leqq a \leqq a_2 \leqq a \cup b, \quad a \cap b \leqq b_1 \leqq b \leqq b_2 \leqq a \cup b$$

and also $a_1 \cup b_1 \leqq a_2 \cap b_2$. If $a_1 \cup b_1 < a_2 \cap b_2$ and no two elements would be equal in (2), then the sublattice of the lattice S, generated by the elements a, b, a_1, b_1, a_2, b_2, would have the diagram of Figure 1. Therefore $a_1 \cup b_1 = a_2 \cap b_2$. If some two elements are equal in (2), then it is easy to prove that $a_1 \cup b_1 = a_2 \cap b_2$.

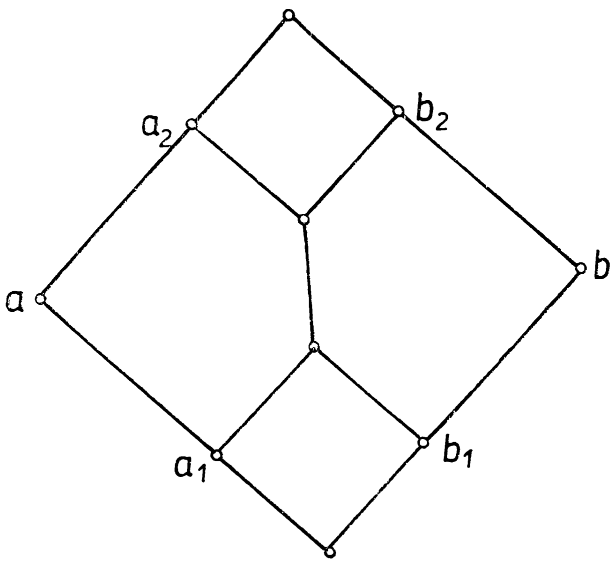

Fig. 1

Theorem 5. *A lattice is p-modular if and only if it satisfies one of the following conditions.*

(i) *For every $a, b, a_1, b_1 \in S$, $a \parallel b$: If $\{a, a_1, b_1, b\}$ is a line with endelements a, b, then $a(a_1 \cup b_1)b$.*

(ii) *For every* $a, b, a_1, b_1 \in S$, $a \parallel b$: *If* $\{a, a_1, b_1, b\}$ *is a line with endelements* a, b, *then* $a(a_1 \cap b_1)b$.

(iii) *For every* $a, b, a_1, b_1, a_2, b_2 \in S$, $a \parallel b$: *If*

$a_2 \cap b = b_1, b_2 \cap a = a_1, a_1 \cup b = b_2, b_1 \cup a = a_2$, *then* $a_1 \cup b_1 = a_2 \cap b_2$.

Proof. Clearly, a lattice is p-modular if and only if it satisfies the condition (iii) (see the proof of the Theorem 4).

We shall prove that the conditions (iii) and (ii) are equivalent. Let lattice S satisfy the condition (ii) and let the elements $a, b, a_1, b_1, a_2, b_2 \in S$, $a \quad b$ satisfy the conditions

$$a_2 \cap b = b_1, \quad b_2 \cap a = a_1, \quad a_1 \cup b = b_2, \quad b_1 \cup a = a_2. \tag{1}$$

If $a_1 = a$, then $b_2 = a_1 \cup b = a \cup b \geqq a_2$. Hence $a_2 \cap b_2 = a_2 = a \cup b_1 = a_1 \cup b_1$ and condition (iii) is fulfilled. The case $b_1 = b$ is analogous. Suppose now that $a_1 \neq a$, $b_1 \neq b$. We show that $\{a, a_2, b_2, b\}$ is a line withend elements a, b. According to the suppositions (1) there holds $a_2 \leqq (a \cup a_2) \cap (b \cup a_2) = a_2 \cap (b \cup a_2) = a_2$ and $a_2 \geqq (a \cap a_2) \cup (b \cap a_2) = a \cup b_1 = a_2$. Therefore aa_2b. In a similar manner it can be shown that ab_2b. Therefore the set $\{a, a_2, a \cup b, b_2, b\}$ forms a line by the dual statement to Lemma 3. Clearly, the set $\{a, a_2, b_2, b\}$ forms a line, hence $a(a_2 \cap b_2)b$. From this it follows

$$a_2 \cap b_2 = (a \cap a_2 \cap b_2) \cup (a_2 \cap b_2 \cap b) =$$

$$= (a \cap (b_1 \cup a) \cap b_2) \cup (a_2 \cap (a_1 \cup b) \cap b) = (a \cap b_2) \cup (a_2 \cap b) =$$

$$= a_1 \cup b_1.$$

(We have applied the relations (1)). We get $a_2 \cap b_2 = a_1 \cup b_1$, as claimed.

