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A NOTE ON A THEOREM OF A. D. ALEXANDROFF

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In the present paper we generalize the well-known Alexandroff theorem stating that any regular additive measure on a ring is countably additive. The problem to prove such a generalization was suggested by L. Mišík and I. Kluvánek in connection with the author's paper [4] (see also [1], § 59—61).

We present here two theorems for non-negative measures and one theorem for vector-valued measures. The paper contains also some remarks concerning paper [3] by E. Marczewski and paper [2] by N. Dinculeanu and I. Kluvánek.

Let T be a set. If $\mathscr{D}$ is a system of subsets of T and μ is a non-negative set-function on $\mathscr{D}$ with finite or infinite values, then we write $\mu : \mathscr{D} \to \langle 0, \infty \rangle$. Let $\mathscr{R}$, $\mathscr{C}$, $\mathscr{U}$ be systems of subsets of T.

Theorem 1. *Let $\mathscr{R}$, $\mathscr{C}$, $\mathscr{U}$ and μ : ® $\to \langle 0, \infty \rangle$ satisfy the following conditions:*

(i) $A, B \in \mathscr{R} \Rightarrow A \cup B \in \mathscr{R}$.

(ii) *If $C \subset \bigcup_{i=1}^{\infty} U_i$, $C \in \mathscr{C}$, $U_i \in \mathscr{U}$ $(i = 1, 2, \ldots)$, then there exists a positive integer n such that $C \subset \bigcup_{i=1}^{n} U_i$.*

(iii) *μ is additive, subadditive and monotone on $\mathscr{R}$.*

(iv) *μ is $(\mathscr{C}, \mathscr{U})$-regular on $\mathscr{R}$, i. e. $\mu(E) = \sup \{\mu(F) : F \in \mathscr{R}$ and there is $C \in \mathscr{C}$ such that $F \subset C \subset E\} = \inf \{\mu(G) : G \in \mathscr{R}$ and there is $U \in \mathscr{U}$ such that $E \subset U \subset G\}$ for any $E \in \mathscr{R}$.*

Then μ is σ-additive on $\mathscr{R}$.

Proof. Let $\{E_i\}_{i=1}^{\infty}$ be a sequence of pairwise disjoint sets of $\mathscr{R}$ and let $E = \bigcup_{i=1}^{\infty} E_i \in \mathscr{R}$.

From (iii) we get

$$\mu(E) \geqq \mu(\bigcup_{i=1}^{n} E_i) = \sum_{i=1}^{n} \mu(E_i) \quad (n = 1, 2, \ldots) .$$

If $\mu(E_j) = \infty$ for some j, then $\mu(E) = \sum_{i=1}^{\infty} \mu(E_i)$ according to (iii).

Let $\mu(E_i) < \infty$ for $i = 1, 2, \ldots$ Let $\varepsilon > 0$ be an arbitrary number. Then according to (iv), there exist $C \in \mathscr{C}$, $F \in \mathscr{R}$, such that $F \subset C \subset E$ and

$$\mu(F) > 2\varepsilon \quad \text{if} \quad \mu(E) = \infty$$

and

$$\mu(F) + \varepsilon > \mu(E) \quad \text{if} \quad \mu(E) < \infty .$$

Further according to (iv) there exist sets $U_i \in \mathscr{U}$, $G_i \in \mathscr{R}$ $(i = 1, 2, \ldots)$ such that $G_i \supset U_i \supset E_i$ and

$$\mu(G_i) - \mu(E_i) < \frac{\varepsilon}{2^i}, \quad i = 1, 2, \ldots$$

According to (ii) there exists a positive integer N such that

$F \subset C \subset \bigcup_{i=1}^{N} U_i \subset \bigcup_{i=1}^{N} G_i$ and according to (iii) we have $\mu(F) \leqq \sum_{i=1}^{N} \mu(G_i)$.

Hence we have

$$\mu(E) < \mu(F) + \varepsilon \leqq \sum_{i=1}^{N} \mu(G_i) + \varepsilon < \sum_{i=1}^{N} \mu(E_i) + 2\varepsilon \leqq$$

$$\leqq \sum_{i=1}^{\infty} \mu(E_i) + \varepsilon , \quad \text{if} \quad \mu(E) < \infty ,$$

and

$$2\varepsilon < \mu(F) \leqq \sum_{i=1}^{N} \mu(G_i) < \varepsilon + \sum_{i=1}^{N} \mu(E_i) , \quad \text{if} \quad \mu(E) = \infty .$$

Note 1. If $\mathscr{C} \cup \mathscr{U} \subset \mathscr{R}$, then the condition (iv) is equivalent to the condition $\mu(E) = \sup \{\mu(C) : E \supset C \in \mathscr{C}\} = \inf \{\mu(U) : E \subset U \in \mathscr{U}\}$ for any $E \in \mathscr{R}$.

