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FINITE VERTEX-TRANSITIVE PLANAR GRAPHS OF THE REGULARITY DEGREE FOUR OR FIVE

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This paper connects onto the study of finite vertex-transitive planar graphs begin in [1]. We shall consider only connected graphs; by the word graph is always meant a connected graph. We admit multiple edges, but not loops.

A vertex-transitive graph is such a graph G that to any two vertices x and y of G there exists an automorphism φ of G such that $\varphi(x) = y$.

A vertex-transitive graph is always regular, i. e. all of its vertices have the same degree, called the regularity degree of the graph. In [1] finite vertex-transitive planar graphs of the regularity degree three were studied. Here we shall continue to study finite vertex-transitive planar graphs by investigating the graphs of the regularity degree four or five. (A finite regular planar graph without multiple edges and loops cannot have a greater regularity degree than five.) As we study planar graphs, we may speak about the faces of a graph.

1. Graphs with multiple edges

In this item we shall state a theorem which will enable us to construct vertex-transitive graphs with multiple edges from vertex-transitive graphs of a smaller regularity degree.

Theorem 1. *Let H be a vertex-transitive graph without multiple edges, let there exist a decomposition of H into pairwise edge-disjoint regular factors $H_1, H_2, \ldots, H_k$ with the property that to any two vertices x, y of H there exists an automorphism φ of H such that $\varphi(x) = y$ and $\varphi(H_i) = H_i$ for $i = 1, \ldots, k$. Let π be a one-to-one mapping of the number set $\{1, \ldots, k\}$ into the set of all positive integers. If to any edge of H_i we adjoin $\pi(i) - 1$ new edges joining the same vertices, we obtain a vertex-transitive graph G. Conversely, every vertex-transitive graph G with multiple edges can be obtained by this method.*

Proof. Let φ be an automorphism of H such that $\varphi(H_i) = H_i$ for $i = 1, \ldots, k$. For the sake of simplicity we consider φ only as a permutation of the vertex set V of H. Any two vertices x, y of G are either non-adjacent, or joined by $\pi(i)$ edges for some i, $1 \leqq i \leqq k$. They are non-adjacent, if and only if they are non-adjacent in H. As φ is an automorphism of H, the vertices $\varphi(x)$, $\varphi(y)$ are non-adjacent in both G and H, if and only if x, y are non-adjacent in both G and H. The vertices x, y are joined by $\pi(i)$ edges in G, if and only if they are joined in H by an edge belonging to H_i ($i = 1, \ldots, k$). As $\varphi(H_i) = H_i$ for $i = 1, \ldots, k$, the vertices $\varphi(x)$, $\varphi(y)$ are joined in H by an edge belonging to H_i if and only if x, y are and thus they are joined in G by $\pi(i)$ edges if and only if x, y are. We have proved that φ is an automorphism of G. As to any two vertices x, y of H there exists an automorphism φ of H such that $\varphi(x) = y$ and $\varphi(H_i) = H_i$ for $i = 1, \ldots, k$, the graph G is vertex-transitive.

Now let G be a vertex-transitive graph containing multiple edges. Let V be its vertex set. Let P be the set of numbers p such that there exists a pair of vertices in G joined exactly by p edges. Let π be a one-to-one mapping of the number set $\{1, 2, \ldots, |P|\}$ onto P. For $i = 1, \ldots, |P|$ let H_i be the graph with the vertex set V in which two vertices are joined by an edge if and only if they are joined exactly by $\pi(i)$ edges in G. Let H be the graph with the vertex set V in which two vertices are joined by an edge if and only if they are joined in G at least by one edge. The graph H is the union of the graphs H_i; the graphs H_i are pairwise edge-disjoint. Now let φ be an automorphism of G. If two vertices x, y are non-adjacent in G, they are non-adjacent also in H and the same holds for $\varphi(x)$, $\varphi(y)$. If the vertices x, y are joined in G by $\pi(i)$ edges, they are joined in H by an edge belonging to H_i and the same holds for $\varphi(x)$, $\varphi(y)$. Therefore each automorphism of G is also an automorphism of H preserving each H_i.

Note that the regularity degree of H is smaller than that of G. Thus we have obtained a method for constructing vertex-transitive graphs with multiple edges from vertex-transitive graphs of a smaller regularity degree.

