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ON DUAL SEMIGROUPS

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The notion of a dual semigroup has been introduced in [1] by Št. Schvarz, who has also given a description of the structure of such semigroups. Based upon these results, Numakura has investigated in [2] the properties of *F-classes* in such semigroups. Using these properties he has proved Theorem 3,1 of paper [1] under weaker assumptions.

The purpose of this paper is to show that if some further properties of *F-classes* are used, also Theorems 3,3; 3,4; 3,4a; 3,4b of [1] can be proved under weaker assumptions. These results are formulated in Theorems 2, 3, 4, 4a below.

The notations and terminology in this paper are the same as in paper [1]. The notions not explicitly defined are the same as in Clifford-Preston [5] and Ljapin [6].

Definition 1. *Let A be a non-vacuous subset of a semigroup S with zero. The left [right] annihilator $\mathbf{L}(A)$ [$\mathbf{R}(A)$] of A is the set of all $x \in S$ with $xA = 0$ [$Ax = 0$].*

Definition 2. *A semigroup $S \neq 0$ is called dual if for every left ideal L of S we have*

$$\mathbf{L}[\mathbf{R}(L)] = L \tag{1}$$

and for every righı ideal R of S we hawe

$$\mathbf{R}[\mathbf{L}(R)] = R. \tag{2}$$

We shall use the following lemmas (see Schwarz [1] Lemma 1,4, p. 204, Lemma 1,3, p. 203, Lemma 1,6, p. 204).

Lemma 1. *Let S be dual. Then*

(a) $\mathbf{L}(S) = \mathbf{R}(S) = 0$; $\mathbf{L}(0) = \mathbf{R}(0) = S$.

(b) *Let $\{L_\nu \mid \nu \in \Lambda\}$ [$\{R_\nu \mid \nu \in \Lambda\}$] be a collection of left [right] ideals of S. We then have*

$$\mathbf{R}(\bigcap_{\nu\in\Lambda} L_\nu) = \bigcup_{\nu\in\Lambda} \mathbf{R}(L_\nu); \quad \mathbf{L}(\bigcap_{\nu\in\Lambda} R_\nu) = \bigcup_{\nu\in\Lambda} \mathbf{L}(R_\nu).$$

Lemma 2. *If S is dual, then $x \in xS$ and $x \in Sx$ for every $x \in S$.*

Green [3] has defined an *F-class* of a semigroup S as the set of all elements x, which generate the same principal two-sided ideal of S. We denote the *F-class* containing a by F_a. If S is dual, then the principal ideal generated by a is $a \cup Sa \cup aS \cup SaS = SaS$.

Lemma 3. (Numakura [2]). *Let S be a dual semigroup without nilpotent ideals and F_a, F_b two F-classes of S. We have:*

(a) *If $F_a \neq F_b$, then $F_aF_b = 0$.*

(b) *If $b \in F_a$, then there exist $c, c' \in F_a$ such that $b = bc = c'b$.*

(c) *For any $a \in F_a$, $a \neq 0$, $F_a \cup \{0\}$ is a minimal two-sided ideal of S.*

Lemma 4. *If S is a dual semigroup without nilpotent ideals, then every two-sided ideal $M \neq 0$ of S contains at least one minimal two-sided ideal of S.*

Proof. Let $M \neq 0$ be a two-sided ideal of S. Choose $a \in M$, $a \neq 0$. According to Lemma 3 $F_a \cup \{0\}$ is a minimal ideal of S. The intersection $M \cap (F_a \cup \{0\})$ is non-empty since it contains at least the elements $\{0\}$, $\{a\}$. Since $F_a \cup \{0\}$ is a minimal ideal, we have

$$0 \neq M \cap (F_a \cup \{0\}) = F_a \cup \{0\}.$$

Hence M contains at least one minimal two-sided ideal, namely $F_a \cup \{0\}$.

Lemma 5. *If for a two-sided ideal J of a dual semigroup we have $F_a \cap J \neq \emptyset$, then $F_a \subset J$.*

Proof. Let $b \in F_a \cap J$. Then $b \in F_a \Rightarrow SbS = SaS$. Further $b \in J \Rightarrow SbS \subset \subset J$. Hence $a \in SaS = SbS \subset J$. Since a is any element $\in F_a$, we have $F_a \subset \subset J$, q.e.d.

