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MAXIMAL INEQUALITIES AND SPACE-TIME REGULARITY OF STOCHASTIC CONVOLUTIONS

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Abstract. Space-time regularity of stochastic convolution integrals

$$J = \int_0^{\cdot} S(\cdot - r)Z(r)\,\mathrm{d}W(r)$$

driven by a cylindrical Wiener process W in an L^2-space on a bounded domain is investigated. The semigroup S is supposed to be given by the Green function of a $2m$-th order parabolic boundary value problem, and Z is a multiplication operator. Under fairly general assumptions, J is proved to be Hölder continuous in time and space. The method yields maximal inequalities for stochastic convolutions in the space of continuous functions as well.

Keywords: stochastic convolutions, maximal inequalities, regularity of stochastic partial differential equations

MSC 1991: 60H15

0. INTRODUCTION

Let us consider an abstract stochastic semilinear parabolic equation

$$\mathrm{d}X = AX\,\mathrm{d}t + f(t,X)\,\mathrm{d}t + \sigma(t,X)\,\mathrm{d}W, \qquad X(0) = \zeta \tag{0.1}$$

in a Hilbert space H, $A\colon \mathrm{Dom}(A) \longrightarrow H$ being an infinitesimal generator of an analytic C_0-semigroup $S(t)$ on H and W an infinite-dimensional Wiener process in

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H. Investigating regularity (or even only sample paths continuity) of mild solutions of (0.1), which are given by the variation of constants formula

$$X(t) = S(t)\zeta + \int_0^t S(t-s)f(s,X(s))\,\mathrm{d}s + \int_0^t S(t-s)\sigma(s,X(s))\,\mathrm{d}W(s), \tag{0.2}$$

one faces the fact that it is the third term on the right hand side of (0.2) that causes the most serious problems. It turns out that a thorough understanding of properties of the stochastic convolution

$$\Psi(t) = \int_0^t S(t-s)\psi(s)\,\mathrm{d}W(s), \qquad t \geqslant 0$$

for an operator-valued process ψ is indispensable when dealing with the equation (0.1). If the Wiener process W has a nuclear covariance operator then a general procedure (the so called factorization method), proposed by G. Da Prato, S. Kwapień and J. Zabczyk, is available, see the paper [2] for the additive noise case, i.e. $\psi = I$ (the identity operator), and [4] or [6], Chapter 7.1, for the general case of an $L(H)$-valued process ψ, where $L(H)$ denotes the space of all bounded linear operators in H. Using this method it is straightforward to show that, under weak restrictions on ψ, the process Ψ has Hölder continuous sample paths in the real interpolation space $(H, \mathrm{Dom}(A))_{\alpha,2}$ for any $\alpha < \frac{1}{2}$, see [11], [21]. In applications to parabolic problems, $H = L^2(D)$ for a domain $D \subseteq \mathbb{R}^d$, and A is given by a $2m$-th order elliptic differential operator in D. Then $(H, \mathrm{Dom}(A))_{\alpha,2}$ is a subspace of the Slobodeckiĭ space $W^{2m\alpha,2}(D)$ and the Sobolev embedding theorem yields that Ψ is Hölder continuous in both the time and space variables provided $2m > d$.

Unfortunately, the situation is much more complicated if W is a standard cylindrical Wiener process (that is, with the covariance operator I). Even in the simple case $H = L^2(]0,1[)$, $\psi = I$ and $A = \Delta$ (the second derivative operator) with homogeneous Dirichlet boundary data it can be shown that $\Psi(t) \in W^{2\alpha,2}(]0,1[)$ if and only if $\alpha < \frac{1}{4}$ (see [13], Example 3.1, cf. also [20] for a different proof), but $W^{2\alpha,2}(]0,1[)$ embeds into $\mathscr{C}([0,1])$ only if $\alpha > \frac{1}{4}$. On the other hand, Hölder continuity of the random field Ψ was established for many particular choices of the operator A and/or the process ψ, see e.g. [7], [26], [16], [3], [9], [10], [14], [1]. Except for the paper [3], where a functional analytic proof (that seems to apply only in the additive noise case) was proposed, all other proofs we know are based on the Kolmogorov test for sample paths continuity. A new version of this argument was used also in [5] (cf. [6], Chapter 5.5) and developed further in [18] to cover stochastic convolutions of the form

$$J(t) = \int_0^t S(t-s)Z(s)\,\mathrm{d}W(s), \qquad t \geqslant 0 \tag{0.3}$$

in $L^2(]0,1[)$, where $A = \Delta$ and $Z(s)$, $s \geqslant 0$, are now generally *unbounded* multiplication operators in $L^2(]0,1[)$. Let us recall that multiplication operator valued processes appear if the diffusion coefficient σ in (0.1) is a superposition operator, which is the most common case.

