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ON A BOUNDARY VALUE PROBLEM FOR A NONLINEAR SECOND-ORDER DIFFERENTIAL EQUATION

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It is shown in the paper by means of Stone's theorem that an assumption concerning the Lipschitz continuity in a theorem on the existence of a solution of a boundary value problem for a nonlinear second order differential equation can be dropped out.

In papers [1], Theorem 6.3, and [2], Theorem 4.18 the following theorem has been proved.

Theorem 1. *Suppose that $a < b$ are two real numbers, $f = f(x, y, z)$ is continuous on $E = \langle a, b\rangle \times R^2$ and is such that*

p) f is nondecreasing in y on E for fixed x, z;

q) there is a constant $k > 0$ such that $|f(x, 0, z) - f(x, 0, 0)| \leqq k\,|z|$ on $\langle a, b\rangle$ for all z;

r) f satisfies a Lipschitz condition with respect to z on each compact subset of E.

Then the boundary value problem

$$y'' = f(x, y, y'),\quad y(a) = y(b) = 0 \tag{1}$$

has a unique solution $y(x) \in C_2(\langle a, b\rangle)$. Furthermore, on $\langle a, b\rangle$

$$|y(x)| \leqq \frac{M}{k^2}\,[e^{k(b-a)} - e^{1/2k(b-a)} - \tfrac{1}{2}k(b-a)] \tag{2}$$

and

$$|y'(x)| \leqq \frac{M}{k}\,[e^{k(b-a)} - 1]\,,$$

where $M = \max\limits_{x\in\langle a,b\rangle} |f(x, 0, 0)|$.

We are going to show that the condition *r*) can be removed and Theorem 1 still remains valid. To that aim we shall need Stone's theorem in the formulation given in paper [3]. We shall also keep the notations from that paper.

Stone's theorem. *Let M be a compact set (in a metric space), $f \in C_0(M)$ and let A be a lattice of continuous functions on M with the following property.*

(a) For every pair x, y, $x \neq y$ of points of M, there exists a function $g \in A$ such that $g(x) = f(x)$, $g(y) = f(y)$. Then there exists a sequence $\{f_n\}$ of functions $f_n \in A$ which uniformly converges to f on M.

By means of that theorem we shall prove Lemma 1.

Lemma 1. *Suppose that $a < b$ are two real numbers, $f = f(x, y, z)$ is continuous on E and satisfies the conditions p) and q). Then there exists a sequence $\{f_n\}$ of functions $f_n \in C_0(E)$ satisfying the conditions p) q) and r) which uniformly converges to f on each compact subset of E.*

Proof. For each natural m, let $N_m = \langle a, b\rangle \times \langle -m, m\rangle \times \langle -m, m\rangle$. Fix an arbitrary N_m. Let B be the set of all functions $g \in C_0(E)$ satisfying the conditions p), q) and r). $g_1(x, y, z) \equiv y + z \in B$, hence $B \neq \emptyset$. When considering the restriction of the functions $g \in B$ on N_m, we shall show that B is a lattice of continuous functions on N_m having the property (a). Hence, by Stone's theorem, this will guarantee that there is a function $f_m \in B$ such that $|f(x, y, z) - f_m(x, y, z)| < 1/m$ for $(x, y, z) \in N_m$. Then $\{f_m\}_{m=1}^{\infty}$ will possess all the required properties.

Let us first prove that B is a lattice of continuous functions. Since the set of all Lipschitz continuous functions forms a lattice of continuous functions ([3], remark b.), $\max(g_1, g_2)$ and $\min(g_1, g_2)$ show the property r) whenever g_1, g_2 do so. As to the property p), if $g_1, g_2 \in B$, $(x, y_1, z) \in N_m$, $(x, y_2, z) \in N_m$ and $g_i(x, y_k, z) = g_{ik}$, $i, k = 1, 2$, then in the case when $g_{11} \leqq g_{21}$, $g_{12} \geqq g_{22}$