Suppose now that the lattice S satisfies the condition (iii). Let the set $\{a, a_1, b_1, b\}$ be a line with endelements a, b. Well shall prove that the elements

$$a, b, (b \cup b_1) \cap a, (a \cup a_1) \cap b, a \cup a_1, b \cup b_1$$

satisfy the conditions (1). Evidently, the first two conditions are fulfilled. Since ab_1b, aa_1b, it follows

$$(b \cup b_1) \cap a = (b \cup b_1) \cap ((b_1 \cup a) \cap a) = b_1 \cap a, \tag{2}$$

$$(a \cup a_1) \cap b = (a \cup a_1) \cap ((a_1 \cup b) \cap b) = a_1 \cap b. \tag{3}$$

(2) gives

$$((b \cup b_1) \cap a) \cup b = (b_1 \cap a) \cup b = (b_1 \cap a) \cup (b_1 \cap b) \cup b = \tag{4}$$

$$= b_1 \cup b.$$

Analogously (3) implies

$$((a \cup a_1) \cap b) \cup a = a_1 \cup a. \tag{5}$$

The relations (4), (5) are the second two conditions of (1) for our elements. Since the lattice S satisfies the condition (iii), we get

$$(a \cup a_1) \cap (b \cup b_1) = ((a \cup a_1) \cap b) \cup ((b \cup b_1) \cap a). \tag{6}$$

(2), (3), (6) yield

$$(a \cup a_1) \cap (b \cup b_1) = (a_1 \cap b) \cup (a \cap b_1). \tag{7}$$

Since aa_1b_1, a_1b_1b and (7) holds, we get $a_1 \cap b_1 = ((a \cup a_1) \cap (a_1 \cup b_1)) \cap$ $\cap ((b \cup b_1) \cap (b_1 \cup a_1)) = (a \cup a_1) \cap (b \cup b_1) \cap (a_1 \cup b_1) = ((a_1 \cap b) \cup$ $\cup (a \cap b_1)) \cap (b_1 \cup a_1) = (a_1 \cap b) \cup (a \cap b_1)$. From this and from (6), (7) it follows $(a \cup a_1) \cap (b \cup b_1) = a_1 \cap b_1 = ((a \cup a_1) \cap b) \cup ((b \cup b_1) \cap a)$. From this relation we get

$$a_1 \cap b_1 \leqq (a \cup (a_1 \cap b_1)) \cap (b \cup (a_1 \cap b_1)) \leqq (a \cup a_1) \cap (b \cup b_1) = a_1 \cap b_1,$$

$$a_1 \cap b_1 \geqq (a \cap a_1 \cap b_1) \cup (b \cap a_1 \cap b_1) = (a \cap (b \cup b_1)) \cup (b \cap (a \cup a_1)) =$$
$$= a_1 \cap b_1.$$

From the two last relations we get $a(a_1 \cap b_1)b$, which proves our assertion.

The equivalency of the conditions (i), (iii) can be proved analogously.

Lemma 9. *If xab, then $x(a \cap b)b$ ($x(a \cup b)b$).*

Proof. From the relation xab it follows

$$x \cup a = x \cup (a \cap x) \cup (a \cap b) = (a \cap b) \cup x,$$

$$a \cap b = (a \cup x) \cap (a \cup b) \cap b = (a \cup x) \cap b.$$

These two relations imply

$$a \cap b \leqq ((a \cap b) \cup x) \cap ((a \cap b) \cup b) = (a \cup x) \cap b = a \cap b.$$

Therefore $a \cap b = ((a \cap b) \cup x) \cap ((a \cap b) \cup b)$. The dual relation is evident, hence $x(a \cap b)b$. The assertion in brackets can be obtained by duality.

Lemma 10. *If the elements x, y, a, b belong to a line K, and xab, xya, then $xy(a \cap b)$ ($xy(a \cup b)$).*

Corollary. *The relation xab implies $xa(a \cap b)$ ($xa(a \cup b)$).*

Proof. From xab and xya it follows that yab by (t_1). The last relation and xya gives xyb by (t_2) (if $y = a$, then xyb, too). By the preceding Lemma from yab it follows that $y(a \cap b)b$. But this and xyb imply $xy(a \cap b)$ by (t_1). The assertion in brackets is dual.

Theorem 6. *Let S be a p-modular lattice. Then every connected line in the lattice S has the property (β), which means that any two neighbouring elements of any connected line are comparable.*

Proof. Let S be a p-modular lattice. Let K be a connected line in S, which has not the property (β). Hence there exist $a, b \in K$, $a \parallel b$, a, b neighbouring elements in the line K. We shall prove that $\{a \cap b\} \cup K$ is a line. To this end, it is sufficient to show:

1. For any $x, y \in K$ one of the relations holds: $x(a \cap b)y$, $xy(a \cap b)$, $yx(a \cap b)$.
2. If the set $\{a \cap b\} \cup K$ contains exactly four elements, then these elements do not form a pseudolinear quadruple.

We first prove the assertion 1. We have considered the following cases. 1. xab, yab, 2. xab, aby, 3. yab, abx, 4. abx, aby. In view of the symmetry it suffices to consider the cases 1. and 2. In the first case if xya, then the relations xab, xya imply $xy(a \cap b)$ by Lemma 10. If yxa, then from yab it follows by Lemma 10 that $yx(a \cap b)$. If xay, then $x \prec a \prec y$ or $y \prec a \prec x$. From the suppositions $a \neq b$ and 1 it follows that $x \prec a \prec b$ and $y \prec a \prec b$ or $b \prec a \prec x$ and $b \prec a \prec y$. Hence $x = a$ or $y = a$. Therefore yxa or xya, which was considered. In the case 2 the set $\{x, a, b, y\}$ forms a line. Since the lattice S is p-modular and $x \parallel y$ (if $x \leqq y$ or $y \leqq x$, then $\{x, a, b, y\}$ is a chain contrary to $a \parallel b$) we get $x(a \cap b)y$ by Theorem 5, (ii).

