Ferdinand Gliviak

On Certain Edge-Critical Graphs of a Given Diameter

Matematický časopis, Vol. 25 (1975), No. 3, 249--263

Persistent URL: http://dml.cz/dmlcz/126397

Mat. Čas. 25, 1975, No. 3, 249—263

ON CERTAIN EDGE-CRITICAL GRAPHS OF A GIVEN DIAMETER

FERDINAND GLIVIAK

1. Introduction. The graphs considered in this paper are undirected, finite, without loops or multiple edges. A graph G is said to be edge-critical (briefly critical), if the deleting of an arbitrary edge from G increases its diameter. Critical graphs were studied in [5], [4], [8], where many problems appeard to be more simple for the graphs of diameter $d \geq 2$ with a girth at least $d + 2$ called ω_d-graphs. Special classes of ω_d-graphs are studied in [2], [3], [7], [9].

Here we shall prove that for an integer $d \geq 2$, and for any graph G of a girth at least $d + 2$ there exists an ω_d-graph containing G as an induced subgraph. Then we shall prove estimates of the minimum degree, the maximum degree and the number of edges of ω_d-graphs, respectively. For proving these assertions we use notions of a k-covering and a $\nu(k)$-extension.

2. Notations and notions. Let G be a graph. Then $V(G)$ will denote the vertex set of G, $E(G)$ the edge set of G, $\mathrm{d}_G(u, v)$ the distance between vertices $u, v \in V(G)$ in G, $d(G)$ the diameter of G, $e_G(u)$ the eccentricity of a vertex u in G, $\deg_G u$ the degree of a vertex u in G, $\delta(G)$ the minimum degree of G, $\Delta(G)$ the maximum degree of G and $N_G(u)$ the neighbourhood of a vertex u (the set of vertices adjacent to u) in G. (Sometimes these symbols are abbreviated to $d(u, v)$, $e(u)$, $\deg u$ and $N(u)$.)

In addition, we denote by $\varkappa(G)$ the vertex-connectivity of G, by $|A|$ the cardinality of a set A, by $[x]$ the greatest integer not exceeding a real number x, by P_r (for an integer $r \geq 2$) the graph generated by a path of the length $r - 1$ and by C_r ($r \geq 3$) the graph generated by a circuit of the length r. Definitions of notions not included here can be found in [6].

The girth of a graph G containing a circuit is the length of a shortest circuit in G and the girth of an acyclic graph is defined as ∞. If K is a circuit of G of the length r and if $d_G(x, y) = d_K(x, y)$ for every two vertices x, y of K, then K is called an exact r-angle of G. The graph G is called irreducible if $N(a) \neq N(b)$ for every $a, b \in V(G)$, $a \neq b$. (This notion arose from studying extensions of ω_d-graphs by one vertex, see [5].) Finally, we define a k-covering and a $\nu(k)$-extension of graphs.

Definition 1. *Let $k \geq 2$ be an integer. A k-covering of a graph G is defined as a set A of vertices of G such that;*

1) $d(a, b) \geq k$ *for every* $a, b \in A$, $a \neq b$;

2) *for every* $x \in V(G)$ *there exists* $y \in A$ *such that* $d(x, y) < k$.

Definition 2. *Let $k \geq 2$ be an integer. By a $v(k)$-extension of a graph G [through a set A] we mean a graph Q that arose from G by adding one new vertex adjacent to every vertex of a k-covering A of G.*

One can see that the notions of a 2-covering and of a kernel of a graph are equivalent. The k-covering and the $v(k)$-extension of a graph of diameter $k \geq 2$ were studied in [5].

Lemma 1. *Let G be an ω_r-graph ($r \geq 2$) and let u be its vertex. Then the set $N_G(u)$ is an r-covering of the graph $G - u$.*

Proof. For every $x, y \in N_G(u)$ we have $d_{G-u}(x, y) \geq r$, because in the reverse case the graph G would contain a circuit of a length $k \leq r + 1$. For every $x \in V(G - u) - N_G(u)$ there exists $z \in N_G(u)$ such that $d(x, z) < r$, because otherwise it would be $d_G(u, x) > r$, which is a contradiction. The lemma follows.

Corollary 1. *The neighbourhood of every vertex of an ω_2-graph G is a 2-covering of G.*

3. Existence theorem.

If a graph G is an induced subgraph of some ω_d-graph, $d \geq 2$, then the girth of G is at least $d + 2$. In this part we shall prove the converse implication.

Theorem 1. *Let $d \geq 2$ be an integer and let G be a graph of a girth at least $d + 2$. Then there exists an irreducible ω_d-graph containing G as an induced subgraph.*

Corollary 2. *Any graph without triangles is isomorphic to an induced subgraph of a graph of diameter two without triangles.*

Now we prove two lemmas and then Theorem 1.

Lemma 2. *Let $k, d \geq 2$ be given integers. Let G be a graph of diameter d and let G_1 be its $v(k)$-extension through a k-covering A. Then we have;*

a) *if* $2 \leq k \leq d$, *then* $\left\lceil \frac{k+2}{2} \right\rceil \leq d(G_1) \leq d$.

b) *if* $2 \leq d < k$, *then* $|A| = 1$ *and* $d \leq d(G_1) \leq d + 1$. *Moreover if we denote* $A = \{a\}$, *then* $d(G_1) = d + 1$ *if and only if the eccentricity of* a, $e_G(a) = d$.