$$\min(g_{12}, g_{22}) = g_{22} \geqq g_{21} \geqq \min(g_{11}, g_{21})$$

and

$$\max(g_{11}, g_{21}) = g_{21} \leqq g_{22} \leqq \max(g_{12}, g_{22})\,.$$

The same result will be obtained in the other cases. Thus with $g_1, g_2 \in B$ also $\min(g_1, g_2)$, $\max(g_1, g_2)$ possess the property p). Now to prove q). Fix an $x \in \langle a, b\rangle$ and denote $g_i(x, 0, z) = g_{iz}$, $g_i(x, 0, 0) = g_{i0}$, $i = 1, 2$. If $g_{10} = g_{20}$, the proof is trivial. Suppose, next, $g_{10} < g_{20}$. By q) we have $g_{i0} - kz \leqq g_{iz} \leqq g_{i0} + kz$. When $g_{2z} > g_{10} + kz$, then $\min(g_{1z}, g_{2z}) = g_{1z}$ and

$$\min(g_{1z}, g_{2z}) \leqq \min(g_{10}, g_{20}) + kz\,. \tag{3}$$

When $g_{2z} \leqq g_{10} + kz$, then (3) is true again. Since $g_{20} - kz > g_{10} - kz$, both $g_{iz} \geqq g_{10} - kz$ and thus $\min(g_{10}, g_{20}) - kz \leqq \min(g_{1z}, g_{2z})$. Similar results can be obtained in the case when $g_{10} > g_{20}$ and for $\max(g_{1z}, g_{2z})$. Thus $\min(g_1, g_2) \in B$, $\max(g_1, g_2) \in B$.

It remains to prove that B satisfies the condition (a). Consider the functions h defined on E by

(4) $$h(x, y, z) = \varphi(x) + \omega(z)\psi(y) + \chi(z)$$

where $\psi(0) = \chi(0) = 0$, $\varphi \in C_0(\langle a, b\rangle)$, $\psi \in C_0((-\infty, \infty))$ and is nondecreasing while ω and χ are defined in $(-\infty, \infty)$, $\omega \geqq 0$, both of them satisfy a Lipschitz condition on each compact interval and $|\chi(z)| \leqq k|z|$ for each z. Clearly each $h \in B$.

Let $(x_i, y_i, z_i) \in N_m$, $(x_1, y_1, z_1) \neq (x_2, y_2, z_2)$ and denote $f(x_i, y_i, z_i) = c_i$, i 1, 2. We shall show that there is a function h given by (4) which satisfies the conditions

(5) $$\varphi(x_i) + \omega(z_i)\psi(y_i) + \chi(z_i) = c_i, \ i = 1, 2 .$$

If the constants $\varphi(x_i)$, $\omega(z_i)$, $\psi(y_i)$, $\chi(z_i) = c_i - \varphi(x_i) - \omega(z_i)\psi(y_i)$, i -- 1, 2, are such that $\omega(z_i) \geqq 0$, $|c_i - \varphi(x_i) - \omega(z_i)\psi(y_i)| \leqq k\,|z_i|$ or $(c_i - \varphi(x_i) - k\,|z_i|)/\omega(z_i) \leqq \psi(y_i) \leqq (c_i - \varphi(x_i) + k\,|z_i|)/\omega(z_i)$ and $\psi(y_i) \leqq \psi(y_k)$ if $y_i \leqq \leqq y_k$, $i, k = 1, 2, 3$, where $y_3 = 0$, $\psi(0) = 0$, then they form an admissible solution of the system (5), from which the functions φ, ψ, χ and ω with the above mentioned properties can be constructed by linear intra- and extrapolation. Hence B satisfies the condition (a).

If $x_1 \neq x_2$, we choose $\varphi(x_i)$, $i = 1, 2$, such that $|c_i - \varphi(x_i)| \leqq k\,|z_i|$. Then $\varphi(x_i)$, $\omega(z_i) = 1$, $\psi(y_i) = 0$, $\chi(z_i) = c_i - \varphi(x_i)$, $i = 1, 2$, gives an admissible solution of (5).