We show the validity of 2. Since $a \parallel b$, it cannot be $ab(a \cap b)$ or $ba(a \cap b)$. Hence we have $a(a \cap b)b$. Therefore the elements $a, b, a \cap b, c$ of the set $\{a \cap b\} \cup K$ can form a pseudolinear quadruple only in this way:

$$a(a \cap b)b, \ (a \cap b)bc, \ bca, \ ca(a \cap b).$$

The relation bca contradicts the fact that the elements a, b are neighbouring in K.

Since $a \parallel b$, we have $K \cup \{a \cap b\} \supsetneqq K$, which is a contradiction to the supposition that K is a connected line.

Jordan—Hölder Theorem for Lines

Lemma 11. *If abx, aby and there exist an element u such that $x \leqq u \leqq y$, then abu.*

Proof. $b \leqq (a \cup b) \cap (b \cup u) \leqq (a \cup b) \cap (b \cup y) = b,$

$$b \geqq (a \cap b) \cup (b \cap u) \geqq (a \cap b) \cup (b \cap x) = b.$$

Theorem 7. *Let S be a p-modular lattice. Let K be a finite connected line with endelements $a, b \in S$, $a \parallel b$. Then the line $K^{\cap}$ is connected.*

Proof. If $K^{\cap}$ is not connected then there exist elements $a_1, b_1, c \in S$ such that $a_1, b_1 \in K^\cap$, $c \notin K^\cap$, $a_1 c b_1$ and $K^\cap \cup \{c\}$ is a line. Hence $a \prec a_1 \prec c \prec$ $\prec b_1 \prec b$, whence

$$acb. \tag{1}$$

Since the line $K^\cap$ is finite, there exist $x, y \in K^\cap$ such that x, y are neighbouring elements of the line $K^\cap$ and xcy. Let $x, y \in K^{\cap\prime\prime}$ and let $x < y$. Then

$$x < c < y \leqq b. \tag{2}$$

We shall show that there exists an element $u \in S$, $u \notin K$, such that aub and $K \cup \{u\}$ is a line, hence the line K is not connected, which is a contradiction with the supposition. Therefore the hypothesis that the line $K^\cap$ is not connected is contradictory.

Since the lattice S is p-modular and the elements x, y form a simple pair $\langle x, y\rangle$, by Theorem 3 there exists $\langle x_1, y_1\rangle \in K^*$ such that $\varphi(\langle x_1, y_1\rangle) = \langle x, y\rangle$. From the construction of the map φ it follows that

$$x_1 < y_1, \ x = x_1 \cap b, \ y = y_1 \cap b. \tag{3}$$

Let $u = x_1 \cup c$. Since $x_1 < y_1$ and $c < y \leqq y_1$ we get

$$x_1 \leqq u \leqq y_1. \tag{4}$$

Further, we shall show that

$$x_1 \cup c = u = (a \cup c) \cap y_1. \tag{5}$$

Since axy, we get by Corollary of Lemma 10 that $(a \cup x)xy$. Therefore $\{a \cup x, x, y\}$ is a line. The relation axy implies $x = (a \cup x) \cap (x \cup y) =$ $= (a \cup x) \cap y$ and this implies $a \cup x \parallel y$. If $(a \cup x)cy$, then $\{a \cup x, x, c, y\}$ is a line by Lemma 3. But axy and xcy imply axc by (t_1), axc and xcy ($x \neq c$, $a, x, c, y \in K^\cap \cup \{c\}$) imply acy by (t_2), acy implies by Corollary of Lemma 10 $(a \cup c)cy$. Finally, acy, $(a \cup c)cy$ and $a \leqq a \cup x \leqq a \cup c$ imply $(a \cup x)cy$ by Lemma 11. Proving $x \leqq x_1 \leqq a \cup x$ and $(a \cup x)x_1y$ we get that $\{a \cup x, x_1, x, c, y\}$ is a line. But, since $x = x_1 \cap b$ (see (3)) and ax_1b ($x_1 \in K$), we have

$$x \cup a = (x_1 \cap b) \cup ((x_1 \cap a) \cup a) = x_1 \cup a, \tag{6}$$

hence $x_1 \leqq a \cup x$. (Clearly $x \leqq x_1$). Since $\langle x_1, y_1\rangle$ is a simple pair, we have x_1y_1b, which yields, by Lemma 9, $x_1(y_1 \cap b)b$, hence x_1yb (see (3)). The relations ax_1b, x_1yb imply ax_1y by (t_1), whence by Corollary of Lemma 10 $(a \cup x_1)x_1y$. Hence

$$(a \cup x)x_1y$$

by (6). Since $\{a \cup x, x_1, x, c, y\}$ is a line, the set $\{a \cup x, x_1, c, y\}$ is a line too. Since the lattice S is p-modular it follows by Theorem 5 (1) that

$$(a \cup x)\,(x_1 \cup c)y.$$

In view of this and of (2), (4), (6), x_1y_1b we have

$$x_1 \cup c = (a \cup x \cup x_1 \cup c) \cap (x_1 \cup c \cup y) = (a \cup x \cup c) \cap (x_1 \cup y) =$$
$$= (a \cup c) \cap ((x_1 \cap y_1) \cup (y_1 \cap b)) = (a \cup c) \cap y_1.$$

This proves (5).