Proof. Let $w = V(G_1) - V(G)$.

a) It is clear that $d_{G_1}(x, y) \leq d_G(x, y)$ for every $x, y \in V(G)$. Further, $d_{G_1}(w, x) \leq k$ holds for every $x \in V(G)$, because either $x \in A$ and then $d_{G_1}(w, x) = 1$ or $x \notin A$ and then there exists $z \in A$ such that $d_G(z, x) \leq k - 1$ so that $d_{G_1}(w, x) \leq k \leq d$. Hence $d(G_1) \leq d$.

If $|A| = 1$, then $d(G_1) = d$, because $d_{G_1}(x, y) = d_G(x, y)$ for all $x, y \in V(G)$ and moreover $d_{G_1}(w, x) \leq k \leq d$ for any $x \in V(G)$. If $|A| \geq 2$, then G_1 contains at least one exact s-angle, $s \geq k + 2$, because it is a $\nu(k)$-extension of G through A. It follows that $d(G_1) \geq \left[\frac{s}{2}\right] \geq \left[\frac{k+2}{2}\right]$ and a) holds.

b) Let $a \in A$. Then $d_G(a, x) \leq d < k$, for every $x \in V(G)$. Hence $|A| = 1$, so that $d_G(x, y) = d_{G_1}(x, y)$, for every $x, y \in V(G)$. Thus we have $d \leq d(G_1) \leq$ $\leq d + 1$. It is clear that $e_G(a) \leq d$. If $e_G(a) < d$, then $d(G_1) = d$, because $d_{G_1}(w, x) \leq d$ for every $x \in V(G)$. If $e_G(a) = d = d(a, z)$, then $d(G_1) = d + 1 =$ $= d(w, z)$. The lemma follows.

Lemma 3. *Let $k \geq 2$ be an integer. From any (irreducible) graph an (irreducible) graph of diameter k can be obtained by a finite number of $\nu(k)$-extensions.*

Proof. Let $d(G) > k$. Let us construct a sequence of graphs

$$G = G_1, G_2, \ldots, G_s \tag{1}$$

(where s is a natural number) in the following manner: G_{i+1} is a $\nu(k)$-extension of G_i through a k-covering X_i of G_i, $1 \leq i \leq s - 1$. This k-covering X_i of G_i is constructed in such a way that X_i contains at least one pair of vertices a, b of G_i such that $d_{G_i}(a, b) > k$. If such a pair does not exist, then we put $s = i$ and the sequence (1) is constructed.

The set X_i, $1 \leq i \leq s - 1$, is not the neighbourhood of a vertex of G_i, because in the reverse case $d_{G_i}(x, y) \leq 2$ for every $x, y \in X_i$. Thusif G_s is irreducible, then G_s is irreducible, too. According to Lemma 2 and the construction of sequence (1) we have $d(G_i) \geq d(G_{i+1})$, $1 \leq i \leq s - 1$, and $d(G_s) \leq k$. If $d(G_s) = k$, then the lemma holds. If $d(G_s) = d(u, v) = r < k$, then we get the required graph Q of diameter k by joining the vertex u with one endpoint of a new path of length $k - r - 1$, which can be done by $k - r$ suitable $\nu(k)$-extensions, too. Thus we proved the part of Lemma 3.

If $d(G) = k$, then the lemma holds. If $d(G) = r < k$, then we construct the required graph analogously as in the case of $d(G_s) = r < k$. The lemma follows.

Proof of Theorem 1. If G is an irreducible graph, then we put $G_1 = G$. If $N_G(u) = N_G(v)$ for some vertices $u \neq v$ of G, then we join one of them with a new vertex. By a successive application of this procedure we obtain an irreducible graph G_1 containing G as an induced subgraph.

Let us construct to G_1 a sequence of graphs $G_1, G_2, \ldots, G_s$ and then we construct to G_s the graph Q of diameter d in such a way as in the proof of Lemma 3, by $\nu(d)$-extensions.

The graph Q is irreducible according to Lemma 3, because G_1 is an irreducible graph. Directly from the construction of Q it follows that G is an induced subgraph of Q. The graph Q is an C_d-graph, because the girth of G is at least $d + 2$ and by $\nu(d)$-extensions a circuit of a length $r \leq d + 1$ does not arise. Hence the theorem holds.

4. Estimates of the minimum and the maximum degree

We shall prove here that if G is an ω_d-graph ($d \geq 2$) with p vertices, then $1 \leq \delta(G) \leq \left(\frac{p}{2}\right)^{\frac{2}{d}}$. It is well known that $2 \leq \Delta(G) \leq p - d + 1$, for any ω_d-graph with p vertices and these bounds are attained. In Theorem 3 we shall prove stronger estimates of the maximum degree of irreducible ω_d-graphs.

