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RIGHT PRIME IDEALS AND MAXIMAL RIGHT IDEALS IN SEMIGROUPS

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In [1] Št. Schwarz studies some properties of prime ideals and of maximal ideals in a semigroup. In this note we shall study analogous properties of right prime ideals and of maximal right ideals.

A two-sided ideal Q of a semigroup S is said to be prime if $AB \subset Q$ implies that $A \subset Q$ or $B \subset Q$, A, B being two-sided ideals of S.

Theorem 1. *A two-sided ideal Q of a semigroup S is a prime ideal of S if and only if $AB \cap BA \subset Q$ implies that $A \subset Q$ or $B \subset Q$, A, B being two-sided ideals of S.*

Proof. Let Q be a prime two-sided ideal of S. Let A, B be two-sided ideals of S and $AB \cap BA \subset Q$. Clearly AB, BA are two-sided ideals of S and $(AB)(BA) \subset AB \cap BA \subset Q$. From this it follows that $AB \subset Q$ or $BA \subset Q$. Hence $A \subset Q$ or $B \subset Q$.

Let Q be a two-sided ideal of S and let $AB \cap BA \subset Q$ imply that $A \subset Q$ or $B \subset Q$, A, B being two-sided ideals of S. If A, B are two-sided ideals of S and $AB \subset Q$, then $AB \cap BA \subset AB \subset Q$. Thus we have $A \subset Q$ or $B \subset Q$. Hence Q is a prime ideal.

There is an analogous definition for right ideals of S.

Definition 1. *A right ideal Q of a semigroup S is said to be right prime if $AB \cap BA \subset Q$ implies that $A \subset Q$ or $B \subset Q$, A, B being right ideals of S.*

Remark. If S is a commutative semigroup, then every prime ideal is a right prime ideal and conversely.

Example 1. The following example shows that a right prime ideal need not be necessarily a prime ideal.

Let $S_1 = \{a, b\}$ be a semigroup in which $xy = x$ for every $x, y \in S_1$. Evidently $\{a\}$, $\{b\}$ and S_1 are all right ideals of S_1. Thus $Q_1 = \{a\}$ is a right prime ideal of S_1. But Q_1 is not a left ideal of S_1. Hence Q_1 is not a prime ideal of S_1.

Example 2. The following example shows that a prime ideal need not be necessarily a right prime ideal.

Let $S_2 = S_1 \cup \{0\}$, where $xO = O = Ox$ for every $x \in S_2$ (S_1 is as in Example

1). Clearly $\{O\}$, S_2 are all two-sided ideals of S_2. Thus $Q_2 = \{O\}$ is a prime ideal of S_2. Put $A = \{a, O\}$, $B = \{b, O\}$. Evidently A, B are right ideals of S_2. Since $AB = A$, $BA = B$, we have $AB \cap BA = A \cap B = Q_2$. But $A \not\subset Q_2$ and $B \not\subset Q_2$. Thus Q_2 is not a right prime ideal of S_2.

Definition 2. *A right ideal R of a semigroup S is called maximal if $R \subset S$ and there does not exist a right ideal R_1 of S such that $R \subsetneq R_1 \subsetneq S$.*

Example 3. The following example shows that a maximal right ideal of S with $S = S^2$ need not be necessarily a right prime ideal. (See Theorem 1 in [1].)

Let $S_3 = \{(i, n - i) \,/\, \text{for all positive integers } n \text{ and for } i = 0, 1\}$. Define in S_3 a multiplication by

$$xy = (i, n + m)$$

if $x = (i, n) \in S_3$ and $y = (j, m) \in S_3$. Then S_3 is a semigroup and $S_3^2 = S_3$. Put $P_3 = \{p\}$, where $p = (0, 1)$. Clearly $R_3 = S_3 - P_3$ is a maximal right ideal of S_3. Put $A = \{p^n \,/\, \text{for all positive integers } n\}$. Evidently A is a right ideal of S_3 and $AA \cap AA = A^2 \subset R_3$. But $p \in A \not\subset R_3$. Thus R_3 is not a right prime ideal of S_3.

Theorem 2. *If R is a maximal right ideal of a semigroup S such that $P \cap P^2 \neq \emptyset$ where $P = S - R$, then R is a right prime ideal of S.*

Proof. Let R be a maximal right ideal of S. If R is not a right prime ideal of S, then there exist two right ideals A, B of S such that $AB \cap BA \subset R$ and $A \not\subset R$, $B \not\subset R$. Since R is maximal, we have $A \cup R = S = B \cup R$, hence $P \subset A$ and $P \subset B$. Thus $P^2 \subset AB \cap BA \subset R$. Since $P \cap P^2 \neq \emptyset$ we have $P \cap R \neq \emptyset$. This is a contradiction. Consequently R is a right prime ideal of S.

Corollary. *If R is a maximal right ideal of a semigroup S such that $S - R$ contains an idempotent, then R is a right prime ideal of S.*

Example 4. The following example shows that a maximal right ideal R of S where $\text{card}\,(S - R) \geqq 2$ need not be necessarily a right prime ideal. (See Theorem 1a in [1].)

Let G be an arbitrary group. Let $S_4 = S_3 \times G$, $P_4 = P_3 \times G$, $R_4 = R_3 \times G = S_4 - P_4$ and $B = A \times G$, where S_3, P_3, R_3 and A are as in Example 3. Then R_4, B are right ideals of the semigroup S_4, $S_4^2 =$ and $\text{card}\, P_4 = \text{card}\, G$. Clearly $B \not\subset R_4$ and $BB \cap BB = B^2 \subset R_4$. Thus R_4 is not a right prime ideal of S_4. Finally, we prove that R_4 is a maximal right ideal of S_4. Let R' be a right ideal of S_4 such that $R_4 \subsetneq R' \subset S_4$. Then there exists $g \in G$ such that $(p, g) \in R'$, where $p \in P_3$. If $h \in G$, then $(p, h) = (p, g)(m, g^{-1}h) \in R'$ where $m \in S_3$ and $m = (1, 0)$. Thus $R' = S_4$.