Here we shall not construct all possible finite planar vertex-transitive graphs of the regularity degree four or five with multiple edges. We shall show only some examples. In Fig. 1 we have some of these graphs with the regularity degree four. In Fig. 1a we see a graph obtained from a circuit by doubling all edges, in Fig. 1b a graph obtained from a circuit of even length by tripling edges of one linear factor, in Figs. 1c and 1d we see graphs obtained from the graph of the cube by doubling the edges of one linear factor.

In the following we shall study only graphs without multiple edges.

2. The regularity degree four

Let G be a finite vertex-transitive planar graph of the regularity degree four. Let v be a vertex of G. The vertex v is incident with four faces of G. By the degree of a face we mean the number of edges belonging to the boundary of this face. Denote the degrees of the faces incident with v by d_1, d_2, d_3, d_4. These values are evidently the same at each vertex.

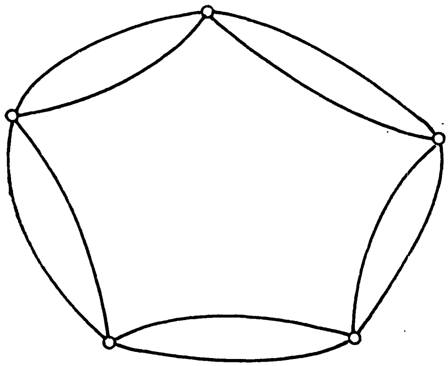

Fig. 1a

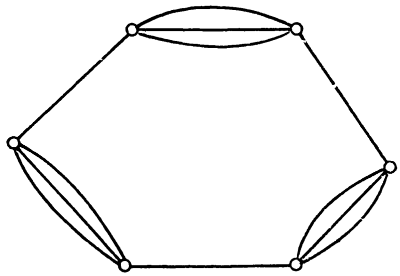

Fig. 1b

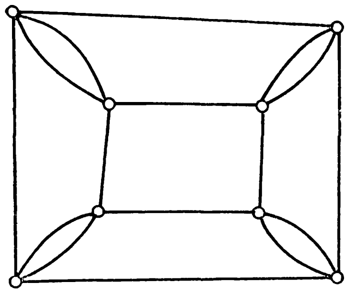

Fig. 1c

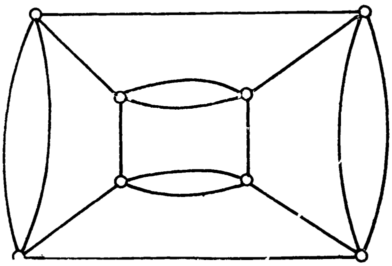

Fig. 1d

Lemma 1. *Let G be a finite vertex-transitive planar graph of the regularity degree four, let n be the number of its vertices. Let any of its vertices be incident with the faces of the degrees d_1, d_2, d_3, d_4. Then*

$$n = 2/(d_1^{-1} + d_2^{-1} + d_3^{-1} + d_4^{-1} - 1)\,.$$

Proof. If d_1, d_2, d_3, d_4 are pairwise different, then each vertex is incident exactly with one face of the degree d_i for $1 \leqq i \leqq 4$. Therefore the number of the faces of the degree d_i is n/d_i and the number of all faces is $n(d_1^{-1} +$

$+ d_2^{-1} + d_3^{-1} + d_4^{-1})$. If $d_1 \neq d_2 \neq d_3 = d_4 \neq d_1$, then each vertex is incident exactly with one face of the degree d_1, one face of the degree d_2 and two faces of the degree d_3. Then the number of the faces of the degree d_1 (or d_2) is n/d_1 (or n/d_2 respectively), the number of the faces of the degree d_3 is $2n/d_3 = n/d_3 + n/d_4$. The number of all faces is again $n(d_1^{-1} + d_2^{-1} + d_3^{-1} + d_4^{-1})$.

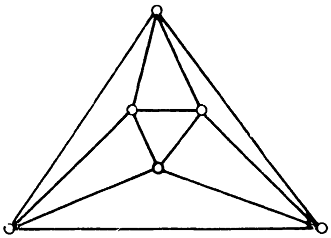

Fig. 2a

Analogously in all other cases we can prove that the number of all faces is equal to this expression. The number of edges of G is $2n$, because G has the regularity degree 4. Substituting into Euler's formula for planar graphs we obtain

$$n = 2/(d_1^{-1} + d_2^{-1} + d_3^{-1} + d_4^{-1} - 1)\,. \tag{1}$$

This lemma implies that in our study it is sufficient to restrict our considerations to the values of d_1, d_2, d_3, d_4 for which the expression (1) equals a positive integer. As we study only graphs without multiple edges, we take $d_i \geqq 3$ for $i = 1, \ldots, 4$.

