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NOTE ON THE SET OF NILPOTENT ELEMENTS AND ON RADICALS OF SEMIGROUPS

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In the present paper we consider some properties of nilpotent elements and radicals in semigroups.

Let S be a semigroup. Under an *ideal* of S we understand a two-sided ideal of S. Let $x(T)$ $[J, I]$ be an element (subsemigroup) [ideals] of S.

An element x [subsemigroup T] is called *nilpotent* with respect to the ideal J if there exists a natural number n such that $x^n \in J[M^n \subseteq J]$.

An ideal I is called *locally nilpotent* with respect to J if every finitely generated subsemigroup $T \subseteq I$ is nilpotent with respect to J.

An ideal I is called *nil-ideal* with respect to J if every element $x \in I$ is nilpotent with respect to J.

An ideal P of S is called *prime* if $S\backslash P$ is an m-system of S (a set $H \subseteq S$ is called an m-*system* of S, if for every two elements $a, b \in H$ there exists such an element $x \in S$ that $axb \in H$; we take the empty set also as an m-system).

An ideal P of S is called *completely prime* if $S\backslash P$ is a face of S (the non-empty subset T of S is called a *face* of S if $ab \in T$ if and only if $a \in T$, $b \in T$; the empty set is also considered a face).

The set of all the nilpotent elements of S with respect to J will be denoted by $N(J)$.

The ideal $R(J)$ $[L(J)]$, which is the union of all the nilpotent [locally nilpotent] ideals of S with respect to J is called the Schwarz [Ševrin] *radical* of S with respect to J.

The ideal $R^*(J)$, which is the union if all nil-ideals of S with respect to J is called the Clifford *radical* of S with respect to J.

Let M be a non-empty subset of S. By $\mathbf{C}(M)$ $[\mathbf{M}(M)]$ we denote the set of all such elements $r \in S$ that the intersection of every face [of every m-system] of the semigroup S which contains r with M is non-empty.

It is known (see [4]) that $\mathbf{M}(M)$ $]\mathbf{C}(M)]$ is the intersection of all prime ideals [complete prime ideals] of S which contain M.

The set $M(J)$ $[C(J)]$ is called the McCoy [Jiang Luh] radical of S with respect to J.

The direct product of the semigroups S_1 and S_2 will be denoted by $S_1 \times S_2$. We use the remaining notions in this paper in their current sense.

Theorem 1. *Let I be the minimal ideal of the semigroup $S_1 \times S_2$.*

Then we have:

(a₁) $$N(I) = N(I') \times N(I''),$$

(a₂) $$R(I) = R(I') \times R(I''),$$

(a₃) $$M(I) = M(I') \times M(I''),$$

(a₄) $$L(I) = L(I') \times L(I''),$$

(a₅) $$R^*(I) = R^*(I') \times R^*(I''),$$

(a₆) $$C(I) = C(I') \times C(I''),$$

where I' $[I'']$ is the projection of I into S_1 $[S_2]$.

Proof. For every minimal ideal of $S_1 \times S_2$ we have:

$$I = I' \times I'',$$

where $I'[I'']$ is the projection of I into $S_1[S_2]$ (see [2]). Wherefrom with respect-to Theorem 3 of [1] we obtain the assertion of Theorem 1.

Theorem 2. *Let M_i $(i = 1, 2)$ be an arbitrary non-empty subset of the semigroup S_i. Then the following holds*

(b₁) $$\mathbf{M}(M_1 \times M_2) = \mathbf{M}(M_1) \times \mathbf{M}(M_2),$$

(b₂) $$\mathbf{C}(M_1 \times M_2) = \mathbf{C}(M_1) \times \mathbf{C}(M_2).$$

The proof can be given in the same way as the proof of Theorem 3, (c) and (f) in [1].

Theorem 3. *Let J_1, J_2 be ideals of the semigroup S. Then we have:*

(c₁) $$N(J_1J_2) = N(J_1) \cap N(J_2),$$

(c₂) $$R(J_1J_2) = R(J_1) \cap R(J_2),$$

(c₃) $$M(J_1J_2) = M(J_1) \cap M(J_2),$$

(c₄) $$L(J_1J_2) = L(J_1) \cap L(J_2),$$

(c₅) $$R^*(J_1J_2) = R^*(J_1) \cap R^*(J_2),$$

(c₆) $$C(J_1J_2) = C(J_1) \cap C(J_2),$$

Proof. I. Let J_1, J_2 be arbitrary ideals of S. We know that the following holds:

$$(\alpha) \qquad J_1 \subseteq \mathscr{S}(J_1) \text{ and } \mathscr{S}(J_1 \cap J_2) = \mathscr{S}(J_1) \cap \mathscr{S}(J_2),$$

where instead of $\mathscr{S}$ we can put any of the signs N, R, L, M, R^*, C (see [3], [5]). As $J_1J_2 \subseteq J_1 \cap J_2$, then from ($\alpha$) we have $\mathscr{S}(J_1J_2) \subseteq \mathscr{S}(J_1) \cap \mathscr{S}(J_2)$, where $\mathscr{S} = N, R, M, L, R^*, C$.