Next we show that

$$(7) \qquad aub.$$

Since

$$c \geqq (x_1 \cap c) \cup (c \cap b) = (x_1 \cap c) \cup c = c$$

and, with respect to (5), (3), (2) and acy,

$$c \leqq (x_1 \cup c) \cap (c \cup b) = (a \cup c) \cap y_1 \cap b = (a \cup c) \cap y =$$
$$= (a \cup c) \cap (c \cup y) = c,$$

we get x_1cb. From ax_1b ($x_1 \in K$) and x_1cb it follows that ax_1c by (t_1). Since ax_1c, x_1cb, ax_1b and acb (see (1)) and the elements a, x_1, c, b do not form a pseudolinear quadruple (the relation cba is not possible) the set $\{a, x_1, c, b\}$ is a line. By Theorem 5, (i) $a(x_1 \cup c)b$, therefore the relation (7) is proved.

We shall now show that

$$(8) \qquad x_1 \neq u, \quad u \neq y_1.$$

If $u = x_1$, then $c \leqq c \cup x_1 = u = x_1$ and $c \leqq b$ (see (2)), hence $c \leqq x_1 \cap b = x$, contrary to (2). If $u = y_1$, then $(c \cup a) \cap y_1$ (see (5)) hence $y_1 \leqq c \cup a$. In view of acb we have $c = (a \cup c) \cap (b \cup c) \geqq y_1 \cap b = y$, thus $c \geqq y$, which contradicts (2).

It remains to show that $K \cup \{u\}$ is a line, that is:

A) For any $e, f \in K$ one of the relations euf, efu, feu holds.

B) If the set $K \cup \{u\}$ contains exactly four elements, then these elements do not form a pseudolinear quadruple.

A) We have to consider the following cases a) ex_1y_1, fx_1y_1, b) ex_1y_1, x_1y_1f. The other two cases are symmetrical.

a) Let ex_1y_1, fx_1y_1 and let efx_1. Then efy_1. The relations efx_1, efy_1 and $x_1 \leqq u \leqq y_1$ (see (4)) imply efu by Lemma 11. If fex_1 then, analogously, feu.

b) Let ex_1y_1, x_1y_1f. Since ax_1y_1, there exists a linear ordering of the line K such that $a \prec e \prec x_1 \prec y_1 \prec f \prec b$. This implies aex_1 and aey_1. Since $x_1 \leqq$

$\leqq u \leqq y_1$ by (4), *aeu* by Lemma 11 holds. Analogously it can be shown that *ufb*. These two relations imply, by Lemma 4,

(9) $\quad u \cap f \geqq u \cap b,\ u \cup f \leqq u \cup b,\ u \cup e \geqq u \cap a,\ u \cup e \leqq u \cup a.$

This and the relation *aub* (see (7)) imply

$$u \leqq (u \cup f) \cap (u \cup e) \leqq (u \cup a) \cap (u \cup b) = u,$$

$$u \geqq (u \cap e) \cup (u \cap f) \geqq (u \cap a) \cup (u \cap b) = u.$$

This proves that *euf*.

B) The set $K \cup \{u\}$ contains the elements x_1, y_1, u and let it contain the element $t \in K$, $t \neq x_1$, $t \neq y_1$. Since $x_1 \leqq u \leqq y_1$ and the elements x_1, y_1 form a simple pair, the elements x_1, y_1, u, t can form a pseulinear quadruple only in this way

$$x_1uy_1,\ uy_1t,\ y_1tx_1,\ tx_1u.$$

The relation y_1tx_1 contradicts the supposition.

Definition 6. *A lattice S satisfies the condition (γ), if to any two elements $a, b \in S$, $a \parallel b$ and to any connected finite line with endelements a, b there exists a connected line with the same length and the same endelements, containing the element $a \cap b$.*

Theorem 8. *A lattice S is p-modular if and only if any its sublattice satisfies the condition (γ).*

Proof. If the lattice S is not p-modular, then it contains a sublattice with the diagram of Figure 1, by Theorem 4. The line $\{a, a_1, b_2, b\}$ and the line $\{a, a_1, a \cap b, b_1, b\}$ are connected and they have different lengths.

Let the lattice S be p-modular. Let K be a finite connected line with endelements $a, b, a \parallel b$. The line $K^\cap$ is connected by Theorem 7. The line K has the property (β) according to Theorem 6, hence $\mathrm{d}K = \mathrm{d}K^\cap$ by Theorem 3. We found to K a connected line $K^\cap$ with endelements $a, b, a \cap b \in K^\cap$ and with the same length.