Without a loss of generality we may distinguish five possible cases:

(α) $d_1 = d_2 = d_3 = d_4$.
(β) $d_1 \neq d_2 = d_3 = d_4$.
(γ) $d_1 = d_3 \neq d_2 = d_4$.

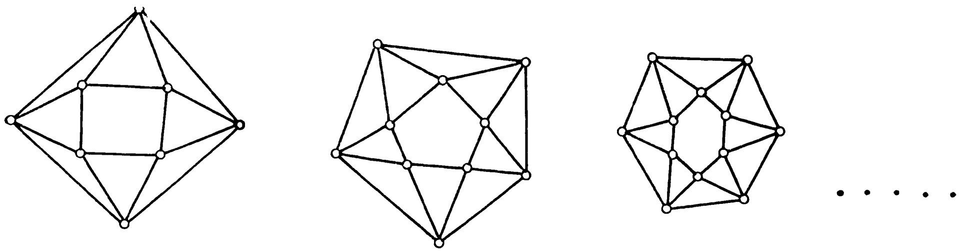

Fig. 2b

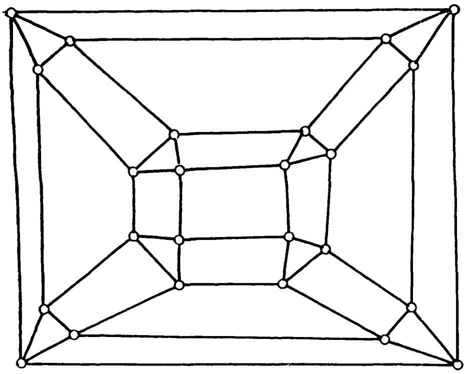

Fig. 2c

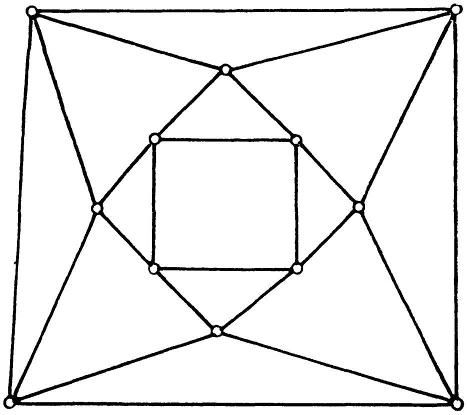

Fig. 2d

(δ) $d_1 \neq d_2 \neq d_3 = d_4 \neq d_1$.
(ε) d_1, d_2, d_3, d_4 are pairwise different.

In the case of (α) evidently all faces of G have equal degrees. It is well known that there exists only one finite graph of the regularity degree four without multiple edges satisfying this, namely the graph of the regular octahedron (Fig. 2a). In this case $d_1 = 3$ and this is also the unique value of d_1 in the case of (α) for which the expression (1) equals a positive integer.

In the case of (β) we have two possibilities:

(a) d_1 arbitrary, $d_2 = 3$;
(b) $d_1 = 3$, $d_2 = 4$.

In the case of (a) we have $n = 2d_1$; we obtain infinitely many graphs and their construction is evident from the examples in Fig. 2b for values of d_1 equal to 4, 5, 6. In the case of (b) we have $n = 24$; the corresponding graph is in Fig. 2c. In these cases, as well as in the following ones, we start the construction from one face and each moment we know the degrees of faces incident with each vertex and their cyclic order; thus the constructed graphs are uniquely possible. The proof of the vertex-transitivity can be made by finding corresponding automorphisms for all pairs of vertices; as the drawing has a high degree of symmetry, it is not too difficult.

In the case (γ) we have again two possibilities:

(a) $d_1 = 3$, $d_2 = 4$;
(b) $d_1 = 3$, $d_2 = 5$.