In the present paper, we aim at establishing the space-time Hölder continuity of the random field J defined by (0.3) if A is a general $2m$-th order elliptic differential operator in a bounded domain $\mathscr{O} \subseteq \mathbb{R}^d$. We assume that $2m > d$ as otherwise the operators $S(t)$, $t > 0$, are not Hilbert-Schmidt and the process J need not be well-defined in $L^2(\mathscr{O})$ (see Theorem 2.1 below for a precise statement). We show that $J(t,\cdot) \in W^{s,p}(\mathscr{O})$ for certain $s > 0$ and $p \in]2,\infty[$ with p sufficiently large for the embedding theorem to imply Hölder continuity (see Theorem 2.2). Moreover, under stronger assumptions, the same method yields the differentiability of $J(t,\cdot)$ and makes it possible to describe the behaviour of $J(t,\cdot)$ on the boundary $\partial\mathscr{O}$ (Theorem 2.3). Finally, we establish Hölder continuity of J in time (Theorem 2.4); as a consequence, a maximal inequality for stochastic convolutions in the space of continuous functions follows. This result seems to be new even in the case $d = m = 1$, $\mathscr{O} =]0,1[$, $Z \equiv 1$, where we obtain

$$\boldsymbol{E} \sup_{0\leqslant t\leqslant T} \left| \int_0^t S(t-s)\,\mathrm{d}W(s) \right|^p_{\mathscr{C}([0,1])} \leqslant \text{const.}\, T^{\frac{p}{4}-2-\varepsilon}$$

for any $p > 8$ and $\varepsilon \in \left]0, \frac{p}{4} - 2\right[$.

Our results are closely related to those obtained by P. Kotelenez in [14]. He considered more general pseudo-differential operators in not necessarily bounded domains; on the other hand, the results of [14] apply only to stochastic convolutions with an L^∞-valued process Z, and no maximal inequalities are established there. The case of unbounded multiplication operators Z may be of some importance in investigating stochastic parabolic equations, cf. the paper [18] for some applications of this kind. Furthermore, we believe our proofs to be more straightforward and lucid. In particular, we avoid the use of the Kolmogorov test, but instead we rely directly on the Sobolev embedding theorem, as was first proposed in [12] (cf. also the proof of Theorem 3.4 in [6]).

The paper is organized as follows. In the first section we introduce some notation and recall a few facts about Sobolev-Slobodeckiĭ spaces, Green functions and infinite-dimensional Itô integrals, whilst in Section 2 the main results are stated. In Section 3, some useful estimates of the Green functions are derived, and Section 4 provides proofs of our theorems.

1. Notation and preliminaries

In this section we will introduce some notation and quote some results that are frequently used in what follows.

First, λ_N will stand for Lebesgue measure on $\mathbb{R}^N$. Let $Q \subseteq \mathbb{R}^N$ be an open bounded set, let $s = k + \lambda$ for some $k \in \mathbb{N}$ and $\lambda \in]0,1[$. By $\mathscr{C}^s(\overline{Q})$ we denote the space of all functions on $\overline{Q}$ having k-th order derivatives which are λ-Hölder continuous on $\overline{Q}$. The space $\mathscr{C}^s(\overline{Q})$ is equipped with the norm

$$|u|_{\mathscr{C}^s(\overline{Q})} \equiv \sum_{|\nu| \leqslant k} \sup_{x \in \overline{Q}} |D^\nu u(x)| + \sum_{|\nu|=k} \sup_{\substack{x,y \in \overline{Q} \\ x \neq y}} \frac{|D^\nu u(x) - D^\nu u(y)|}{|x-y|^\lambda}.$$