Example 1. Let $\mathscr{R}$ be the ring generated by all intervals of the form $\langle a, b)$, where $-\infty < a \leqq b < \infty$. Let F be a continuous to the left, non-decreasing, finite real-valued function defined on the real line. Every set E in $\mathscr{R}$ can be written in the form $E = \bigcup_{i=1}^{r} \langle a_i, b_i)$, where the intervals $\langle a_i, b_i)$ are mutually disjoint. One may define the set function μ on $\mathscr{R}$ by the formula $\mu(E) = \sum_{i=1}^{r} [F(b_i) - F(a_i)]$. Evidently μ is additive and non-negative on $\mathscr{R}$. From the continuity to the left of F the $(\mathscr{C}, \mathscr{U})$-regularity of μ on $\mathscr{R}$ follows, where $(\mathscr{C}, \mathscr{U})$ is the system consisting of $\emptyset$ and of all finite sums of bounded closed (open) intervals.

Example 2. Let T be a Hausdorff topological space. Let $\mathscr{R}$ be a system of subsets of T closed under finite unions. Let $\mathscr{C}$ ($\mathscr{U}$) be any system of compact (open) subsets of T. Let $\mu : \mathscr{R} \to \langle 0, \infty \rangle$ be additive subadditive, monotone and ($\mathscr{C}$, $\mathscr{U}$)-regular on $\mathscr{R}$. Then μ is σ-additive according to Theorem 1.

Theorem 2. *Let $\mathscr{R}$, $\mathscr{C}$ and $\mu : \mathscr{R} \to \langle 0, \infty \rangle$ satisfy the following conditions:*

(v) *$\mathscr{R}$ is a ring.*

(vi) *If $C_i \in \mathscr{C}$ ($i = 1, 2, \ldots$) and $\bigcap_{i=1}^{n} C_i \neq \emptyset$ ($n = 1, 2, \ldots$), then $\bigcap_{i=1}^{\infty} C_i \neq \emptyset$.*

(vii) *If $A \in \mathscr{R}$, $A \subset C \in \mathscr{C}$, then $\mu(A) < \infty$.*

(ii) *μ is additive on $\mathscr{R}$.*

(viii) *μ is inner $\mathscr{C}$-regular on $\mathscr{R}$ i. e. $\mu(E) = \sup \{\mu(F) : F \in \mathscr{R}$ and there is $C \in \mathscr{C}$ such that $F \subset C \subset E\}$.*

Then μ is σ-additive on $\mathscr{R}$.

Proof. Let $\{A_i\}_{i=1}^{\infty}$ be a sequence of pairwise disjoint sets of $\mathscr{R}$ such that $A = \bigcup_{i=1}^{\infty} A_i \in \mathscr{R}$.

If $\mu(A) < \infty$, then the sets $E_n = A - \bigcup_{i=1}^{n} A_i$ ($n = 1, 2, \ldots$) form a non-increasing sequence of sets of $\mathscr{R}$ with $\bigcap_{n=1}^{\infty} E_n = \emptyset$ and $\mu(E_n) < \infty$. Similarly as in [3] § 4, Theorem (i) it can be proved that $\lim_{n\to\infty} \mu(E_n) = 0$ and hence $\lim_{n\to\infty} [\mu(A) - \sum_{i=1}^{n} (A_i)] = 0$.

Let $\mu(A = \infty$ and $K > 0$ be an arbitrary number. According to (vii) and (viii) there exists sets $C \in \mathscr{C}$, $F \in \mathscr{R}$ such that $F \subset C \subset A$ and $\infty > \mu(F) > K$. Hence $F = \bigcup_{i=1}^{\infty} (A_i \cap F)$ and by the foregoing we have $K < \mu(F) = \sum_{i=1}^{\infty} \mu(A_i \cap F) \leqq \sum_{i=1}^{\infty} \mu(A_i)$.

Note 2. If $\mathscr{C} \subset \mathscr{R}$, then the condition (vii) is equivalent to the condition $\mu(C) < \infty$ for every $C \in \mathscr{C}$ and the condition (viii) to the condition $\mu(E) = \sup \{\mu(C) : E \subset C \in \mathscr{C}\}$ for every $E \in \mathscr{R}$.