Theorem 3. *If S is a semigroup with $S = eS$ for some $e \in S$, then every maximal right ideal of S is a right prime ideal of S.*

Proof. Let R be a maximal right ideal of S. Denote $P = S - R$. First we prove that $xS = S$ (for some $x \in S$) implies $x \in P$. Indeed, if $x \in R$, then $S = xS \subset RS \subset R$. This contradicts $R \neq S$. Now $eS = S$ implies $e \in P$ and $e^2 \in P^2$. Since $e^2S = eS = S$, hence $e^2 \in P$. Then $e^2 \in P \cap P^2 \neq \emptyset$ and it follows from Theorem 2 that R is a right prime ideal of S.

Corollary. *If S is a semigroup with a left identity element, then every maximal right ideal of S is a right prime ideal of S.*

Remark. Example 3 shows that the semigroup S_3 has a right identity element $m = (1, 0)$ and the maximal right ideal R_3 of S_3 is not a right prime ideal of S_3.

Theorem 4. *Let $\{R_\alpha \,/\, \alpha \in \Lambda\}$ be the set of all different maximal right ideals of a semigroup S. Suppose* $\text{card}\,\Lambda \geqq 2$ *and denote $P_\alpha = S - R_\alpha$ and $R^* = \bigcap_{\alpha \in \Lambda} R_\alpha$. We then have:*

a) $P_\alpha \cap P_\beta = \emptyset$ *for* $\alpha \neq \beta$.

b) $S = [\bigcup_{\alpha \in \Lambda} P_\alpha] \cup R^*$.

c) *For every $\alpha \neq \beta$ we have $P_\alpha \subset R_\beta$.*

d) *If A is a right ideal of S and $A \cap P_\alpha \neq \emptyset$, then $P_\alpha \subset A$.*

e) *For α we have $P_\alpha S \subset \bigcap_{\beta \in \Lambda,\, \beta \neq \alpha} R_\beta$.*

Remark. The case $\text{card}\,\Lambda = 1$ is trivial.

Proof. a)—d). The proof is similar to the proof of Theorem 2 in [1].

e) If $\beta \neq \alpha$ ($\alpha, \beta \in \Lambda$), then from c) it follows that $P_\alpha \subset R_\beta$. Thus $P_\alpha S \subset R_\beta S \subset R_\beta$. Hence $P_\alpha S \subset \bigcap_{\beta \in \Lambda,\, \beta \neq \alpha} R_\beta$.

Remark. Example 1 gives a semigroup in which $R^* = \{a\} \cap \{b\} = \emptyset$. (See Theorem 2d in [1].)

Let $\mathbf{R} = \{R_\alpha \,/\, \alpha \in \Lambda\}$ be the set of all maximal right ideals of S and (as above) $R^* = \bigcap_{\alpha \in \Lambda} R_\alpha$.

Theorem 5. *Let S be a semigroup containing maximal right ideals. Then every right prime ideal of S containing R^* and different from S is a maximal right ideal of S.*

Proof. The proof is an easy adaptation of the proof of Theorem 3 in [1]. Let Q be a right prime ideal of S and $R^* \subset Q \neq S$. We use the notations of Theorem 4. By b), d) we have

$$Q = S - [\bigcup_{\alpha \in H} P_\alpha] = \bigcap_{\alpha \in H} (S - P_\alpha) = \bigcap_{\alpha \in H} R_\alpha,$$

where $\emptyset \neq H \subset \Lambda$.

If $\text{card}\, H \geqq 2$, then $Q = R' \cap R_\beta$, where $R' = \bigcap_{\alpha \in H,\, \alpha \neq \beta} R_\alpha$. Thus $R'R_\beta \cap$

$\cap R_\beta R' \subset R' \cap R_\beta = Q$. Since Q is right prime, we have $R' \subset Q$ or $R_\beta \subset Q$. Thus $R' \subset R_\beta$ or $R_\beta \subset R'$. If $R' \subset R_\beta$, then by Theorem 4c we have $P_\beta \subset \bigcap_{\alpha \in H, \alpha \neq \beta} R_\alpha = R'$. Hence $P_\beta \subset R_\beta$, a contradiction with $P_\beta \cap R_\beta = \emptyset$. If $R_\beta \subset R'$, then it follows from Definition 2 that $R' = R_\beta$. Thus $P_\beta \subset R' = R_\beta$. This is a contradiction. It follows that card $H = 1$. Thus $Q = R_\alpha$, i. e. Q is a maximal right ideal of S and our Theorem is proved.

Theorem 6. *Let S be a semigroup containing maximal right ideals. A rihgt prime ideal $Q \neq S$ is a maximal right ideal of S if and only if $R^* \subset Q$.*

Proof follows from Theorem 5.

Let now be $\mathbf{Q} = \{Q_\alpha \mid \alpha \in \Lambda\}$ the set of all right prime ideals of S and different from S and $Q^* = \bigcap_{\beta \in \Lambda} Q_\alpha$.

Theorem 7. *Let S be a semigroup containing maximal right ideals. Then every right prime ideal of S (and $\neq S$) is a maximal right ideal of S if and only if $R^* \subset Q^*$.*

Proof follows from Theorem 6.

Theorem 8. *Let S be a semigroup with $S = eS$ for some $e \in S$, containing maximal right ideals. Then $\mathbf{Q} = \mathbf{R}$ if and only if $Q^* = R^*$.*

Proof. This follows from Theorem 3 and Theorem 7.

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