II. (c_1) Let x be an element of $N(J_1) \cap N(J_2)$, then x is nilpotent with respect to $J_1(x^{n_1} \in J_1)$ and $J_2(x^{n_2} \in J_2)$. Let $n = n_1 + n_2$, then $x^n \in J_1J_2$. This means that $N(J_1) \cap N(J_2) \subseteq N(J_1J_2)$.

(c_2) Let x be an element of $R(J_1) \cap R(J_2)$, then x is the element of a nilpotent ideal I_2 with respect to $J_2(I_2^{n_2} \in J_2)$ and of a nilpotent ideal I_1 with respect to $J_1(I_1^{n_1} \subseteq J_1)$. The ideal $I_1 \cap I_2$ is nilpotent with respect to J_1J_2, because $(I_1 \cap I_2)^n \subseteq J_1J_2$, where $n = n_1 + n_2$. This means that $R(J_1) \cap \cap R(J_2) \subseteq R(J_1J_2)$.

(c_3) Let x be an element of $M(J_1) \cap M(J_2)$. An arbitrary m-system H, which contains x, contains also an element $x_1 \in J_1$ and an element $x_2 \in J_2$. Because H is an m-system of S, there exists at least one element $h \in S$ such that $x_1hx_2 \in H$, but the element $x_1hx_2 \in J_1J_2$. It follows that $M(J_1) \cap M(J_2) \subseteq M(J_1J_2)$.

(c_4) Let $x \in L(J_1) \cap L(J_2)$; then the element x is from a locally nilpotent ideal I_1 with respect to J_1 and from a locally nilpotent ideal I_2 with respect to J_2. Let H be an arbitrary finitely generated subsemigroup of $I_1 \cap I_2$; then there exist natural numbers n_1 and n_2 such that $H^{n_1} \subseteq J_1$ and $H^{n_2} \subseteq J_2$. Therefore for $n = n_1 + n_2$ we have $H^n \subseteq J_1J_2$. Then $L(J_1) \cap L(J_2) \subseteq L(J_1J_2)$.

(c_5) Let x be an arbitrary element of $R^*(J_1) \cap R^*(J_2)$. This means that x is in a nil-ideal I_1 with respect to $J_1(x^{n_1} \in J_1)$ and in a nil-ideal I_2 with respect to $J_2(x^{n_2} \in J_2)$. We will prove that $I_1 \cap I_2$ is a nil-ideal with respect to the ideal J_1J_2. It is clear that $x \in I_1 \cap I_2$ and for $n = n_1 + n_2$ we have $x^n \in J_1J_2$. Thus $R^*(J_1) \cap R^*(J_2) \subseteq R^*(J_1J_2)$.

(c_6) We will prove the assertion (c_6) similarly as (c_3). It is nessesary to take instead of an m-system H a face T of S. From I and II the assertion of Theorem 3 follows.

It is known that the set S_J of all ideals in the sense of multiplication of complexes is a semigroup.

Theorem 4. *Let S be a semigroup and S_J the smigroup of all ideals of S. Then we have:*

(a) *the mapping $J \to N(J)$ is a homomorphism of the semigroup S_J into the semilattice of all subsets of S.*

(b) *the mapping $J \to S(J)$ in an endomorphism of the semigroup S_J into the semilattice of all ideals of S_J, where we can put instead of S any of the signs R, L, M, R, C.*

In (a) [(b)] we understand under the semilattice operation $\cap$ the intersection of two subsets [ideals] of S. The proof follows from Theorem 3.

R. Šulka in his paper [3] proved the following assertions.

$$R(J_1) \cup R(J_2) \subseteq R(J_1 \cup J_2)\,, \tag{d$_1$}$$

$$R^*(J_1) \cup R^*(J_2) \subseteq R^*(J_1 \cup J_2)\,, \tag{d$_2$}$$

$$M(J_1) \cup M(J_2) \subseteq M(J_1 \cup J_2)\,, \tag{d$_3$}$$

where J_1, J_2 are ideals of S. In paper [3] it is shown that there exist such semigroups for which the equality in (d_1), (d_2) and (d_3) does not hold.