Note 3. If $\mathscr{R}$ is an algebra of subsets of T and $\mu(T) = 1$, then the assertion of Theorem 2 is identical with the assertion of Theorem (i), § 4 of [3].

Note 4. The assertion of Theorem 2 need not hold if we replace the assumption (v) by the assumption (i) even if μ is finite additive, subadditive and monotone (see example 3).

Example 3. Let $T = \{1, 2, \ldots\}$. Let $\mathscr{B}_0 = \{E \subset T$: either $E = \emptyset$ or E is finite$\}$; $\mathscr{B}_1 = \{E \subset T : T - E \in \mathscr{B}_0,\ 1 \in E\}$; $\mathscr{B} = \mathscr{B}_0 \cup \mathscr{B}_1$. Then $\mathscr{B}$ satisfies the properties (i) and (vi). Let $\mu : \mathscr{B} \to \langle 0, \infty)$ be defined in the following way: $\mu(E) = 0$ if $E \in \mathscr{B}_0$ and $\mu(E) = 1$ if $E \in \mathscr{B}_1$. Evidently μ is additive subadditive, monotone and inner $\mathscr{B}$-regular on $\mathscr{B}$. But μ is not σ-additive on $\mathscr{B}$ since $1 = \mu(T) \neq \sum_{n=1}^{\infty} \mu(\{n\}) = 0$.

Note 5. If the systems $\mathscr{C}$ and $\mathscr{U}$ satisfy the property (ii), then $\mathscr{C}$ need not satisfy the property (vi). If, e. g. $\mathscr{C}$ is an arbitrary system and $\mathscr{U}$ is a finite system of subsets of T.

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Let T be a set, $\mathscr{R}$, $\mathscr{C}$, $\mathscr{U}$ be systems of subsets of T. Let Φ be an operator defined on $\mathscr{U}$ such that $\Phi(U) \subset U$ for every $U \in \mathscr{U}$. Let X be a locally convex space with the topology defined by the family $\{|.|_p\}_{p \in P}$ of seminorms. Let μ be a vector-valued set function on $\mathscr{R}$ with values in X (write $\mu : \mathscr{R} \to X\}$. We say that $\mu : \mathscr{R} \to X$ is σ-additive on $\mathscr{R}$ if $\lim_{n \to \infty} |\mu(E) - \sum_{i=1}^{n} \mu(E_i)|_p = 0$ for every sequence $\{E_i\}_{i=1}^{\infty}$ of pairwise disjoint sets from $\mathscr{R}$ such that $E = \bigcup_{i=1}^{\infty} E_i \in \mathscr{R}$ and for every $p \in P$.

Theorem 3. *Let $\mathscr{R}$, $\mathscr{C}$, $\mathscr{U}$ and $\mu : \mathscr{R} \to X$ satisfy the following conditions:*

(a) *$\mathscr{R}$ is a ring and $\mathscr{U} \subset \mathscr{R}$.*

(b) *If $U_1, U_2 \in \mathscr{U}$, then $U_1 \cap U_2 \in \mathscr{U}$ and $\Phi(U_1 \cap U_2) \subset \Phi(U_1) \cap \Phi(U_2)$.*

(c) *If $C \in \mathscr{C}$, $U_i \in \mathscr{U}$ $(i = 1, 2, \ldots)$ and $C \subset \bigcup_{i=1}^{\infty} \Phi(U_i)$, then there exists a positive integer N such that $C \subset \bigcup_{i=1}^{N} \Phi(U_i)$.*

(d) *μ is additive on $\mathscr{R}$.*

(e) *μ is $(\mathscr{C}, \mathscr{U})$-regular on $\mathscr{R}$, i. e. to any set $E \in \mathscr{R}$ and any neighbourhood V of O in X there exist sets $C \in \mathscr{C}$ and $U \in \mathscr{U}$ such that $C \subset E \subset \Phi(U)$ and $\mu(B) \in V$ for every $B \in \mathscr{R}$, $B \subset U - C$.*

Then μ is σ-additive on $\mathscr{R}$.

Proof. Let $\{E_i\}_{i=1}^{\infty}$ be a sequence of pairwise disjoint sets of $\mathscr{R}$ and let $E = \bigcup_{i=1}^{\infty} E_i \in \mathscr{R}$.