The $L^q(Q)$-spaces, $1 \leqslant q \leqslant \infty$, are defined in the standard way, let us denote by $L(L^q(Q), L^p(Q))$ the space of all bounded linear operators from $L^q(Q)$ into $L^p(Q)$. The Hilbert-Schmidt norm of an operator $Y \in L(L^2(Q))$ is denoted by $\|Y\|_{(\mathrm{HS})}$. If $s \in \mathbb{N}$ and $q \in [1,\infty[$ then $W^{s,q}(Q)$ stands for the standard Sobolev space. If $s = k + \lambda$, $k \in \mathbb{N}$, $\lambda \in]0,1[$ then by $W^{s,q}(Q)$ we denote the Sobolev-Slobodeckiĭ space (see e.g. [15], §8.3). Namely, $W^{s,q}(Q)$ is the space of all $u \in W^{k,q}(Q)$ such that

$$|u|_{W^{s,q}(Q)} \equiv \left(|u|^q_{W^{k,q}(Q)} + \sum_{|\nu|=k} \int_Q \int_Q \frac{|D^\nu u(x) - D^\nu u(y)|^q}{|x-y|^{N+\lambda q}} \, \mathrm{d}x \, \mathrm{d}y \right)^{1/q} < \infty.$$

For further references we recall the Sobolev embedding theorem (see e.g. [25], Theorem 4.6.1).

Theorem 1.1. *Let $Q \subseteq \mathbb{R}^N$ be a bounded domain with a Lipschitz boundary. Let $s \in]0,\infty[$, $q \in]1,\infty[$ and $\lambda \in [0,\infty[$. Then $W^{s,q}(Q) \hookrightarrow \mathscr{C}^\lambda(\overline{Q})$ provided $s > \lambda + N/q$.*

Throughout the paper $\mathscr{O} \subseteq R^d$ will be a fixed bounded domain, and

$$\mathscr{A} = \mathscr{A}(x,D) = \sum_{|\nu| \leqslant 2m} a_\nu(x) D^\nu, \qquad x \in \mathscr{O},$$

a fixed $2m$-th order elliptic differential operator. Let $\{B_j\,;\, j = 1, \ldots, m\}$ be a system of boundary operators,

$$B_j = B_j(x,D) = \sum_{|\nu| \leqslant r_j} b_{j\nu}(x) D^\nu, \qquad j = 1, \ldots, m, \quad x \in \partial\mathscr{O}.$$

We assume:

(i) $\partial\mathscr{O}$ is of the class $\mathscr{C}^{2m+\Lambda}$ for a $\Lambda > 0$.

(ii) The coefficients $\{a_\nu\}$ are Hölder continuous functions on $\overline{\mathscr{O}}$.

(iii) $\mathscr{A}$ is uniformly elliptic on $\overline{\mathscr{O}}$; that is, there exists a $\delta > 0$ such that

$$(-1)^m \sum_{|\nu|=2m} a_\nu(x)\xi^\nu \leqslant -\delta|\xi|^{2m} \qquad \text{for all } x \in \overline{\mathscr{O}} \text{ and } \xi \in \mathbb{R}^d.$$

(iv) One has $0 \leqslant r_j \leqslant 2m-1$ and $\{b_{j\nu}\} \subseteq \mathscr{C}^{2m-r_j+\eta}(\partial\mathscr{O})$ for an $\eta > 0$.

(v) The system $\{B_j\}$ fulfils uniformly the complementarity condition on $\partial\mathscr{O}$ (see [22], §1, for the definition).

We will employ many times the following results (see [8], Theorem 1.1, cf. also [23], Theorem 2): Under the above assumptions, there exists a Green function G for the system $\{\mathscr{A}, B_1, \ldots, B_m\}$. That is, $G\colon\ ]0,\infty[\times\mathscr{O}\times\mathscr{O} \longrightarrow \mathbb{R}$ is a continuous function, continuously differentiable with respect to the first variable, and has continuous derivatives of orders less than or equal to $2m$ with respect to the second variable. Further, G fulfils

$$\begin{aligned} \left(\frac{\partial}{\partial t} - \mathscr{A}\right) G(\cdot,\cdot,y) &= 0 \qquad \text{in }]0,\infty[\times\mathscr{O}, \\ B_j G(\cdot,\cdot,y) &= 0 \qquad \text{on }]0,\infty[\times\partial\mathscr{O} \end{aligned} \tag{1.1}$$

for any $y \in \mathscr{O}$, and

$$\lim_{t\to 0+} \int_{\mathscr{O}} G(t,x,y) f(y)\,\mathrm{d}y = f(x), \qquad x \in \mathscr{O}, \quad f \in \mathscr{C}(\overline{\mathscr{O}}).$$