In both these cases $d_1 = 3$. Therefore each vertex of G must be incident exactly with two triangular faces. We shall prove that these two faces cannot

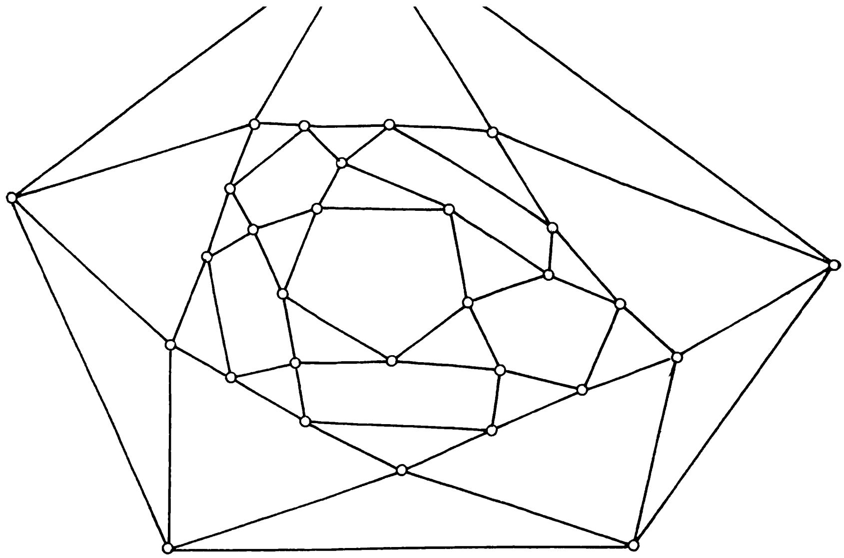

Fig. 2e

have a common edge. Let T_1, T_2 be two triangular faces incident with a vertex u; let them have a common edge uv. Let w be the vertex of T_1 different from u and v. There exists an automorphism of G mapping w onto u, therefore w must be also an end vertex of an edge belonging to two triangular faces. As w cannot be incident with more than two triangular faces, this edge must be either uw, or vw. If it is uw (or vw), then u (or v respectively) is incident with three triangular faces, which is impossible. Therefore no two triangular faces have a common edge. Let us construct a graph $F(G)$ whose vertices are the triangular faces of G and in which two vertices are joined by an edge if and only if the corresponding faces have a common vertex in G. The graph $F(G)$ is planar and regular of the degree three and all of its faces have the degree d_2 (i. e. 4 in (a) or 5 in (b)). In the case of (a) the graph $F(G)$ must be the graph of the cube, in the case of (b) the graph of the regular dodecahedron. We go from $F(G)$ back to G constructing the line graph of $F(G)$. We obtain the graphs in Figs. 2d and 2e.

Now consider the case of (δ). First we prove that the case of $d_3 = 3$ is impossible. Analogously to the case of (γ) we prove that no two triangular faces have a common edge. Now let T be a triangular face of such a graph