Lemma 4. *Let $d \geq 2$ be an integer and let G be an ω_d-graph with p vertices and minimum degree m. Then we have;*

a) *If $m = 1$, then $p \geq d + 1$.*

b) *If $m = 2$, then $p \geq 2d$.*

c) *If $m \geq 3$ and $d = 2$, then $p \geq 2m$.*

d) *If $m \geq 3$ and $d \geq 3$, then* $p \geq 2\dfrac{m(m-1)^{\left[\frac{d}{2}\right]} - 2}{m-2} + x$, *where*

$$x = \begin{cases} m(m-2)\,(m-1)^{\left[\frac{d}{2}\right]-1} & \text{if } m \text{ is odd;} \\ -2(m-1)^{\left[\frac{d}{2}\right]-1} & \text{if } m \text{ is even.} \end{cases}$$

Proof. Parts a) and b) hold, because P_{d+1} and C_{2d} are the smallest ω_d-graphs with minimum degrees 1 and 2, respectively.

c) Suppose that for $u \in V(G)$ we have $\deg u = m$. Every vertex $w \in N(u)$ is adjacent to at least $m - 1$ vertices not belonging to $N(u) \cup \{u\}$, because G does not contain a triangle and $\deg w \geq m$. Thus $p \geq 2m$.

d) Let us put $A_i(z) = \{x | x \in V(G) \wedge d(z, x) = i\}$, where $z \in V(G)$ and $i = 1, 2, \ldots, d$. Let $d(a, b) = d$ for $a, b \in V(G)$. Then the sets $A_i(a)$ and $A_i(b)$ are non-empty for $i = 1, 2, \ldots, d$ and moreover $|A_1(a)| \geq m$ and $|A_1(b)| \geq m$. We have $|A_i(z)| \geq m(m-1)^{i-1}$ for $z = a, b$ and for $i = 2, 3, \ldots, \left[\frac{d}{2}\right]$, because any vertex from $A_{i-1}(z)$ is adjacent to at least $m - 1$ vertices of the set $A_i(z)$ and in addition different vertices of $A_{i-1}(z)$ to different vertices of $A_i(z)$, since the girth of G is at least $d + 2$. Therefore the sets $\bigcup_{i=1}^{[\frac{d}{2}]} A_i(a)$ and $\bigcup_{i=1}^{[\frac{d}{2}]-1} A_i(b)$ are disjoint, as $d(a, b) = d$. Hence

$$p \geq \left| \bigcup_{i=1}^{[\frac{d}{2}]} A_i(a) \right| + \left| \bigcup_{i=1}^{[\frac{d}{2}]-1} A_i(b) \right| \geq 2(1 + m + m(m-1) + \ldots + m(m-1)^{[\frac{d}{2}]-1}) +$$

$$+ m(m-1)^{[\frac{d}{2}]-1} = f(m, d).$$

Let $d = 2s + 1$, $s \geq 1$. Then $\left[\frac{d}{2}\right] = s$ and $|A_{s+1}(a)| \geq (m-1)\,|A_s(a)| \geq \geq m(m-1)^s$, since G does not contain a circuit of length $k \leq 2s + 2$. It follows that $A_{s+1}(a) \cap (\bigcup_{i=1}^{s-1} A_i(b)) = \emptyset$, because $d(a, b) = 2s + 1$. Thus we can add the number $m(m - 1)$ to the foregoig estimate and then we have

$$p \geq f(m, d) + m(m-1)^s = 2\,\frac{m(m-1)^s - 2}{m-2} + m(m-2)\,(m-1)^{s-1}.$$

Thus the assertion of the Lemma holds.

Let $d = 2s$, where $s \geq 2$. Then $\left[\frac{d}{2}\right] = s$. Every vertex $u \in A_s(a)$ is adjacent to at most one vertex from the set $A_{s-1}(b)$, because in the reverse case G contains a circuit of length $k \leq 2s$. Thus the vertex u is adjacent to at least $m - 2$ vertices of $A_{s+1}(a)$ not belonging to $A_{s-1}(b)$. Let W be the set of vertices from $A_{s+1}(a)$ not counted so far. Obviously there exist at least $(m-2)|A_s(a)| \geq \geq m(m-2)\,(m-1)^{s-1}$ edges with one endpoint in the set $A_s(a)$ and the second in W. Every vertex $x \in W$ is adjacent to at most m vertices of the set $A_s(a)$, because otherwise G would contain a circuit of length $k \leq 2s$. Hence $|W| \geq (m-2)\,(m-1)^{s-1}$ and then we have

$$p \geq f(m, d) + (m-2)\,(m-1)^{s-1} = 2\,\frac{m(m-1)^s - 2}{m-2} - 2(m-1)^{s-1}.$$

This completes the proof.

These estimates are reached e. g. for $d = 2$, $m \geq 2$ in the complete bipartite graph $K_{r,r}$ $(r \geq 2)$.

Theorem 2. *Let G be an ω_d-graph, $d \geq 2$ natural, with p vertices and minimum degree m. Then*

$$(2) \qquad 1 \leq m < \left(\frac{p}{2}\right)^{\frac{2}{d}} + 1\,.$$

Proof. It is clear that $m \geq 1$ and the equality holds in any tree of diameter d. Now we prove the upper estimate. If $d = 2$, then from Lemma 4 it follows that $m < \frac{p}{2} + 1$.