Theorem 5. *Let J_1 and J_2 be ideals of S. Then we have:*

$$R^*(R^*(J_1) \cup R^*(J_2)) = R^*(J_1 \cup J_2)\,, \tag{e$_1$}$$

$$M(M(J_1) \cup M(J_2)) = M(J_1 \cup J_2)\,. \tag{e$_2$}$$

Proof. I. From (d_2) we have: $R^*(R^*(J_1) \cup R^*(J_2)) \subseteq R^*(R^*(J_1 \cup J_2)) = R^*(J_1 \cup J_2)$ (see [3]).

II. As $J_1 \subseteq R^*(J_1)$ and $J_2 \subseteq R^*(J_2)$, then $J_1 \cup J_2 \subseteq R^*(J_1) \cup R^*(J_2)$. It follows that $R^*(J_1 \cup J_2) \subseteq R^*(R^*(J_1) \cup R^*(J_2))$. The proof of ($e_2$) is similar ($e_1$).

If we suppose that the suppositions of Theorem 5 are fulfilled, we have

$$R^*(J_1 \cup R^*(J_2)) = R^*(J_1 \cup J_2)\,, \tag{f$_1$}$$

$$M(J_1 \cup M(J_2)) = M(J_1 \cup J_2)\,. \tag{f$_2$}$$

The equalities (e_2) (f_2) are fulfilled even in the case when $J_1[J_2]$ is an arbitrary non-empty subset of S.

There exists a semigroup S in which the following is not fulfilled $R(R(J_1) \cup R(J_2)) = R(J_1 \cup J_2)$, where J_1 and J_2 are ideals of S (see [4], Example 2.). Let S be the semigroup generated by the set $\{0, a, b_1, b_2, \dots\}$ subject to the generating relations

$0x = x0 = 0$ for every $x \in S$;

$a^2 = 0$;

$b_i b_j = 0$ for $i, j = 1, 2, \dots$;

$b_i a b_j = 0$ for $i = j$; $i, j = 1, 2, \dots$;

$(ab_i)^{i+1} = (b_i a)^{i+1} = 0$ for $i = 1, 2, \dots$.

Then $R(R\{0\}) \neq R(\{0\})$ (see [4]). We put $J_1 = J_2 = 0$. Then $R(R(\{0\}) \cup (\{0\})) \neq R(\{0\} \cup \{0\})$.

Let us denote by $\mathscr{P}$ the system of complete prime ideals of S (we take an empty set as a complete prime ideal, too).

Lemma 1. *A non-empty subsystem $\mathcal{U}$ of the system $\mathcal{P}$ of S is linearly ordered with respect to $\subseteq$ if and only if for arbitrary $P \in \mathcal{U}$, $Q \in \mathcal{U}$ there is $P \cap Q \in \mathcal{U}$*

Proof. I. Let $\mathcal{U}$ be a linearly ordered subsystem, then it is clear that for every $P \in \mathcal{U}$, $Q \in \mathcal{U}$ is $P \cap Q \in \mathcal{U}$.

II. Let there for an arbitrary $P \in \mathcal{U}$, $Q \in \mathcal{U}$ be $P \cap Q \in \mathcal{U}$; then either $P \subseteq Q$ or $Q \subseteq P$. Let us suppose the reverse, i. e. $P \nsubseteq Q$ and $Q \nsubseteq P$. Then there exist elements $y \in Q$, $y \notin P$ and $x \notin Q$, $x \in P$. Hence $x, y \in S \backslash (P \cap Q)$ and $xy \in P \cap Q$. Because $P \cap Q \in \mathcal{U}$, then $S\backslash(P \cap Q)$ is a face of S. It means $xy \in S\backslash(P \cap Q)$. It is a contradiction of $xy \in P \cap Q$.

Corollary 1. *The set $\mathcal{P}$ of all complete prime ideals of S with respect to $\subseteq$ is linearly ordered if and only if for an arbitrary $P \in \mathcal{P}$ and $Q \in \mathcal{P}$ there is $P \cap \cap Q \in \mathcal{P}$.*

Corollary 2. *If in the semigroup S every ideal is a complete prime ideal, then the set $\mathcal{P}$ is linearly ordered with respect to $\subseteq$* (see [8]).