Definition 7. *A lattice S is upper semimodular if to any three elements $a, b, x \in S$*

(1) $$a \parallel b, \quad a \cup b > x > a$$

there exists at least one t such that

$$a \cup b > t \geqq b \quad \text{and} \quad (x \cap t) \cup a = x.$$

(Definition 7 is from [3]).

Remark. Every modular lattice is p-modular, but the lattice of Figure 2 is p-modular and is neither upper semimodular nor modular. The lattice of Figure 3 is upper semimodular, but is not p-modular.

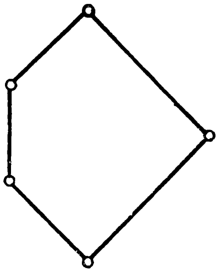

Fig. 2

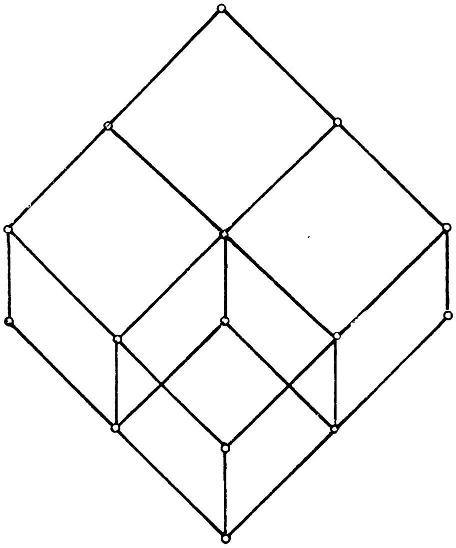

Fig. 3

Theorem 9. *Let S be an upper semimodular lattice. If K is a connected line in S with endelements $a, b \in S$, $a \parallel b$, $a \cap b \in K$, then for any two neighbouring elements $c, d \in K$ either $c \lhd d$ or $d \lhd c$.*

Proof. Let, for instance, $c, d \in K'(= K \cap [a \cap b, a])$ and let $c > d$, where c, d are neighbouring elements in K. Let $[d, c]$ not form a priminterval, thus there exists an element $u \in S$, $c > u > d$. Since $c, d \in K'$, we get

$$a \geqq c > u > d \geqq a \cap b. \tag{1}$$

Since the line K is connected the set $K \cup \{u\}$ does not form a line. The condition (ii) of Lemma 3 is fulfilled, hence the condition (i) of Lemma 3 is not fulfilled and the relation aub does not hold. Hence

$$u < a \cap (u \cup b). \tag{2}$$

Since $c \in K$, we get acb. This and (1) gives

$$c = (a \cup c) \cap (c \cup b) = a \cap (b \cup c) \geqq a \cap (b \cup u).$$

Therefore either a) $c > a \cap (b \cup u)$ or b) $c = a \cap (b \cup u)$.

a) Let $t = a \cap (b \cup u)$, hence $c > t$ and $t > u$ by (2). This and (1) gives

$$c > t > d. \tag{3}$$

We show that atb: $t \geqq (a \cap t) \cup (b \cap t) \geqq (c \cap t) \cup (b \cap t) = t \cup (b \cap t) = t$, $t \leqq (a \cup t) \cap (b \cup t) \leqq (a \cup c) \cap (b \cup (a \cap (b \cup u))) \leqq a \cap (b \cup (b \cup u)) = $ $= a \cap (b \cup u) = t$ (see (1), (3)).

The set $K \cup \{t\}$ is a line by Lemma 3 and $t \notin K$ which contradicts the assumption.

b) Let $c = a \cap (b \cup u)$. We show that the conditions (1) of Definition 7 are fulfilled by the elements $u, b \cup d, c$. We first show that $u \parallel b \cup d$. If $u \leqq$ $\leqq b \cup d$, then $d = (a \cup d) \cap (b \cup d) \geqq a \cap u = u$ (since $d \in K$, we have adb and $a > u$ by (1)), thus $d \geqq u$, which contradicts (1). If $b \cup d \leqq u$, then $b \leqq u$. This and $u < a$ (see (1)) give $b < a$, contrary to the assumption. Consequently, $u \parallel b \cup d$. From $c = a \cap (b \cup u)$ it follows that $c \leqq b \cup u$. If $c = b \cup u$, then $b \leqq c \leqq a$ contrary to the assumption. Since $c < b \cup u$ and $u < c$ ((1)), we get

$$(b \cup d) \cup u = b \cup u > c > u.$$

Since the elements $u, b \cup d, c$ satisfy the conditions (1) of Definition 7 and the lattice S is upper semimodular, there exists an element z such that

$$b \cup u > z \geqq b \cup d \tag{4}$$

and

$$(c \cap z) \cup u = c. \tag{5}$$

Thus $c \geqq c \cap z$. If $c \cap z = c$, then $c \cup z = z$. Combining the relations (4), (1) and $c \cup z = z$ we get

$$b \cup u = (b \cup d) \cup u \leqq z \cup u \leqq z \cup c = z,$$

hence $b \cup u \leqq z$, which contradicts (4).

