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AN EXAMPLE CONCERNING THE PROJECTIVE TENSOR PRODUCT OF VECTOR MEASURES

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If $\Sigma_{n=1}^{\infty} x_n$ and $\Sigma_{n=1}^{\infty} y_n$ are unconditionally (= subseries) convergent series of elements of locally convex linear spaces X and Y, respectively, it is natural to ask whether the series $\Sigma_{n=1}^{\infty} \Sigma_{m=1}^{\infty} x_n \otimes y_m$ is also unconditionally convergent in the space $X \hat{\otimes} Y$, the projective tensor product of X and Y. (See e. g. [1; Chpt. IV] for terminology used.) The aim of this note is to show that the answer is in the negative in general by exhibiting a counter-example in which $X = Y$ is a reflexive Banach space. The same space and technique have been used in [5] for constructing two commuting, strongly complete, Boolean algebras of projections, both of bound 1, but such that the algebra of projections they generate is unbounded. In our example the partial sums of $\Sigma\, \Sigma x_n \otimes y_m$ are unbounded.

Example. For $k = 1, 2, \ldots$ let X_k be the linear space of all functions ξ on $N_k = \{1, 2, \ldots, 2^k\}$. Let the norm on X_k be defined by $|\xi| = \max\{|\xi(s)| : s = 1, 2, \ldots, 2^k\}$. Let X be the space consisting of all sequences $x = (\xi_k)_{k=1}^{\infty}$, where $\xi_k \in X_k$, $k = 1, 2, \ldots$ and $\Sigma_{k=1}^{\infty} |\xi_k|^2 < \infty$. The norm in X is defined by $|x| = (\Sigma_{k=1}^{\infty} |\xi_k|^2)^{\frac{1}{2}}$.

If n is an integer $\geqslant 1$ let k_n be the integer for which

$$\sum_{i=1}^{k_n-1} 2^i < n \leqslant \sum_{i=1}^{k_n} 2^i$$

(we use the convention $\Sigma_{i=1}^{0} = 0$). Put

$$s_n = n - \sum_{i=1}^{k_n-1} 2^i.$$

Let ξ_{k_n} be the element of X_{k_n} such that $\xi_{k_n}(s_n) = 1$ and $\xi_{k_n}(s) = 0$ for $s \in N_{k_n}$, $s \neq s_n$. Let $x_n = (\xi_{nk})_{k=1}^{\infty}$, where $\xi_{nk} = k_n^{-1}\, \xi_{k_n}$ for $k = k_n$ and $\xi_{nk} = 0$ for $k \neq k_n$. Clearly, $x_n \in X$ for $n = 1, 2, \ldots$. The series $\Sigma_{n=1}^{\infty} x_n$ is unconditionally convergent, since $\Sigma_{n=1}^{\infty} k_n^{-2} < \infty$.

Let, for $k = 1, 2, \ldots, Z_k$ be the linear space of all functions ζ on $N_k \times N_k$. Let the norm in Z_k be given by the formula $\|\zeta\| = \inf \Sigma_{i=1}^{p} |\xi_i|\ |\eta_i|$, where the infimum is taken over all representations of ζ in the form $\zeta(s, t) = \Sigma_{i=1}^{p} \xi_i(s)\ \eta_i(t)$ with $\xi_i \in X_k$, $\eta_i \in X_k$. Let Z be the space consisting of all sequences $z = (\zeta_k)_{k=1}^{\infty}$ with $\zeta_k \in Z_k$, $k = 1, 2, \ldots$, such that $\Sigma_{k=1}^{\infty} \|\zeta_k\|^2 < \infty$. Again, the norm is given by $\|z\| = (\Sigma_{k=1}^{\infty} \|\zeta_k\|^2)^{\frac{1}{2}}$.

If $x = (\xi_k)_{k=1}^{\infty}$ and $y = (\eta_k)_{k=1}^{\infty}$ are two elements of X, let $z = xy$ be the element of Z defined by $z = (\zeta_k)_{k=1}^{\infty}$, where $\zeta_k(s, t) = \xi_k(s)\ \eta_k(t)$, $(s, t) \in N_k \times N_k$, $k = 1, 2, \ldots$. The mapping $(x, y) \to xy$ is clearly a bilinear operation on $X \times X$ with values in Z. It is bounded, in fact $\|xy\| \leqslant |x|\ |y|$.

We use a result from [4; pp. 368—369], viz. for $k = 1, 2, \ldots$, there is a function ω_k in Z_k taking values 0 and 1 only such that $\|\omega_k\| \geqslant 2^{\frac{1}{2}k-1}$. Denote by Ω_k the set of all couples (n, m) such that $\Sigma_{i=1}^{k-1} 2^i < n, m \leqslant \Sigma_{i=1}^{k} 2^i$ and $\omega_k(s_n, s_m) = 1$. Since $k^{-2}\ 2^{\frac{1}{2}k-1} \to \infty$ and $\omega_k \in Z_k$ we can construct a sequence $(\bar{x}_n)_{n=1}^{\infty} \subset X$ belonging to a representation of ω_k such that

$$\Big\| \sum_{(n,m)\in\Omega_k} \bar{x}_n \bar{x}_m \Big\| \to \infty ,$$

for $k \to \infty$.

Since the linear mapping $\Sigma x_i \otimes y_i \to \Sigma x_i y_i$ from a dense subset of $X \hat{\otimes} X$ into Z is bounded, the partial sums of $\Sigma\Sigma\ \bar{x}_n \otimes \bar{x}_m$ are not bounded in $X \hat{\otimes} X$.

Remarks. 1. Let X and Y be locally convex topological linear spaces, $\mathscr{S}$ and $\mathscr{T}$ σ-algebras of subsets of sets S and T, respectively, and $\mu : \mathscr{S} \to X$ and $\nu : \mathscr{T} \to Y$ σ-additive measures. If we put

$$\lambda(E \times F) = \mu(E) \otimes \nu(F),\ E \in \mathscr{S},\ F \in \mathscr{T},$$

then the additive extension of λ onto the algebra generated by the sets $E \times F$ need not be bounded. In fact, it is enough to put $S = T = \{1, 2, \ldots\}$, $\mathscr{S} = \mathscr{T} =$ = set of all subsets of S and $\mu(E) = \nu(E) = \Sigma_{n\in E}\, x_n$, $E \in \mathscr{S} = \mathscr{T}$, where Σx_n is the series constructed in the Example.

2. If $\Sigma_{n=1}^{\infty} x_n$ and $\Sigma_{n=1}^{\infty} y_n$ are unconditionally convergent series in normed linear spaces X and Y, respectively, and if one of them possess an absolute basis then $\Sigma\Sigma x_n \hat{\otimes} y_m$ is unconditionally convergent in $X \hat{\otimes} Y$. More generally, if one of the spaces X, Y is an ‚admissible factor' then the last series is convergent unconditionally (see [2]).

3. If Σx_n and Σy_n are unconditionally convergent series in locally convex topological linear spaces X and Y, respectively, then $\Sigma\Sigma x_n \otimes y_m$ is unconditionally convergent in $X \check{\otimes} Y$, the inductive tensor product of X and Y (see [3]).

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