Theorem 6. *Let $\mathcal{U}$ be an arbitrary non-empty subsystem of the system $\mathcal{P}$. $\bigcap_{P \in \mathcal{U}} P \in \mathcal{P}$ if and only if for every $P \in \mathcal{P}$ and $Q \in \mathcal{P}$ is $P \cap Q \in \mathcal{P}$.*

Proof. I. If for every non-empty subsystem $\mathcal{U}$ of the system $\mathcal{P}$ $\bigcap_{P \in \mathcal{U}} P \in \mathcal{P}$ holds, then for every two $P, Q \in \mathcal{P}$ there is $P \cap Q \in \mathcal{P}$.

II. Let $\mathcal{U}$ be an arbitrary non-empty subsystem of $\mathcal{P}$ and for every $P \in \mathcal{P}$ and $Q \in \mathcal{P}$ there is $P \cap Q \in \mathcal{P}$. Then the subsystem $\mathcal{U}$ is linearly ordered with respect to $\subseteq$. Let $a, b \in S \backslash \bigcap_{P \in \mathcal{U}} P$, then there exist $P \in \mathcal{U}$, $Q \in \mathcal{U}$ such that $a \in P$ and $b \in Q$. This means that a, b are not the elements of at least one of P, Q. Let e. g. $a \notin P$, $b \notin P$. Then $ab \in S\backslash P \subseteq S \backslash \bigcap_{P \in \mathcal{U}} P$. When $ab \in S \backslash \bigcap_{P \in \mathcal{U}} P$ then $ab \notin R$, where $R \in \mathcal{U}$. This means that $ab \in S\backslash R$, where $S \backslash R$ is a face of S. It follows that $a, b \in S\backslash R \subseteq S \backslash \bigcap_{P \in \mathcal{U}} P$. Therefore $S \backslash \bigcap_{P \in \mathcal{U}} P$ is a face of the semigroup S and $\bigcap_{P \in \mathcal{U}} P \in \mathcal{P}$.

It is known that the subset H of the semigroup S is a face of the semigroup S if and only if $S\backslash P \in \mathcal{P}$ (see [6]). We denote by $\mathcal{H}$ the set of all faces of the semigroup S.

Lemma 2. *A non-empty subsystem $\mathcal{V}$ of the system $\mathcal{H}$ is linearly ordered with respect to $\subseteq$ if and only if for arbitrary $H \in \mathcal{V}$, $T \in \mathcal{V}$ there is $H \cup T \in \mathcal{V}$.*

Proof. I. Let $\mathcal{V}$ be a linearly ordered subsystem; then it is clear that for every $H \in \mathcal{V}$, $T \in \mathcal{V}$ is $H \cup T \in \mathcal{V}$.

II. Let there for an arbitrary $H \in \mathcal{V}$ and $T \in \mathcal{V}$ be $H \cup T \in \mathcal{V}$. The set $P = S\backslash H$ is a complete prime ideal of the semigroup S for every $H \in \mathcal{V}$.

Let $\mathscr{U} = \{P \mid P = S\backslash H, H \in \mathscr{V}\}$. Then $P \cap Q = (S\backslash H) \cap (S\backslash T) = S\backslash(H \cup T) \in \in \mathscr{U}$, where $P \in \mathscr{U}$, $Q \in \mathscr{U}$. Following Lemma 1 we have either $S\backslash H \subseteq S\backslash T$ or $S\backslash T \subseteq S\backslash H$. It follows that either $H \subseteq T$, or $T \subseteq H$.

Theorem 7. *Let $\mathscr{V}$ be an arbitrary subsystem of the system $\mathscr{H}$.* Then $\bigcup_{H \in \mathscr{V}} \mathscr{V} H \in \in \mathscr{H}$ *if and only if for every $H \in \mathscr{H}$ and $T \in \mathscr{H}$* there *is $H \cup T \in \mathscr{H}$.*

Proof. Let there for an arbitrary $H \in \mathscr{H}$, $T \in \mathscr{H}$ be $H \cup T \in \mathscr{H}$. Let $P \in \mathscr{P}$ and $Q \in \mathscr{P}$; then $P \cap Q = (S\backslash H) \cap (S\backslash T) = S\backslash(H \cup T) \in \mathscr{P}$. The set $P = S\backslash H$ is a complete prime ideal of S for every $H \in \mathscr{V}$. Following Theorem 6 we have $\bigcap_{P \in \mathscr{U}} P \in \mathscr{P}$. Further we have $S\backslash \bigcup_{H \in \mathscr{V}} H = \bigcap_{P \in \mathscr{U}} P$. It follows that $\bigcup_{H \in \mathscr{V}} H$ $H \in \mathscr{H}$. The second part of the theorem is clear.