Therefore $c > c \cap z$. According to (4) $z \geqq b \cup d$, hence $z \geqq d$. This gives $c \cap z \geqq c \cap d = d$. If $c \cap z = d$, the relation (5) would not hold. We have shown that

$$c > c \cap z > d. \tag{6}$$

We next show that $a(c \cap z)b$. Since $b \cup (c \cap z) \leqq (b \cup d) \cup (c \cap z) \leqq z \cup$ $\cup (c \cap z) = z$ and $a \cap (b \cup (c \cap z)) \leqq a \cap (b \cup c) = c$, we get $a \cap (b \cup$ $\cup (c \cap z)) \leqq c \cap z$. (We have used the relations (1), (4) and acb). Then

$$c \cap z \leqq (a \cup (c \cap z)) \cap (b \cup (c \cap z)) = a \cap (b \cup (c \cap z)) \leqq c \cap z.$$

It is easy to prove the second identity. We have proved that the element $c \cap z$ satisfies the suppositions of Lemma 3. Hence $K \cup \{c \cap z\}$ is a line, $c \cap z \notin K$, which contradicts the fact that K is a connected line. Hence the assumption that $[d, c]$ is not a priminterval is contradictory.

Corollary. *If a lattice S is upper semimodular and K is a connected line with endelements $a, b, a \| b, a \cap b \in K$, then $K = K' \cup K''$, where K', K'' are connected chains between $a, a \cap b$ and $b, a \cap b$.*

Remark. Two intervals of a lattice are called *transposes* when they can be written as $[a \cap b, a]$ and $[b, a \cup b]$ for suitable a, b. Likewise, two intervals $[x, y]$ and $[x', y']$ are called *projective* if and only if there exists a finite sequence $[x, y], [x_1, y_1], \ldots, [x', y']$ in which any two successive intervals are transposes. From paper [1] it follows that the following theorem is true.

Let the lattice S be upper semimodular, K, L be connected chains in S with endelements a, b $(a < b)$ and K be a finite chain, then the following holds:

1. *The chain L is finite and has the same length as K.*

2. *There exists a $1 - 1$ mapping of the primintervals of the chain K onto the primintervals of the chain L such that the corresponding primintervals are projective.*

Lemma 12. *Let the lattice S be p-modular and upper semimodular. Let K, L be finite connected lines with endelements a, b. Then there exists a $1 - 1$ correspondence between the set of simple pairs of the line K and the set of simple pairs of the line L such that the corresponding simple pairs are projective.*

Proof. We shall say that the lines are in the relation $\mathscr{P}$, if there exists a $1 - 1$ mapping of the set K^* onto the set L^* such that the corresponding simple pairs are projective. We show that $L \mathscr{P} L^{\cap}$. According to Theorem 3 and Theorem 6 there exists a $1 - 1$ mapping φ of the set L^* onto the set $L^{\cap *}$.

Let $\varphi(\langle x, y\rangle) = \langle b \cap x, b \cap y\rangle$. In view of the definition of the mapping φ in the proof of Theorem 3 we have $x < y$, $b \cap x < b \cap y$. We have

$$x \cap (b \cap y) = x \cap y \cap b = x \cap b .$$

Since $\langle x, y\rangle$ is a simple pair, we get axy. From this it follows that xyb, hence

$$x \cup (b \cap y) = (x \cap y) \cup (y \cap b) = y .$$

Therefore the simple pairs $\langle x, y\rangle$, $\langle b \cap x, b \cap y\rangle$ are transposed. Analogously, if $\varphi(\langle x, y\rangle) = \langle a \cap x, a \cap y\rangle$, then $\langle x, y\rangle$, $\langle a \cap x, a \cap y\rangle$ are transposed, hence $L \mathscr{P} L^{\cap}$.

Since the lattice S is upper semimodular, in view of the Corollary to Theorem 9 $L^{\cap} = L^{\cap\prime} \cup L^{\cap\prime\prime}$, where $L^{\cap\prime}, L^{\cap\prime\prime}$ are connected chains between $a \cap b, a$ and $a \cap b, b$. Analogously, $K^{\cap\prime}$, $K^{\cap\prime\prime}$ are connected chains between $a \cap b, a$ and $a \cap b, b$. According to Remark following Theorem 9, $L^{\cap\prime} \mathscr{P} K^{\cap\prime}$, $L^{\cap\prime\prime} \mathscr{P} K^{\cap\prime\prime}$, hence $L^{\cap} \mathscr{P} K^{\cap}$.

Since $L \mathscr{P} L^{\cap}$ and $K \mathscr{P} K^{\cap}$, $L^{\cap} \mathscr{P} K^{\cap}$ and the relation $\mathscr{P}$ is symmetrical and transitive, we have $L \mathscr{P} K$ as claimed.

