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ON SEMIPRIME IDEALS OF THE DIRECT PRODUCT OF SEMIGROUPS

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The purpose of this note is the proof of the theorem of a necessary and sufficient condition for every left (right) ideal of the direct product of semigroups to be a semiprime ideal.

In the first place we introduce some notions and properties which we shall require. We say that an element $a \in S$ satisfies the Conditon (m, n), if in the semigroup S there exists an element x such that

$$a = a^m \, x \, a^n$$

where m, n are non negative integers and a^0 means the void sympbol. (See [1]). The set of all elements of S, satisfying the Condition (m, n) is called a class of regularity and will be denoted by $\mathscr{R}_S\,(m, n)$. By means of these classes of regularity some properties of semigroups have been studied. We show, how to characterize by means of this notion semiprime ideals of the direct product of semigroups. In our considerations the statements; a semigroup S satisfies the Condition (m, n), or a semigroup $S = \mathscr{R}_S\,(m, n)$, are equivalent.

Let $\{S_i\}$, $i \in I$ be an arbitrary system of semigroups. Denote by S the set of all functions ξ, defined on I such that $\xi(i) \in S_i$. Introduce in S a multiplication in this way; If α, $\beta \in S$ are arbitrary elements of S, then the product $\gamma = \alpha \,.\, \beta$ is given by $\gamma(i) = \alpha(i)\, \beta(i)$ for every $i \in I$. The set S with this multiplication is a semigroup, called a direct product of semigroups $\{S_i\}$, $i \in I$ and is denoted by $S = \prod_{i \in I} S_i$.

In [3] it is proved that if L_i is a left ideal of a semigroup S_i, $i \in I$, then $L = \prod_{i \in I} L_i$ is a left ideal of the semigroup $S = \prod_{i \in I} S_i$.

A left ideal L of the semigroup S is called a semiprime left ideal of S if for every element $a \in S$ and an arbitrary integer n the relation $a^n \in L$ implies that $a \in L$.

Theorem 1. *Let L_i be a semiprime left ideal of a semigroup S_i for every $i \in I$. Then $L = \prod_{i \in I} L_i$ is a semiprime left ideal of $S = \prod_{i \in I} S_i$.*

Proof. Let $\alpha \in S = \prod_{i \in I} S_i$ be an arbitrary element and let be $\alpha^n \in L = \prod_{i \in I} L_i$. Then $[\alpha(i)]^n \in L_i$ for every $i \in I$. Since L_i is a semiprime left ideal of S_i, we have $\alpha(i) \in L_i$ for every $i \in I$. Hence $\alpha \in L = \prod_{i \in I} L_i$.

Let $N \subseteq S = \prod_{i \in I} S_i$. The set of all elements $x_i \in S_i$, for which there exists at least one element $\xi \in N$ such that $\xi(i) = x_i$, will be denoted by $\mathscr{P}_i(N)$ and called the projection of the set N into the semigroup S_i.

Theorem 2. *Let $L = \prod_{i \in I} L_i$ be a semiprime left ideal of a semigroup $S = \prod_{i \in I} S_i$. Then $\mathscr{P}_i(L)$ is a semiprime left ideal of S_i.*

Proof. Let $L = \prod_{i \in I} L_i$ be a semiprime left ideal of $S = \prod_{i \in I} S_i$. The fact that $\mathscr{P}_i(L)$ is a left ideal of S_i is known from [3]. It is only necessary to prove that it is semiprime. Let $a_i \in S_i$, $a_i^n \in \mathscr{P}_i(L)$, where $i \in I$ is arbitrary, but fixed. It is necessary to show that $a_i \in \mathscr{P}_i(L)$. Since $a_i^n \in \mathscr{P}_i(L)$, it follows that there exists an element $\beta \in L$, such that, $\beta(i) = a_i^n$. Put $\beta(j) = b_j$ for $j \neq i$, $j \in I$. Let $\alpha \in S$ such that $\alpha(i) = a_i$, $\alpha(j) = b_j$ for $j \neq i$, $j \in I$. Since $\beta \in L$, then $\beta(j) = b_j \in \mathscr{P}_j(L)$ for every $j \neq i$. But $\mathscr{P}_j(L)$ is a left ideal, hence $b_j^n \in \mathscr{P}_j(L)$ for $j \neq i$. And according to the assumption $a_i^n \in \mathscr{P}_i(L)$. That is, $[\alpha(i)]^n \in \mathscr{P}_i(L)$ for every $i \in I$, hence $\alpha^n \in L$ (since $L = \prod_{i \in I} L_i$). But since L is a semiprime left ideal of S, then $\alpha \in L$, hence $\alpha(i) = a_i \in \mathscr{P}_i(L)$, that is, $\mathscr{P}_i(L)$ is a semiprime left ideal of S_i.

Theorems 1 and 2 imply;

Corollary. *A left ideal $L = \prod_{i \in I} L_i$ of a semigroup $S = \prod_{i \in I} S_i$ is a semiprime if and only if the left ideal L_i for every $i \in I$ is semiprime.*

Lemma 1 ([2 p. 241]). *Every left ideal of the semigroup S is semiprime if and only if the semigroup S satisfies the Condition* (0,2).

Lemma 2 ([4]). *A semigroup $S = \prod_{i \in I} S_i$ satisfies the Condition (m, n) if and only if the semigroup S_i for every $i \in I$ satisfies this condition.*

Lemmas 1 and 2 imply;

Theorem 3. *Every left ideal of a semigroup $S = \prod_{i \in I} S_i$ is semiprime if and only if every left ideal of S_i for every $i \in I$ is semiprime.*

Theorem 4. *The following statements are equivalent;*

(a) *A semigroup S_i for every $i \in I$ satisfies the Condition* (0,2).

(b) *A semigroup $S = \prod_{i \in I} S_i$ satisfies the Condition* (0,2).

(c) *Every left ideal of* S_i *for every* $i \in I$ *is semiprime.*

(d) *Every left ideal of* $S = \prod_{i \in I} S_i$ *is semiprime.*

Proof. (a) $\Leftrightarrow$ (b) according to Lemma 2, (a) $\Leftrightarrow$ (c) according to Leamma 1, (b) $\Leftrightarrow$ (d) according to Lemma 1 and Lemma 2.

We say that a left (right) ideal $L(R)$ of a semigroup S is complete if $SL = L$ $(RS = R)$.

Lemma 3 ([4]). *Every left ideal of a semigroup* S *is a complete left ideal of* S *if and only if* $S = \mathscr{R}_S(0,1)$.

Theorem 5. *Let every left ideal of a semigroup* $S = \prod_{i \in I} S_i$ *be a semiprime. Then*

(a) *every left ideal of* $S = \prod S_i$ *is complete,*

(b) *every left ideal of* S_i, *for every* $i \in I$ *is complete.*

Proof. (a) The statement follows from Lemma 1, Lemma 3 and from the relation; $\mathscr{R}_S(m_1, n_1) \leqq \mathscr{R}_S(m_2, n_2)$, if $m_1 \geqq m_2, n_1 \geqq n_2$ (see [2], pp. 111—112).

(b) The statement follows from Lemma 1, Lemma 2, Lemma 3 and again from the relation cited in the proof of (a).

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