Let $d \geq 3$. If $m = 2$, then $p \geq 2d$ according to Lemma 4. If we write $p = 2d + x$, where $x \geq 0$, then we have $\left(\frac{p}{2}\right)^{\frac{2}{d}} + 1 = (d + x)^{\frac{2}{d}} + 1 \geq d^{\frac{2}{d}} + + 1 > 1 + 1 = 2$, because $d^{\frac{2}{d}} = e^{\frac{2}{d}\ln d} > 1$ for an integer $d \geq 3$. Thus the assertion of Theorem holds.

Let $m \geq 3$ and $d = 2s$, where $s \geq 2$. Then according to Lemma 4 we have $\frac{p}{2} \geq \frac{m(m-1)^s - 2}{m-2} - (m-1)^{s-1} = (m-1)^s + \frac{2(m-1)^s - 2}{m-2} (m-1)^{s-1} > > (m-1)^s$, because the inequality

$$\frac{2(m-1)^s - 2}{m-2} - (m-1)^{s-1} > 0 \tag{3}$$

holds. Hence $\frac{p}{2} > (m-1)^{\frac{d}{2}}$ and then $m < \left(\frac{p}{2}\right)^{\frac{2}{d}} + 1$.

Let $m \geq 3$ and $d = 2s + 1$, where $s \geq 1$. Then according to Lemma 4 we have $p \geq 2\,\frac{m(m-1)^s - 2}{m-2} + (m(m-2)\,(m-1)^{s-1} = 2(m-1)^s + + m(m-2)\,(m-1)^{s-1} + 2\,\frac{2(m-1)^s - 2}{m-2}$. Consequently we have

$$p \geq (m-1)^{s-1}\,(m^2 - 2) + 4\,\frac{(m-1)^s - 1}{m-2}. \tag{4}$$

It can be easily verified that $m^2 - 2 > (m-1)^e$ and also $\frac{(m-1)^s - 1}{m-2} \geq 1$. Thus we have $p \geq (m-1)^{s+1} + 4 = (m-1)^{\frac{d+1}{2}} + 4$. If $m \geq 5$, then $(m-1)^{\frac{d+1}{5}} + 4 > 2(m-1)^{\frac{d}{2}}$ and thus $p \geq 2(m-1)$.

If $m = 3$ or 4, then from the formula (4) it follows that

$$p \geq (m-1)^{s-1}\,(m-2) + 4\,\frac{(m-1)^s - 1}{m-2} > 2(m-1)^{\frac{2s+1}{2}} = 2(m-1)^{\frac{d}{2}}.$$

Therefore $m < \left(\frac{p}{2}\right)^{\frac{2}{d}} + 1$ and the Theorem holds.

Remark 1. The estimate (2) can be improved in some cases:

a) If $d \geq 4$ and even, $p \geq 10$, then $m < \left(\frac{p-8}{2}\right)^{\frac{2}{d}}$. The proof of this estimate is the same as in Theorem 4, but we use the inequality $\frac{2(m-1)^s - 2}{m-2} - (m-1)^{s-1} \geq 4$ instead of (3).

b) If $d \geq 3$ and odd, then $m < (p-4)^{\frac{2}{d+1}} + 1$. This upper estimate follows directly from the inequality $p > (m-1)^{\frac{d+1}{2}} + 4$, proved in Theorem 4.

Now we prove an estimate of the maximum degree of irreducible ω_d-graphs.

Lemma 5. *Let $d \geq 2$ be an integer. Let G be an irreducible ω_d-graph with p vertices and the maximum degree n. Then we have $d + n - 1 + c \leq p \leq$ $\leq 1 + n \sum_{i=1}^{d} (n-1)^{i-1}$, where*

$$c = \begin{cases} 0, & \text{if } d = 2,\ n = 2; \\ 3, & \text{if } d = 2,\ n \geq 3; \\ \max(0, n-2), & \text{if } d = 3; \\ \max(0, n-3), & \text{if } d \geq 4. \end{cases}$$

Proof. Obviously, the upper estimate holds and is reached in the Moore graphs. We shall prove the lower estimate.

Let $\deg u = n$ for $u \in V(G)$. Let us put $A = N(u)$, $B = V(G) - (A \cup \{u\})$. We have $n \geq 2$, because $d \geq 2$. If $d = 2$ and $n = 2$, then obviously $p \geq$ $\geq 1 + n = 3$.

Let $d = 2$, $n = |A| \geq 3$. Then $B \neq \emptyset$, because G is an irreducible graph. If $B = \{x\}$, then there would be $N_G(x) = N_G(u)$, what is impossible. Hence $|B| \geq 2$.

Let $B = \{x, y\}$, $x \neq y$. If $(x, y) \in E(G)$, then every vertex $a \in A$ is adjacent to exactly one vertex from the set $\{x, y\}$, because G is an ω_2-graph. Thus either x or y is adjacent to at least two vertices of A (because $|A| \geq 3$) and then their neighbourhoods will be equal, which is impossible. If $(x, y) \notin E(G)$, then G contains the edges (a, x) (a, y) for every $a \in A$, because $d(G) = 2$, and then $N_G(x) = N_G(y)$, which is impossible. Hence $|B| \geq 3$ and then $p \geq$ $\geq 1 + n \geq 3$.