Lemma 13. *Let L be an infinite line with endelements $a, b, a \| b$. Then the line $L^\cap$ is infinite too.*

Proof. Let us map any element $x \in L$ onto the ordered pair $(a \cap x, b \cap x)$: $\varphi(x) = (a \cap x, b \cap x)$. We show that the mapping is $1 - 1$. If $\varphi(x) = \varphi(y)$, then

$$(1) \qquad a \cap x = a \cap y, \quad b \cap x = b \cap y.$$

Since $x, y \in L$, we can, for instance, consider that axy, hence xyb.
This and (1) give

$$x = (a \cap x) \cup (x \cap y) = (a \cap y) \cup (x \cap y) \leqq y,$$

$$y = (x \cap y) \cup (y \cap b) = (x \cap y) \cup (b \cap x) \leqq x,$$

hence $x = y$.

If the line L is infinite, then the set of ordered pairs $\{(a \cap x, b \cap x) \mid x \in L\}$ is infinite too, hence $L^\cap = \{a \cap x \mid x \in L\} \cup \{b \cap x \mid x \in L\}$ cannot be finite.

Theorem 10. *Let the lattice S be p-modular and upper semimodular. Let K, L be connected lines with endelements a, b, $a \| b$. Let the line K be finite. Then there exists a $1 - 1$ mapping of the set of simple pairs of the line K onto the set of simple pairs of the line L such that the corresponding simple pairs are projective.*

Proof. If the line L is finite, then the assertion follows from Lemma 12. If the line L is infinite, then the line $L^\cap$ is infinite by Lemma 13 and hence the connected line $\bar{L}$ which contains $L^\cap$ is infinite too. Since the lattice S is upper semimodular, $K^\cap$ is connected and there holds

$$\bar{L}' \mathscr{P} K^{\cap\prime} \quad \text{and} \quad \bar{L}'' \mathscr{P} K^\cap.$$

Hence the chains $\bar{L}'$, $\bar{L}''$ are finite, which contradicts the fact that $\bar{L}$ is infinite. Hence the assumption that the L is infinite is false.

Remark Clearly if a lattice is lower semimodular (a dual definition to Definition 7) and p-modular, then Theorem 10 is true.

Example

Consider the lattice $AG_n(D)$ of affine subspaces of the n-dimensional vector space D^n over a field D which has not the characteristic 2. Affine subspaces are defined as subsets of D^n containing with every two a, b all points of the form $a + \lambda(b - a)$, $\lambda \in D$. It is well known that this lattice is lower semimodular and it is not modular. We shall show that this lattice is p-modular too.

The elements of the lattice $AG_n(D)$ have a form $a + A$ where A is a vector subspace of the D^n and a is an element of D^n.

We first prove

1. a) *The meet of two elements $a + A$, $b + B$ of the lattice $AG_n(D)$ is either $\emptyset$ or $z + (A \cap B)$, where $z \in (a + A) \cap (b + B)$.*

b) *The join of two elements $a + A, b + B$ of $AG_n(D)$ is $a + (\overline{b - a} \oplus A \oplus B)$. where $\overline{b - a}$ is the vector subspace of D^n generated by $b - a$ and $A \oplus B$ is the lattice-join of A and B in the lattice of all vector subspaces of D^n.*

Proof. a) If $(a + A) \cap (b + B) \neq 0$, then there exists an element $z \in a + A$, $z \in b + B$. Hence $a + A = z + A$, $b + B = z + B$. This implies

$$(a + A) \cap (b + B) = (z + A) \cap (z + B) \supset z + (A \cap B).$$

If $x \in z + A$ and $x \in z + B$, then $x = z + a$, $x = z + b$ for some $a \in A$, $b \in B$. Hence $a = b$ and $a \in A \cap B$, which follows $x = z + a \in z + (A \cap B)$.

b) Clearly, $(a + A) \vee (b + B) \subset a + (\overline{b - a} \oplus A \oplus B)$. Let $x \in a + (\overline{b - a} \oplus A \oplus B)$. Then $x = a + \alpha(b - a) + a_1 + b_1$, $a_1 \in A, b_1 \in B$. $x \in D$. If $\alpha = 1$ then $x = a_1 + b_1 + b$. We can write

$$x = (a + a_1) + \tfrac{1}{2}(y - (a + a_1)),$$

where

$$y = (a - a_1) + 2(b + b_1 - (a - a_1)).$$

The point y belongs to $(a + A) \vee (b + B)$ because it belongs to the line which is defined by points $a - a_1 \in a + A$ and $b + b_1 \in b + B$. Since the point x belongs to the line which is defined by points lying in the set $(a + A) \vee (b + B)$, it belongs to the $(a + A) \vee (b + B)$. If $\alpha = 0$, then $x = a + a_1 + b_1$. We can prove that $x \in (a + A) \vee (b + B)$, analogously as in the foregoing case. If $x = a + \alpha(b - a) + a_1 + b_1$ and $\alpha \neq 1$, $\alpha \neq 0$, then $x = a + \alpha(b - a) + (1 - \alpha) . a_1/(1 - \alpha) + \alpha b_1/\alpha$, where $a_2 = a_1/(1 - \alpha) \in A$ and $b_2 = b_1/\alpha \in B$. Hence $x = a + \alpha(b - a) + (1 - \alpha)a_2 + \alpha b_2 = a + a_2 + \alpha(b + b_2 - (a + a_2))$. Therefore x belongs to the line which is defined by $a + a_2 \in a + A$ and $b + b_2 \in b + B$, consequently, $x \in (a + A) \vee (b + B)$.