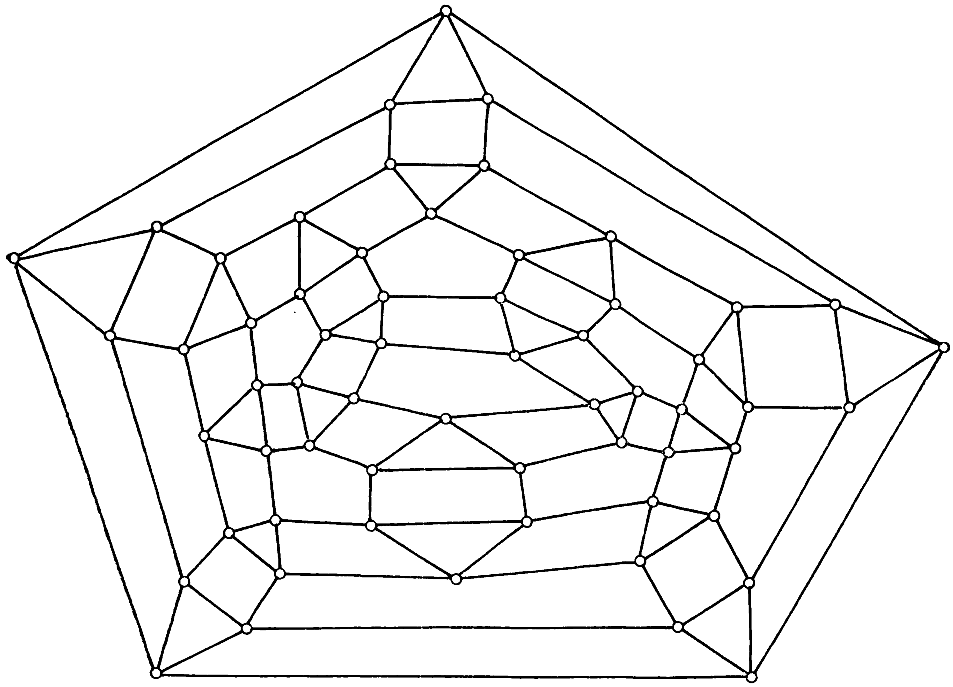

Fig. 2f

with the vertices u, v, w. The edge uv is the common edge of T and some non-triangular face of G; let the degree of this face be d_1 (without a loss of generality). The edge uw is the common edge of T and some other non-triangular face. The degree of this face cannot be d_1, because in that case the vertex u would be incident with two faces of the degree d_1. Thus it is d_2. But the edge vw is also a common edge of T and some non-triangular face. The degree of this face cannot be d_1, because the vertex v would be incident with two faces of the degree d_1, and it cannot be d_2, because the vertex w would be incident with two faces of the degree d_2. We have obtained a contradiction. Thus $d_3 \neq 3$. In the unique case in which the expression (1) equals a positive integer we have $d_1 = 3$, $d_2 = 5$, $d_3 = 4$ (without a loss of generality). We can prove that in a graph corresponding to this case no two tetragonal faces have a common edge. If it were so, then by omitting all edges which belong to two tetragonal faces we would obtain a disconnected regular graph of the degree three in which two connected components would contain one face of the degree equal to the number of vertices of this component and other faces of the degree three and five, and no two faces of the same degree have a common edge.

This is not possible. (The detailed proof is left to the reader.) Therefore two tetragonal faces can have at most one vertex in common. If by $F(G)$ we denote the graph whose vertices are tetragonal faces of G and in which two vertices are joined by an edge if and only if the corresponding tetragonal faces have a vertex in common, then $F(G)$ is a regular graph of the degree four and its faces have degrees three and five (it is obviously planar); each edge is incident

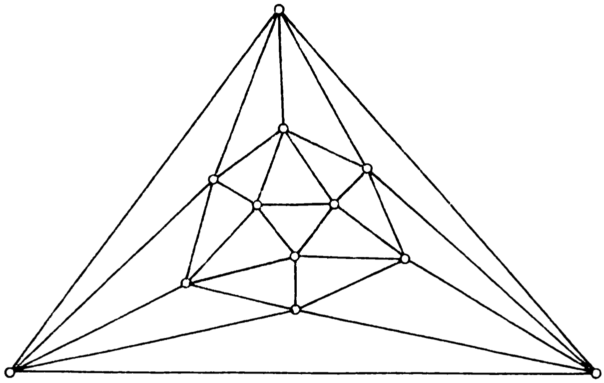

Fig. 3a

with one face of the degree three and one face of the degree five. Therefore $F(G)$ is isomorphic to the graph in Fig. 2e. If we go back to the graph G, we obtain the graph in Fig. 2f.

In the case of (ε), if all of the numbers d_1, d_2, d_3, d_4 are greater than or equal to three, the expression (1) is never equal to a positive integer. Thus this case is impossible.

We have obtained a theorem.

Theorem 2. *Fig. 2 gives a complete list of finite connected vertex-transitive planar graphs of the regularity degree four without multiple edges.*

3. The regularity degree five

For this case a lemma analogous to Lemma 1 holds.

Lemma 2. *Let G be a finite vertex-transitive planar graph of the regularity degree five, let n be the number of its vertices. Let any of its vertices be incident with the faces of the degrees d_1, d_2, d_3, d_4, d_5. Then*