Let $d \geq 3$. It is obvious that $p \geq 1 + n = |A \cup \{u\}|$. The graph G contains at least one path $P(x, y)$ of the length d such that $d_G(x, y) = d$. This path contains at most three vertices from the set $A \cup \{u\}$, the vertex u and two vertices adjacent to them. Hence $p \geq 1 + n + (d + 1 - 3) = d + n - 1$. The set A contains at most one vertex of the degree one because G is irreducible. At most two vertices of A that belong to $P(x, y)$ can have the degree greater than one. Consequently, if $n > 3$, then at least $n - 3$ vertices of A are adjacent to some vertices of the set B and moreover different vertices of A are adjacent to different vertices of B, because G does not contain any 4-angle. It follows that $p \geq d + n - 1 + \max(0, n - 3)$.

If $d = 3$, then at most one of the vertices of A belonging to $P(x, y)$ has the degree greater than one. Thus the proved estimate can be improved by one, because G is irreducible. Hence we have $p \geq d + n - 1 + \max(0, n - 2)$. This estimate is reached in a tree whose construction is clear from the text. Q.E.D.

Lemma 6. *Let T be an irreducible tree of diameter $d \geq 4$, with p vertices and maximum degree n. Then we have;*

$$d + 2n - 4 \leq p \leq \begin{cases} \dfrac{n(n-1)^s - 2}{n-2} + n(n-1)^{s-1}, & \textit{if} \quad d = 2s; \\ \dfrac{n(n-1)^s - 2}{n-2} + (n^2 + n - 3)(n-1)^{s-2}, & \textit{if } d = 2s + 1. \end{cases}$$

Proof. The lower estimate follows directly from Lemma 5. We shall prove the upper estimate. Let $A \subset V(T)$ be the center of T and let $a \in A$. Then the degree of every vertex $x \in V(T)$ such that $d_T(a, x) \leq \left[\dfrac{d}{2}\right] - 2$ can be equal to n.

Let $d = 2s$. Then $\deg x = 2$ for every vertex x of G such that $d(a, x) = s - 1$, because T is irreducible and $\deg x = 1$ for every vertex x of G such that $d(a, x) = s$, because $d(T) = 2s$. Hence we have

$$p \leq 1 + n \cdot \sum_{i=1}^{s} (n-1)^{i-1} + n(n-1)^{s-1} = \frac{n(n-1)^s - 2}{n-2} + n(n-1)^{s-1}.$$

Let $d = 2s + 1$. Then the centre A of T consists of adjacent vertices, according to [6], Theorem 4. 2. Let $A = \{a, b\}$. The branch of the tree T that contains the vertex b and does not contain a has the length s. The endpoints of this branch have the degree one; the vertices adjacent with these endpoints have the degree two and all other vertices of this branch can have the degree n. By adding these vertices we obtain the formula $p \leq 1 + n \sum_{i=1}^{s} (n-1)^{i-1} + n(n-1)^{s-1} - (n-1)^{s-2} + 2(n-1)^{s-1} = \dfrac{n(n-1)^s - 2}{n-2} + (n^2 + n - 3)(n-1)^{s-2}$. Obviously this estimate can be attained. The Lemma follows.

Theorem 3. *Let $d \geq 2$ be an integer. Let G be an irreducible ω_d-graph different from the graphs $P_{d+1}, C_{2d}, C_{2d+1}$, with p vertices and the maximum degree n. Then we have*

$$\text{a)}\quad \left(\frac{p}{3}\right)^{\frac{1}{d}} + 1 < n \leq \begin{cases} p - 4, & \text{if } d = 2; \\ \frac{p}{2}, & \text{if } d = 3; \\ \frac{p-d+4}{2}, & \text{if } d \geq 4. \end{cases}$$

b) *If moreover G is a tree, then* $\left(\frac{p}{4}\right)^{\frac{2}{d}} + 1 < n \leq \frac{p-d+4}{2}$.

Proof. One can easily verify that if G is not isomorphic with P_{d+1}, C_{2d} and C_{2d+1} then $n \geq 3$.

a) The estimates in a) follow from Lemma 5. If $d = 2$, then $n + 4 \leq p$ and hence $n \leq p - 4$. If $d = 3$, then $2n \leq p$ and thus $n \leq \frac{p}{2}$. If $d \geq 4$, then $d + 2n - 4 \leq p$ and thus $n \leq \frac{p-d+4}{2}$.

Further we have $p \leq \frac{n(n-1)^d - 2}{n-2} < \frac{n}{n-2} (n-1)^d \leq 3(n-1)^d$, because $n \geq 3$ and $\frac{n}{n-2} \leq 3$. Thus we have $\left(\frac{p}{3}\right)^{\frac{1}{d}} + 1 < n$.

b) If G is an irreducible tree of diameter d, different from P_{d+1}, then $d(G) \geq 4$. Therefore the inequality $n \leq \frac{p-d+4}{2}$ follows from a).