2. *If $a + A \subset b + B$, then $A \subset B$ and if $a + A = b + B$, then $A = B$.*

Proof. From $a + A \subset b + B$ it follows that $a = b + b_1, b_1 \in B$, hence $a - b \in B$. If $x \in A$, then $a + x = b + b_1$, $b_1 \in B$. Therefore $x = -(a - b) + b_1$, hence $x \in B$.

The second assertion follows from the first.

3. *If the elements $\bar{a}_1, \bar{b}_1, \bar{a}, \bar{b} \in AG_n(D)$ satisfy*

$$(\bar{a}_1 \cup \bar{b}) \cap \bar{a} = \bar{a}_1, \quad (\bar{a} \cup \bar{b}_1) \cap \bar{b} = \bar{b}_1,$$

then $(\bar{a}_1 \cup \bar{b}_1) = (\bar{a}_1 \cup \bar{b}) \cap (\bar{b}_1 \cup \bar{a})$.

Proof. Let $\bar{a}_1 = a_1 + A_1$, $\bar{b}_1 = b_1 + B_1$, $\bar{a} = a + A$, $\bar{b} = b + B$. Since

$a_1 + A_1 \subset a + A$, $b_1 + B_1 \subset b + B$, we get $a_1 \in a + A$, $b_1 \in b + B$ and we have

$$(1) \qquad a + A = a_1 + A, \quad b + B = b_1 + B.$$

From the assumption and (1) it follows that

$$a_1 + A_1 = ((a_1 + A_1) \vee (b_1 + B)) \cap (a + A) =$$
$$(a_1 + (\overline{b_1 - a_1} \oplus A_1 \oplus B)) \cap (a + A) = z + ((\overline{b_1 - a_1} \oplus A_1 \oplus B) \cap A),$$

where

$$z \in (a_1 + (\overline{b_1 - a_1} \oplus A_1 \oplus B)) \cap (a + A).$$

Hence by 2. we get

$$(2) \qquad A_1 = (\overline{b_1 - a_1} \oplus A_1 \oplus B) \cap A$$

and analogously

$$(3) \qquad B_1 = (\overline{b_1 - a_1} \oplus A \oplus B_1) \cap B.$$

Since the lattice of all vector subspaces of D^n is modular, it follows that

$$A_1 \oplus B_1 = ((\overline{b_1 - a_1} \oplus A_1 \oplus B) \cap A) \oplus ((\overline{b_1 - a_1} \oplus A \oplus B_1) \cap B) =$$
$$= (\overline{b_1 - a_1} \oplus A_1 \oplus B) \cap (A \oplus ((\overline{b_1 - a_1} \oplus A \oplus B_1) \cap B)) =$$
$$= (\overline{b_1 - a_1} \oplus A_1 \oplus B) \cap (\overline{b_1 - a_1} \oplus A \oplus B_1) \cup (A \oplus B)$$

and

$$\overline{b_1 - a_1} \oplus A_1 \oplus B_1 =$$
$$= (\overline{b_1 - a_1} \oplus A_1 \oplus B) \cap (\overline{b_1 - a_1} \oplus A \oplus B_1) \cap (\overline{b_1 - a_1} \oplus A \oplus B).$$

Since $a_1 + A_1 \subset a + A$, $b_1 + B_1 \subset b + B$, by 2. $a_1 \subset A$, $B_1 \subset B$.

Hence $\overline{b_1 - a_1} \oplus A_1 \oplus B_1 = (\overline{b_1 - a_1} \oplus A_1 \oplus B) \cap (\overline{b_1 - a_1} \oplus A \oplus B_1)$.

From this, (1) and 1. it follows that

$$(\bar{a}_1 \cup \bar{b}) \cap (\bar{b}_1 \cup \bar{a}) =$$
$$= (a_1 + (\overline{b_1 - a_1} \oplus A_1 \oplus B)) \cap (a_1 + (\overline{b_1 - a_1} \oplus A \oplus B_1)) =$$
$$= a_1 + ((\overline{b_1 - a_1} \oplus A_1 \oplus B) \cap (\overline{b_1 - a_1} \oplus A \oplus B_1) =$$
$$= a_1 + (\overline{b_1 - a_1} \oplus A_1 \oplus B_1) = (a_1 + A_1) \vee (b_1 + B_1) = \bar{a}_1 \cup \bar{b}_1.$$

REFERENCES

[1] FELSCHER, W.: Jordan-Hölder Sätze und modular geordnete Mengen. Math. Z. *75*, 1961, 83–114.
[2] KOLIBIAR, M.: Linien in Verbänden. Univ. Iaşi, Sec. *1* 1965, 90–98.
[3] SZÁSZ, G.: Einführung in die Verbandstheorie. Budapest 1962.
[4] BUMCROT, R. J.: Betweenes geometry in lattices. Rend. Circolo mat. Palermo, Serie II, Tomo *XII*, 1964, 11–28.

Received May 4, 1970

Katedra algebry a teórie čísel
Prírodovedeckej fakulty Univerzity Komenského
Bratislava