Let $p \in P$ and $\varepsilon > 0$ be an arbitrary number. According to (e) there exist sets $C \in \mathscr{C}$, $U \in \mathscr{U}$ such that $C \subset E \subset \Phi(U)$ and

$$(1) \qquad |\mu(B)|_p < \frac{\varepsilon}{4} \text{ for every set } B \in \mathscr{R},\ B \subset U - C.$$

According to (b) and (e) there exist sets $U_i \in \mathscr{U}$ $(i = 1,2,\ldots)$ $U_i \subset U$ such that $E_i \subset \Phi(U_i)$ and

$$(2) \qquad |\mu(B)|_p < \frac{\varepsilon}{2^{i+1}} \text{ for every set } B \in \mathscr{R},\ B \subset U_i - E_i\ (i = 1, 2, \ldots)$$

According to (c) there exists a positive integer N such that $C \subset \bigcup_{i=1}^{N} U_i$ and hence

$$(3) \qquad E - \bigcup_{i=1}^{n} U_i \subset U - C,\ \bigcup_{i=1}^{n} U_i - E \subset U - C,\ \bigcup_{i=1}^{n} U_i \in \mathscr{R}$$

for every $n \geqq N$. From the condition (d) and from (1)—(3) we get

$$(4) \quad |\mu(E) - \mu(\bigcup_{i=1}^{n} U_i)|_p = |\mu(E) - [\mu(\bigcup_{i=1}^{n} U_i - E) + \mu(\bigcup_{i=1}^{n} U_i \cap E)]|_p =$$

$$= |\mu[E - (\bigcup_{i=1}^{n} U_i)] - \mu(\bigcup_{i=1}^{n} U_i - E)|_p \leqq$$

$$\leqq |\mu[E - (\bigcup_{i=1}^{n} U_i)]|_p + |\mu(\bigcup_{i=1}^{n} U_i - E)|_p < \frac{\varepsilon}{2}.$$

From (d) and from (2) we obtain

$$(5) \qquad |\mu(\bigcup_{i=1}^{n} U_i) - \mu(\bigcup_{i=1}^{n} E_i)|_p = |\mu[U_1 - (\bigcup_{i=1}^{n} E)]|_i +$$

$$\sum_{j=2}^{n} |\mu\{[U_j - (\bigcup_{.=1}^{n} E_i)] - \bigcup_{K=1}^{j-1} [U_k - (\bigcup_{i=1}^{n} E_i)]\}|_p < \frac{\varepsilon}{2}.$$

From (4) and (5) and the condition (d) we obtain

$$|\mu(E) - \sum_{i=1}^{n}(E_i)|_p \leqq |\mu(E) - \mu(\bigcup_{i=1}^{n} U_i)|_p + |\mu(\bigcup_{i=1}^{n} U_i) - \mu(\bigcup_{i=1}^{u} E_i)|_p < \varepsilon$$

for every $n \geqq N$.

Example 4. Let T be a Hausdorff topological space, $\mathscr{R}$ be a ring of subsets of T, $\mathscr{C}$ be the system of all compact subsets of T, $\mathscr{U} = \mathscr{R}$ and $\Phi(U) = \operatorname{Int} U$ for $U \in \mathscr{U}$. Then the $(\mathscr{C}, \mathscr{U})$-regularity of a vector-valued function $\mu : \mathscr{R} \to X$ means:

(R_2') To any set $E \in \mathscr{R}$ and any neighbourhood V of 0 in X there exists a compact set C and a set $U \in \mathscr{R}$ such that $C \subset E \subset \operatorname{Int} U$ and $\mu(B) \in V$ for every set $B \in \mathscr{R}$, $B \subset U - C$.

The following holds: If $\mu : \mathscr{R} \to X$ is additive and (R_2') regular on $\mathscr{R}$ then μ is σ-additive on $\mathscr{R}$.

This assertion is a strengthening of Theorem 3 of [2] in two directions: 1. T need not be locally compact. 2. The (R_2') regularity is weaker than the (R_2) regularity, which is assumed in Theorem 3 of [2].

Example 5. Let T be a set and $\mathscr{R}$ be a ring of subsets of T. Let $\mu : \mathscr{R} \to X$ be a vector-valued additive set function on $\mathscr{R}$. Let there for any $E \in \mathscr{R}$ and any neighbourbood V of 0 in X exist a finite set $C \subset T$ such that $C \subset E$ and $\mu(B) \in V$ for any $B \in R$, $B \subset E - C$. Then μ is σ-additive on $\mathscr{R}$ according to Theorem 3. It suffices to put $\mathscr{C}$ = the system of all finite subsets of T and $\mathscr{U} = \mathscr{R}$.

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