Let $d = 2s$, $s \geq 2$. Then from Lemma 6 it follows that

$$p \leq \frac{n(n-1)^s - 2}{n-2} + n(n-1)^{s-1} < \frac{n}{n-2} \cdot (n-1)^s + (n-1)^s \leq 4(n-1)^s,$$

because for $n \geq 3$ we have $\frac{n}{n-2} \leq 3$. Hence $n > \left(\frac{p}{4}\right)^{\frac{2}{d}} + 1$.

Let $d = 2s + 1$, $s \geq 2$. Then $s = \frac{d-1}{2}$ and from Lemma 6 we obtain

$$p \leq \frac{n(n-1)^s - 2}{n-2} + (n^2 - n - 3)(n+1)^{s-2} < \frac{n}{n-2} (n-1)^{\frac{d-1}{2}} + (n-1) \times$$

$$\times (n^2 + n - 3) = (n-1)^{\frac{d}{2}} \cdot \frac{1}{\sqrt{n-1}} \cdot \left(\frac{n}{n-2} + \frac{n^2+n-3}{n-1}\right) \leq (n-1)^{\frac{d}{2}} \times$$

$\times \frac{1}{\sqrt{n-1}}\left(3+1+\frac{3}{n-1}\right) \leq 4(n-1)^{\frac{d}{2}}$, where we used the inequalities $n \geq 3$ and $\frac{n}{n-2} \leq 3$. Consequently, $\left(\frac{p}{4}\right)^{\frac{2}{d}} + 1 < n$. This completes the proof.

5. The estimate of the number of edges

The maximum number of edges among all graphs with p vertices and no triangles is $\frac{p^2}{4}$, according to the well-known Turán's theorem. In this section we shall prove that the number of edges of an ω_d-graph ($d \geq 2$) with p vertices is at most $\min\left(\frac{p^2}{4}, \frac{p(p-1)}{d}\right)$. First of all we given an estimate of the cardinality of a k-covering ($k \geq 2$) of a graph that will be useful later.

Obviously, the cardinality of a k-covering of a graph G is at least the number of components of G and at most the number of vertices of G. If $d(G) < k$, then any vertex of G forms a k-covering of G.

Theorem 4. *Let k, d be given integers such that $2 \leq k \leq d$. Let A be a k-covering of a graph G of diameter d. Then we have;*

$$\text{a) } 1 \leq |A| \leq \begin{cases} \dfrac{2p-2}{k} & \text{if } k \text{ is even;} \\[2ex] \dfrac{2p}{k+1} & \text{if } k \text{ is odd.} \end{cases}$$

b) *If moreover $\varkappa(G) \geq 2$, then*

$$1 \leq |A| \leq \begin{cases} \dfrac{p-2}{k-1} & \text{if } k \text{ is even;} \\[2ex] \dfrac{p}{k} & \text{if } k \text{ is odd.} \end{cases}$$

Proof. Obviously $|A| \geq 1$. This estimate is reached (in both cases) in a graph that arises from the graph C_r, $r \geq 4$, by adding one new vertex w adjacent to every vertex of C_r. It is clear that G is a connected graph and $p \geq d+1 \geq k+1 \geq 3$. Let $A = \{a_1, a_2, \ldots, a_s\}$ be a k-covering of G.

a) If $s = |A| = 1$, then the estimate holds, because

$$\frac{2p-2}{k} \geq \frac{2(k+1)-2}{k} \geq 1 \text{ and also } \frac{2p}{k+1} \geq \frac{2(k+1)}{k+1} > 1.$$

Let $s \geq 2$. Let $P(a_i, a_j)$ be a path between the vertices $a_i, a_j \in A$ in G. Its length is at least k. Put

$$Z(a_1) = \left\{ x \in V(G) | x \in P(a_1, a_2) \wedge d(a_1, x) \leq \left[\frac{k+1}{2}\right] - 1 \right\};$$

$$Z(a_i) = \left\{ x \in V(G) | x \in P(a_1, a_i) \wedge d(a_i, x) \leq \left[\frac{k+1}{2}\right] - 1 \right\},$$

where $i = 2, 3, \ldots, s$. We have $Z(a_i) \cap Z(a_j) = \emptyset$, for $i \neq j$, $1 \leq i, j \leq s$, because otherwise it would be $d(a_i, a_j) < k$. Obviously $|Z(a_i)| = \left[\frac{k+1}{2}\right]$, for $i = 1, 2, \ldots, s$. Thus we have $p \geq |\bigcup_{i=1}^{s} Z(a_i)| = s\left[\frac{k+1}{2}\right]$.