In the case $x_1 = x_2$, $z_1 = z_2$, by the property p) of f, $y_i \leqq y_k$ implies $c_i \leqq c_k$, i, k 1, 2. Then by putting $\omega(z_i) = 1$ and choosing $\varphi(x_i)$ properly, we can obtain that $\psi(y_i) \leqq \psi(y_k)$ is satisfied for $y_i \leqq y_k$ also if $i, k = 1, 2, 3$.

The case $x_1 = x_2$, $z_1 \neq z_2$ remains to be investigated Here some subcases are possible: 1. If $\operatorname{sgn} y_1 = \operatorname{sgn} y_2$, then by a suitable choice of $\varphi(x_i)$ we can get $\operatorname{sgn} \psi(y_1) = \operatorname{sgn} \psi(y_2) = \operatorname{sgn} y_i$ and then, by taking $\omega(z_i)$ properly we get that $y_i \leqq y_k$ implies $\psi(y_i) \leqq \psi(y_k)$ even for $i, k = 1, 2, 3$. 2. Consider now the case $\operatorname{sgn} y_1 \neq \operatorname{sgn} y_2$, e. g. $y_1 \leqq 0 \leqq y_2$. Then again, if $c_1 \leqq 0 \leqq c_2$ or $c_1c_2 \geqq 0$, everything can be properly arranged. In the case $c_1 > 0 > c_2$ we must have that $s = \langle c_1 - k\,|z_1|, c_1 + k\,|z_1|\rangle \cap \langle c_2 - k\,|z_2|, c_2 + k\,|z_2|\rangle \neq \emptyset$. If not, then by p) $f(x_1, 0, z_1) \geqq c_1$ and so by q) we would have $f(x_1, 0, 0) \geqq \geqq c_1 - k\,|z_1|$ and at the same time $f(x_1, 0, z_2) \leqq c_2$ and thus, $f(x_1, 0, 0) \leqq \leqq c_2 + k\,|z_2|$. Since $s \neq \emptyset$, we also get by a suitable choice of $\varphi(x_i)$ and $\omega(z_i) \geqq 0$ that $\psi(y_i)$ satisfies the required conditions.

From Theorem 1, by Lemma 1, we shall prove

Theorem 2. *If all hypotheses of Theorem* 1 *are satisfied except the condition r), then the boundary value problem* (1) *has at least one solution* $y(x) \in C_2(\langle a, b\rangle)$ *satisfying the inequalities* (2) *where* M *and* k *are as in Theorem* 1.

Proof. By Lemma 1, there is a sequence $\{f_n\}$ of functions $f_n \in C_0(E)$ satisfying the conditions p), q), and r) which uniformly converges to f on each compact subset of E. Theorem 1 gives the existence of a sequence $\{y_n\}$ of the solutions of the boundary value problem

$$y'' = f_n(x, y, y'), \; y(a) = y(b) = 0 \,. \tag{6}$$

Each y_n satisfies the inequalities (2) where instead of M the constant $M_n = \max_{x \in \langle a,b \rangle} |f_n(x, 0, 0)|$ appears. Therefore the set of all points $(x, y_n(x), y_n'(x))$, $a \leqq x \leqq b$, $n = 1, 2, 3, \ldots$, lies in a compact subset of E. Hence the sequence of y_n'' is uniformly bounded and thus both sequences $\{y_n\}$, $\{y_n'\}$ satisfy the hypotheses of Ascoli's Lemma. Therefore there is a subsequence $\{y_{n_k}\}_{k=1}^{\infty}$ which is uniformly convergent, together with $\{y_{n_k}'\}_{k=1}^{\infty}$ to a function y and its derivative y', respectively. From (6) it follows that y_{n_k}'' converge uniformly to y'' and y satisfies the conditions (1) and (2).

REFERENCES

[1] BEBERNES, J. W.: A subfunction approach to boundary value problems for ordinary differential equations. Pacif. J. Math., *13*, 1963, 1053—1066.

[2] JACKSON, L. K.: Subfunctions and differential inequalities. Advances in Math., *2*, 1968, 307—363.

[3] ŠEDA, V.: On an application of Stone's theorem in the theory of differential equations. Časop. pěstov. mat. (to appear).

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