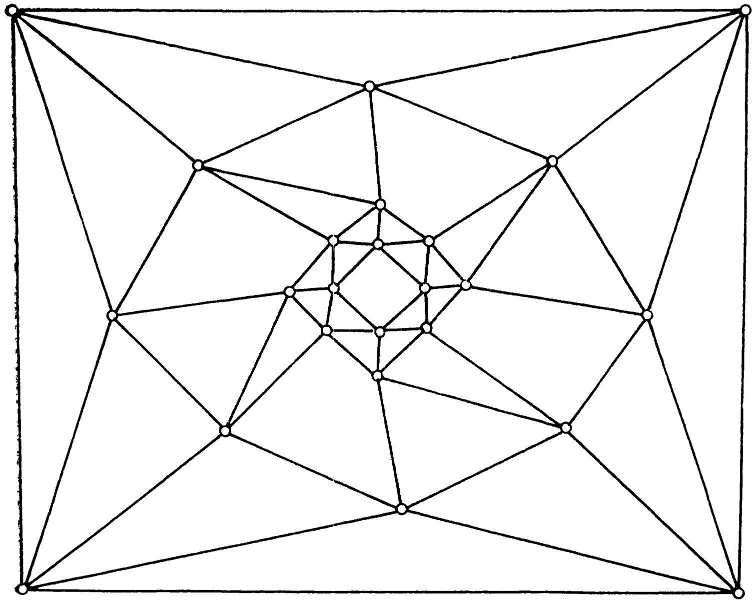

Fig. 3b

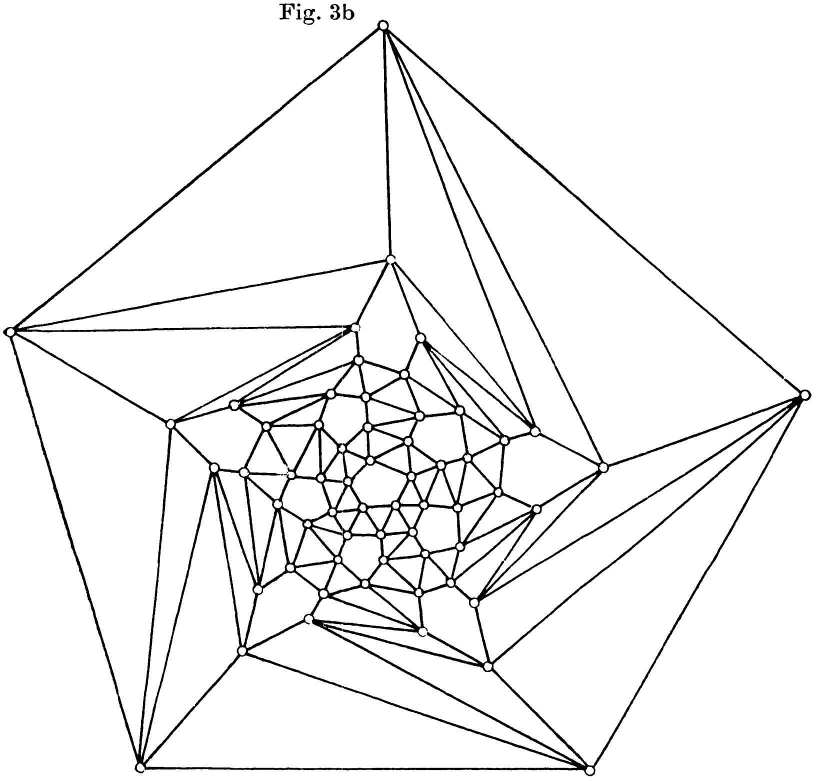

Fig. 3c

$$n = 2/\left(d_1^{-1} + d_2^{-1} + d_3^{-1} + d_4^{-1} + d_5^{-1} - \frac{3}{2}\right).$$

The proof is analogous to the proof of Lemma 1.

Again we restrict our considerations to graphs without multiple edges; therefore $d_i \geqq 3$ for $i = 1, \ldots, 5$. There are three possible cases:

(a) $d_1 = d_2 = d_3 = d_4 = d_5 = 3$;
(b) $d_1 = d_2 = d_3 = d_4 = 3,\ d_5 = 4$;
(c) $d_1 = d_2 = d_3 = d_4 = 3,\ d_5 = 5$.

In the case of (a) we have $n = 12$; the required graph is the graph of the regular icosahedron (Fig. 3a). In the case of (b) we have $n = 24$, in the case of (c) we have $n = 60$. The corresponding graphs are in Figs. 3b, 3c. For their vertex-transitivity and uniqueness see a remark in § **2**, case of (β). Thus we have a theorem.

Theorem 3. *The Fig.* 3 *gives a complete list of finite connected vertex-transitive planar graphs of the regularity degree five without multiple edges.*

Finite vertex-transitive planar graphs of the regularity degree greater than five must contain multiple edges; thus they are obtained by the method described in Theorem 1.

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