If k is odd, then $p \geq s\frac{k+1}{2}$ and hence $s \leq \frac{2p}{k+1}$. If k is even, then the vertex w of the path $P(a_1, a_2)$, such that $d(a_1, w) = \frac{k}{2}$, does not belong to any set $Z(a_i)$, where $1 \leq i \leq s$, because in the opposite case either $d(a_1, a_2) < k$ or $d(a_j, a_1) < k$, where $j = 2, 3, \ldots, s$. Hence $p \geq \frac{sk}{2} + 1$ and then $s \leq \frac{2p-2}{k}$. This bound is reached in the graph in Fig. 1 and Fig. 2 if k is

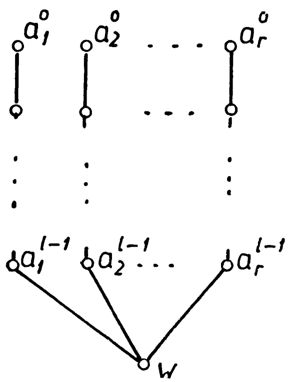

Fig. 1

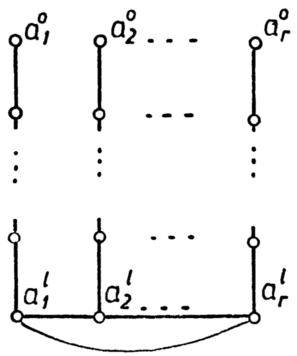

Fig. 2

even and odd, respectively. In both examples $r \geq 1$ is an integer, $A = \{a_1^0, a_2^0, \dots, a_r^0\}$ and the subgraph induced by the set $\{a_1^l, \dots, a_r^l\}$ is complete.

b) Let $\varkappa(G) \geq 2$. The vertices u, v of G such that $d(u, v) = d$ belong to some circuit of the length at least $2d$, because $p \geq 3$, $d(G) = d$ and $\varkappa(G) \geq 2$. Hence $p \geq 2d$. If $s = |A| = 1$, then the estimate holds, since

$$\frac{p-2}{k-1} \geq \frac{2d-2}{k-1} \geq \frac{2d-2}{d-1} \geq 1 \text{ and also } \frac{p}{k} \geq \frac{2d}{k} \geq 1.$$

Let $s \geq 2$. Let $C(a_i, a_j)$, where $i \neq j$ be the circuit of G containing the vertices a_i, a_j of A. Its length is at least $2k$. Put

$$X(a_1) = \left\{ x \in V(G) \mid x \in C(a_1, a_2) \wedge d(a_1, x) \leq \left\lceil \frac{k+1}{2} \right\rceil - 1 \right\};$$

$$X(a_i) = \left\{ x \in V(G) \mid x \in C(a_1, a_i) \wedge d(a_i, x) \leq \left[\frac{k+1}{2} \right] - 1 \right\},$$

where $i = 2, 3, \dots, s$. Then $a_i \in X(a_i)$ and $|X(a_i)| = 2\left[\frac{k+1}{2}\right] - 1$ for $i = 1, 2, \dots, s$. Moreover, $X(a_i) \cap X(a_j) = \emptyset$ for $i \neq j$, $1 \leq i, j \leq s$, as otherwise there would be $d(a_i, a_j) < k$. Hence we have $p \geq |\bigcup_{i=1}^{s} X(a_i)| = s\left(2\left[\frac{k+1}{2}\right] - 1\right)$. If k is odd, then $p \geq sk$ and then $s \leq \frac{p}{k}$. Let k be even and let w_1, w_2 be two vertices of the circuit $C(a_1, a_2)$ such that $d(a_1, w_1) = d(a_1, w_2) = \frac{k}{2}$. The vertices w_1, w_2 do not belong to any set $X(a_i)$, $i = 2, \dots, s$, because in the reverse case there would be $d(a_i, a_1) < k$. It follows that $p \geq |\bigcup_{i=1}^{s} X(a_i)| + 2 = (k-1)s + 2$ and then $s \leq \frac{p-2}{k-1}$. This upper estimate is reached for k even in the graph in Fig. 3 and for k odd in the graph in Fig. 4, where $A = \{a_1^l, a_2^l, \dots, a_r^l\}$ and the subgraphs induced by the sets A_0 and A_{2l} are complete. This completes the proof.

Corollary 3. *Let A be a k-covering of a graph G of diameter d with p vertices, where $2 \leq k \leq d$. Then $|A| \leq \frac{2(p-1)}{k}$. In addition, if $\varkappa(G) \geq 2$, then $|A| \leq \frac{p-2}{k-1}$.*

Proof. The corollary follows from Theorem 4, because we have $p \geq k + 1$ and then $\frac{2p-2}{k} \geq \frac{2p}{k+1}$. If $\varkappa(G) \geq 2$, then $p \geq 2d \geq 2k$ and then we have $\frac{p-2}{k-1} \geq \frac{p}{k}$.

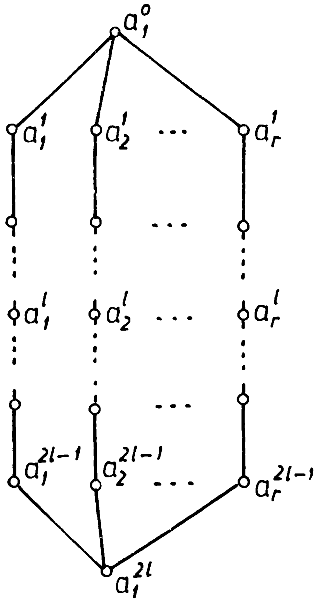

Fig. 3

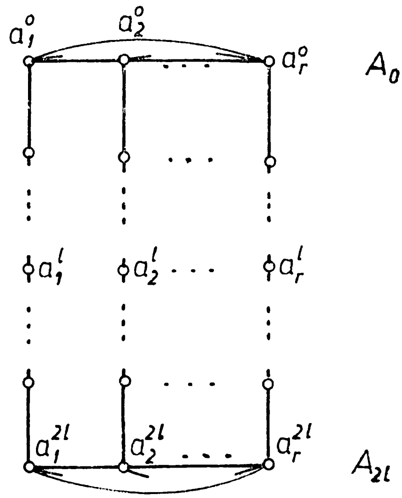

Fig. 4

Corollary 4. *Let G be an w_d-graph ($d \geq 2$) with p vertices and q edges. Then we have;*