Let S_1, S_2 be semigroups and $S = S_1 \times S_2$ their direct product.

Theorem 8. *Let $\mathscr{P}_1[\mathscr{P}_2, \mathscr{P}]$ be the set of all complete prime ideals of $S_1[S_2, S = S_1 \times S_2]$. $\mathscr{P}$ is linearly ordered with respect to $\subseteq$ if and only if every one of the sets $\mathscr{P}_1$ and $\mathscr{P}_2$ is linearly ordered with respect to $\subseteq$ and at least one of the semigroups S_1 and S_2 does not contain its proper non-zero complete prime ideal.*

Proof. I. Let $\mathscr{P}$ be linearly ordered with respect to $\subseteq$ and let $P_1 \in \mathscr{P}_1[P_2 \in \in \mathscr{P}_2]$ such that $P_1 \neq \emptyset$, $P_1 \neq S_1$ $[P_2 \neq \emptyset$, $P_2 \neq S_2]$. Then it follows that $P = S_1 \times P_2$ and $P' = P_1 \times S_2$ are complete prime ideals of S and $P \nsubseteq P'$, $P' \nsubseteq P$. This is a contradiction. Further let $\mathscr{P}_1 = \{\emptyset, S_1\}$. Let $\mathscr{P}_2$ be linearly non ordered. Then there exist $P_2, P_2' \in \mathscr{P}_2$ such that $P_2 \nsubseteq P_2'$, $P_2' \nsubseteq P_2$. The ideal $P = S_1 \times P_2[P' = S_1 \times P_2']$ is a complete prime ideal of S and $P \nsubseteq P'$, $P' \nsubseteq P$.

II. Let $\mathscr{P}_1$, $\mathscr{P}_2$ be linearly ordered with respect to $\subseteq$ and let $\mathscr{P}_2 = \{\emptyset, S_2\}$.

For an arbitrary $P, P' \in \mathscr{P}$ we have:

$$P = (P_1 \times S_2) \cup (S_1 \times P_2), \quad P' = (P_1' \times S_2) \cup (S_1 \times P_2') .$$

The following cases may arise:

$P_2 = P_2' = S_2$;
$P_2 = \emptyset$, $P_2' = S_2$;
$P_2 = S_2$, $P_2 = \emptyset$;
$P_2 = P_2 = \emptyset$.

As $P_1 \subseteq P_1'$ or $P_1' \subseteq P_1$, we have $P \subseteq P'$ or $P' \subseteq P$.

We denote by $\mathscr{T}[\mathscr{T}_1, \mathscr{T}_2]$ the topology on $S = S_1 \times S_2$ $[S_1, S_2]$; the base is $\mathscr{H}[\mathscr{H}_1, \mathscr{H}_2]$, where $\mathscr{H}[\mathscr{H}_1, \mathscr{H}_2]$ is the set of faces or $S[S_1, S_2]$ (see [7]). We denote by $\mathscr{T}_1 \times \mathscr{T}_2$ the topology of the semigroup S, the base of which is $\mathscr{H}_1 \times \mathscr{H}_2$ (see [3]).

Theorem 9. *Let S_1, S_2 be semigroups and let $S = S_1 \times S_2$. Then for the topology $\mathscr{T}$ and $\mathscr{T}_1 \times \mathscr{T}_2$ we have $\mathscr{H} = \mathscr{H}_1 \times \mathscr{H}_2$.*

Proof. It is clear that $\mathscr{H}_1 \times \mathscr{H}_2 \subseteq \mathscr{H}$ holds (see [3]).

Let $H \in \mathscr{H}$, then $P = S \backslash H \in \mathscr{P}$ and $P = (P_1 \times S_2) \cup (S_1 \times P_2)$ (see [4], [1]). Further $H = S \backslash P = (S_1 \times S_2) \backslash [(P_1 \times S_2) \cup (S_1 \times P_2)] = [(S_1 \times S_2) \backslash \backslash (P_1 \times S_2)] \cap [(S_1 \times S_2) \backslash (S_1 \times P_2)] = [(S_1 \backslash P_1) \times S_2] \cap [S_1 \times (S_2 \backslash P_2)] = (S_1 \backslash P_1) \times (S_2 \backslash P_2) = H_1 \times H_2 \in \mathscr{H}_1 \times \mathscr{H}_2$. It follows that $\mathscr{H}_1 \subseteq \mathscr{H}_1 \times \mathscr{H}_2$.

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