Analogously we have:

Lemma 6. *Let S be a dual semigroup without nilpotent ideals. Then every two-sided ideal of S is contained in a maximal two-sided ideal of S.*

Remark. Lemma 6 trivially holds if there is a unique *F-class* different from $\{0\}$. Then $S - \{0\}$ is an *F-class*, $M_\alpha^* = \{0\}$ and S is a simple semigroup with zero.

Proof. The semigroup S can be expressed as the union of disjoint *F-classes*. We may suppose that there exist at least two *F-classes* different from $\{0\}$. Let M be a proper two-sided ideal of S. Then there exists at least one class F_a with $M \cap F_a = \emptyset$. We prove that the set $S - F_a$ is a maximal two-sided ideal of S.

We first show that $S - F_a$ is a two-sided ideal. Write $S = \bigcup_{\xi \in S} (F_\xi \cup \{0\})$. Then $S(S - F_a) = S[\bigcup_{\xi \neq a} (F_\xi \cup \{0\})] = \bigcup_{\xi \neq a} S(F_\xi \cup \{0\}) = \bigcup_{\xi \neq a} (F_\xi \cup \{0\} = S - F_a$.

Analogously $(S - F_a) S = S - F_a$.

Here we used the fact that $F_\xi \cup \{0\}$ is a minimal two-sided ideal of S.

To prove that $S - F_a$ is a maximal two-sided ideal of S suppose that M' is a two-sided ideal of S such that $S - F_a \subset M' \subseteq S$. Since $M' \cap F_a \neq 0$ by Lemma 5 we have $F_a \subset M'$, hence $M' = S$. This proves our assertion.

Theorem 1. (Numakura [2]). *Let S be a dual semigroup without nilpotent ideals. We then have $S = \bigcup_{\nu \in \Lambda} M_\nu$, where $M_\alpha M_\beta = M_\alpha \cap M_\beta = 0$ for $\alpha \neq \beta \in \Lambda$ and M_ν are simple dual semigroups.*

The converse statement is given by

Theorem 2. *Let $\{M_\nu \mid \nu \in \Lambda\}$ be a collection of simple dual semigroups with $M_\alpha \cap M_\beta = \emptyset$ for $\alpha \neq \beta \in \Lambda$. Let us identify the zero elements of all M_ν, $\nu \in \Lambda$. The set $S = \bigcup_{\nu \in \Lambda} M_\nu$ with the multiplication * defined as follows*

$$a * b = \begin{cases} ab \text{ if } a, b \text{ belong to the same } M_\nu, \\ 0 \text{ if } a \in M_\alpha,\ b \in M_\beta,\ \alpha \neq \beta \in \Lambda, \end{cases}$$

is a dual semigroup without nilpotent ideals.

The proof that S is dual is given in paper [1], Theorem 3,2, p.210. The fact that the semigroup S has no nilpotent ideals is evident from the construction of the semigroup S.

Combining Theorem 1 and Theorem 2 we get:

Theorem 3. *Let S be a semigroup with zero and without nilpotent ideals. Then S is dual if and only if S is the union of its minimal two-sided ideals and each of these minimal ideals is a dual semigroup.*

Another criterion for the duality of a semigroup is given in Theorem 4. To this end we need the following lemma:

Lemma 7. *Let M^* be a maximal two-sided ideal of a semigroup S. Then $S - M^*$ is an F-class.*

Proof. Let x be any element $\in S - M^*$ and F_x the corresponding F-class. We have $F_x \cap M^* = \emptyset$, for otherwise we would have $F_x \subset M^*$, in particular $x \in M^*$, contrary to the assumption.

Take any element $y \in S - M^*$. Then $[y] = y \cup Sy \cup yS \cup SyS$ is an ideal $\neq M^*$, hence with respect to the maximality $M^* \cup [y] = S$. Therefore $x \in [y]$ and this implies $[x] \subset [y]$. Symetrically we can prove $[y] \subset [x]$. Hence $[x] = [y]$ and therefore $F_x = F_y$, q.e.d.