a) *if* $\varkappa(G) \geq 2$, *then* $q \leq \frac{p(p-1)}{d}$;

b) *if* $\varkappa(G) \geq 3$, *then* $q \leq \frac{p(p-2)}{2(d-1)}$.

Proof. The neighbourhood $N_G(u)$ of any vertex u of G is a d-covering of $G - u$, according to Lemma 1. If $\varkappa(G) \geq 2$, then the graph $G - u$ is connected and $d(G - u) \geq d(G) = d$. According to Corollary 3 we have $|N_G(u)| = \deg u \leq \frac{2(p-1)}{d}$ and then $\deg u \leq \frac{p-2}{d-1}$ according to Corollary 3. Hence we have $q \leq \frac{p(p-2)}{2(d-1)}$. Q.E.D.

Next, the following lemma, proved in [8], will be used.

Lemma 7. *In an edge-critical graph there exists at most one block containing a circuit.*

Theorem 5. *Let $d \geq 2$ be an integer. Let G be an ω_d-graph with p vertices and q edges. Then* $q \leq \min\left(\frac{p^2}{4}, \frac{p(p-1)}{d}\right)$.

Proof. The inequality $q \leq \frac{p^2}{4}$ holds, see e. g. [5]. One can verify that G is an edge-critical graph and according to Lemma 7 it has at most one block containing a circuit. If G is a tree, then the estimate holds, because $q = p - 1$ and $p \geq d + 1 \geq 3$. Let B be a block of G containing at least one circuit. Let $p_0 = |V(B)|$, $q_0 = |E(G)|$. The number $r = p - p_0 \geq 0$ is equal to the number of vertices of G not belonging to B, i. e. the number of vertices of all acyclic branches of G and hence r is equal to the number of edges of G not belonging to B.

Let $u \in V(B)$. Then $|N_B(u)| \geq 2$ holds, because B is a block. The neighbourhood $N_G(u)$ is a d-covering of $G - u$, according to Lemma 1. One can verify that the set $N_B(u)$ is a d-covering of $B - u$. The graph $B - u$ is connected and moreover $d(B - u) \geq d$, because $d_{B-u}(x, y) = d_{G-u}(x, y) \geq d$ for every $x, y \in N_B(u)$, $x \neq y$. According to Corollary 3 of Theorem 4 we have $|N_B(u)| \leq \frac{2(p_0 - 1)}{d}$ and then $q_0 \leq \frac{p_0(p_0 - 1)}{d}$. We have

$$q = q_0 + r \leq \frac{p_0(p_0 - 1)}{d} + r \leq \frac{(p_0 + r)(p_0 + r - 1)}{d} = \frac{p(p-1)}{d}.$$

The theorem follows.

The proved estimate is for $d \geq 4$ better than the estimate $q \leq \frac{p^2}{4}$. It is reached for an integer $r \geq 2$ in the complete bipartite graph $K_{r,r}$.

REFERENCES

[1] BOLLOBÁS, B.: Graphs with given diameter and maximal valency and with a minimal number of edges. In: Combinatorial Mathematics and its Applications, Proc. Conf. Oxford, 1969, Academic Press, London 1971, 25—37.

[2] BOSÁK, J., KOTZIG, A., ZNÁM, Š.: Strongly geodetic graphs. J. Comb. Theory 5, 1968, 170—176.

[3] GEWIRTZ, A.: Graphs with maximal even girth. Canad. J. Math. 21, 1969, 915—935.

[4] GLIVIAK, F.: On certain classes of graphs of diameter two without superfluous edges. Acta F.R.N. Univ. Com., Math. 21, 1968, 39—48.
[5] GLIVIAK, F., KYŠ, P., PLESNÍK, J.: On the extension of graphs with a given diameter without superfluous edges. Mat. Čas. 19, 1969, 92—101.
[6] HARARY, F.: Graph theory. Addison-Wesley Publ. Comp., Reading, 1969.
[7] HOFFMAN, A. J., SINGLETON, R. R.: On Moore graphs with diameter 2 and 3. IBM J. Res. and Develop. 4, 1960, 497—504.
[8] PLESNÍK, J.: Critical graphs of given diameter. Acta F.R.N. Univ. Com., Math. 30, 1975, 71—93.
[9] ZNÁM, Š.: On the existence and regularity of graphs with certain properties. Submitted to Discrete Mathematics.

Received September 18, 1973

Laboratórium EEG
Výzkumný ústav lekárskej bioniky
883 46 Bratislava-Kramáre