Moreover, G satisfies heat kernel type estimates:

Theorem 1.2. *For every $T > 0$ there exist constants $C, c > 0$ such that*

$$|D^\nu G(t,x,y)| \leqslant C t^{-(d+|\nu|)/(2m)} \exp\left(-c\left|\frac{x-y}{t^{1/(2m)}}\right|^{2m/(2m-1)}\right)$$

for all $t \in]0,T]$, $x, y \in \mathscr{O}$, and for any multi-index ν, $|\nu| \leqslant 2m$.

Here and in the sequel, $D^\nu G$ refers to the partial derivatives of the function $x \longmapsto G(t,x,y)$.

Let $q \in]1,\infty[$, define

$$\begin{aligned} \mathrm{Dom}(A_q) &\equiv \left\{u \in W^{2m,q}(\mathscr{O});\ B_j u = 0 \text{ on } \partial\mathscr{O} \text{ for } j = 1,\ldots,m\right\}, \\ A_q u &\equiv \mathscr{A}u \quad \text{for} \quad u \in \mathrm{Dom}(A_q). \end{aligned}$$

Then A_q is an infinitesimal generator of a C_0-semigroup S_q on $L^q(\mathscr{O})$ (see e.g. [24], Theorem 3.8.2), and

$$S_q(t)u(x) = \int_{\mathscr{O}} G(t,x,y)u(y)\,\mathrm{d}y, \qquad t \in \,]0,\infty[\,, \quad x \in \mathscr{O}, \quad u \in L^q(\mathscr{O}). \tag{1.2}$$

The formula (1.2) defines a C_0-semigroup S_1 on $L^1(\mathscr{O})$ as well, note that $S_q(t) = S_p(t)$ on $L^q(\mathscr{O})$ for all $q \geqslant p$, so we will omit the subscript with no danger of confusion. Moreover, (1.2) together with Theorem 1.2 yield that the operators $S(t)$, $t > 0$, map the space $L^1(\mathscr{O})$ into $L^\infty(\mathscr{O})$.

Let $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geqslant 0}, \boldsymbol{P})$ be a filtered probability space. Let $W(t)$ be a cylindrical Wiener process on $L^2(\mathscr{O})$. That is, $W(t) \in L(L^2(\mathscr{O}), L^2(\Omega))$, $\boldsymbol{E}\big(W(t)(h)W(t)(g)\big) = t\langle h, g\rangle$ for any $t \geqslant 0$, $h, g \in L^2(\mathscr{O})$, and $(W(t)(h))_{t\geqslant 0}$ is a real valued $(\mathscr{F}_t)$-adapted Wiener process with covariance $|h|^2_{L^2(\mathscr{O})}$. Let us fix an orthonormal basis $\{e_k\}_{k=1}^\infty$ in $L^2(\mathscr{O})$ and set $W_k(t) = W(t)(e_k)$. Then $\{W_k\}$ is a sequence of independent real valued $(\mathscr{F}_t)$-adapted Wiener processes and formally $W(t) = \sum W_k(t)e_k$. This series does not converge in $L^2(\mathscr{O})$ but converges almost surely in any Hilbert space containing $L^2(\mathscr{O})$ with a Hilbert-Schmidt embedding.

If $\xi\colon \Omega \times [0,T] \longrightarrow L(L^2(\mathscr{O}))$ is an $(\mathscr{F}_t)$-adapted measurable stochastic process satisfying

$$\boldsymbol{P}\bigg\{\int_0^T \|\xi(t)\|^2_{(\mathrm{HS})}\,\mathrm{d}t \,<\, \infty\bigg\} = 1$$

then the stochastic Itô integral

$$\int_0^t \xi(s)\,\mathrm{d}W(s), \qquad t \in [0,T],$$

is a well-defined $L^2(\mathscr{O})$-valued process, and

$$\int_0^t \xi(s)\,\mathrm{d}W(s) = \sum_{k=1}^\infty \int_0^t \xi(s)e_k\,\mathrm{d}W_k(s) \qquad \text{in } L^2(\mathscr{O}),$$

where the series converges in probability. We refer the reader to the books [17], §15 and §16, and [6], Chapter 4.3, for a systematic exposition of stochastic integration with respect to a cylindrical Wiener process.