Remark. If S is a dual semigroup without nilpotent ideals and M^* is a maximal two-sided ideal of S, then by Lemma 3 $(S - M^*) \cup \{O\}$ is a minimal two-sided ideal of S.

Lemma 8. (Schwarz [1], Theorem 2,1, p.206). *Let S be a dual semigroup and J a two-sided ideal of S which does not contain a nilpotent subideal of S. Then J and $\mathbf{R}(J)$ are dual semigroups.*

Theorem 4. *Let S be a semigroups with zero and without nilpotent ideals. Suppose that there exist at least two maximal two-sided ideals of S. Let $\{M_\alpha^* \mid \alpha \in \Lambda\}$ be the set of all maximal ideals of S. Then S is dual if and only if*

(a) $\bigcap_{\alpha\in\Lambda} M_\alpha^* = 0$;

(b) *Every semigroup M_α^*, $\alpha \in \Lambda$ is dual.*

Proof. 1. Suppose that S is dual. Condition (b) is satisfied according to Lemma 8. The duality implies according to Lemma 4 that every two-sided ideal J of S contains a minimal two-sided ideal of S. By Theorem 1 we have $S = \bigcup_{\nu\in\Lambda} M_\nu$, where $\{M_\nu \mid \nu \in \Lambda\}$ is the set of all minimal two-sided ideals of S. Now, since S is dual, we have $O = \mathbf{R}(S) = \mathbf{R}(\bigcup_{\nu\in\Lambda} M_\nu) = \bigcap_{\nu\in\Lambda} \mathbf{R}(M_\nu)$. The set $\{\mathbf{R}(M_\nu) \mid \nu \in \Lambda\}$ is exactly the set of all maximal two-sided ideals of S. Hence the first part of our Theorem is proved.

2. Suppose that the conditions (a) and (b) are satisfied. We show that S is dual. According to [1] Lemma 3,1c we can write $S = M_\alpha^* \cup \mathbf{L}(M_\alpha^*)$ with $M_\alpha^* \cap \mathbf{L}(M_\alpha^*) = 0$. The two-sided ideal $\mathbf{L}(M_\alpha^*)$ is contained in a maximal two-sided ideal M_β^* of S; $\mathbf{L}(M_\alpha^*)$ is also a two-sided ideal of M_β^*. According to Lemma 8 $\mathbf{L}(M_\alpha^*)$ is therefore a dual semigroup. The condition (a) implies (see [1] Lemma 3,1d) that S is a union of minimal two-sided ideals of S, each of which is a dual semigroup. According to Theorem 3 S is dual.

This proves our Theorem.

Similarly we can prove:

Theorem 4a. *Suppose that the suppositions of Theorem 4 are satisfied. Then S is dual if and only if*

(a) $\bigcap_{\alpha\in\Lambda} M_\alpha^* = 0$;

(b) *There is a pair of two-sided ideals M_1, M_2 which are themselves dual semigroups and for which we have $S = M_1 \cup M_2$ with $M_1 M_2 = 0$.*

Theorem 4b. *Let S be a semigroup with zero and without nilpotent ideals. Let $\{M_\alpha^* \mid \alpha \in \Lambda\}$ be the set of all maximal ideals of S. Then S is dual if and only if*

(a) $\bigcap_{\alpha \in \Lambda} M_\alpha^* = 0$.

(b) *Each of the semigroups* $\mathbf{L}(M_\alpha^*)$ *is dual.*

REFERENCES

[1] Schwarz Š., *On dual semigroups*, Czechosl. Math. J. *10* (85) (1960), 201—230.
[2] Numakura K., *Compact dual semigroups without nilpotent ideals*, Duke Math. J. *31* (1964), 555—573.
[3] Green J. A., *On the structure of semigroups*, Ann. Math., *54* (1951), 163—172.
[4] Clifford H., *Semigroups without nilpotent ideals*, Amer. J. Math. *71* (1949), 834—844.
[5] Clifford H., Preston G. B., *The algebraic theory of semigroups*, V. 1, Amer. Math. Society, Providence, R. I., 1961.
[6] Ляпин Е. С., *Полугруппы*, Москва, 1960.

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