From now on, we take a fixed $T > 0$. Let $\delta \in [1,\infty[$, $r \in [1,\infty]$ and $q \in [1,\infty]$. By $\mathscr{P}_{\delta,r,q}$ we denote the space of all measurable $(\mathscr{F}_t)$-adapted $L^q(\mathscr{O})$-valued stochastic processes $Z = (Z(t), 0 \leqslant t \leqslant T)$ such that

$$\|Z\|_{\delta,r,q} \equiv \begin{cases} \bigg[\boldsymbol{E}\Big(\displaystyle\int_0^T |Z(s)|^r_{L^q(\mathscr{O})}\,\mathrm{d}s\Big)^{\delta/r}\bigg]^{1/\delta} < \infty & \text{if } r < \infty, \\ \Big[\boldsymbol{E}\operatorname*{ess\,sup}_{0\leqslant s\leqslant T} |Z(s)|^\delta_{L^q(\mathscr{O})}\Big]^{1/\delta} < \infty & \text{if } r = \infty. \end{cases}$$

We say that $Z \in \mathscr{P}_{0,r,q}$ if Z is a measurable $(\mathscr{F}_t)$-adapted $L^q(\mathscr{O})$-valued stochastic process fulfilling

$$\boldsymbol{P}\left\{\int_0^T |Z(s)|^r_{L^q(\mathscr{O})}\,\mathrm{d}s < \infty\right\} = 1.$$

Any function $g \in L^q(\mathscr{O})$, $q \in [2,\infty]$, may be viewed as a multiplication operator

$$g\colon L^2(\mathscr{O}) \longrightarrow L^{2q/(q+2)}(\mathscr{O}), \quad u \longmapsto gu.$$

Note that the Hölder inequality implies

$$|g|_{L(L^2(\mathscr{O}),L^{2q/(q+2)}(\mathscr{O}))} \leqslant |g|_{L^q(\mathscr{O})}. \tag{1.3}$$

(Here we set $2\infty/(\infty+2) = 2$.) Consequently, if $Z \in \mathscr{P}_{0,r,2}$ then

$$s \longmapsto S(t-s)Z(s), \qquad s \in [0,t],$$

is an $L(L^2(\mathscr{O}))$-valued process for each $t \in [0,T]$ due to (1.3) and (1.2).

Finally, throughout the paper we adopt the convention that, for any number $q \in [2,\infty]$, $\overline{q}$ stands for the dual index to $q/2$. That is, $\overline{q} = q/(q-2)$ and $\overline{\infty} = 1$. Moreover, we set $q' = 2\overline{q}$. Occasionally, we will denote $\mu \equiv 1/(2m)$, thus $2m/(2m-1) = 1/(1-\mu)$. Let $k \geqslant 0$ be an integer and let $2m > d + 2k$, we define

$$\theta(k) = \frac{2d}{2m-d-2k}, \qquad \beta(q,k) = \frac{d}{m}\left(1 - \frac{1}{q'}\right) + \frac{k}{m}, \qquad \gamma(q,k) = \frac{2}{1-\beta(q,k)},$$
$$\theta = \theta(0), \qquad \beta(q) = \beta(q,0), \qquad \gamma(q) = \gamma(q,0).$$

Note that

$$\beta(q,k) = \frac{d}{2m}\left(1 + \frac{2}{q}\right) + \frac{k}{m},$$

moreover, $\beta(q,k) \in \,]0,1[$ provided $q > \theta(k)$.

2. Main results

In the first theorem, we present a sufficient condition for the stochastic convolution integral to be an $L^2(\mathscr{O})$-valued stochastic process.

Theorem 2.1. *Assume that $2m > d$ and suppose that $Z \in \mathscr{P}_{0,r,2}$ for an $r \in \left]\frac{4m}{2m-d}, \infty\right]$. Then*

$$\int_0^t \|S(t-s)Z(s)\|^2_{(\mathrm{HS})}\,\mathrm{d}s < \infty \qquad \boldsymbol{P}\text{-almost surely}$$

for all $t \in [0,T]$. Consequently, the stochastic convolution

$$J^Z(t) = \int_0^t S(t-s)Z(s)\,\mathrm{d}W(s), \qquad t \in [0,T],$$

is a well-defined $L^2(\mathscr{O})$-valued stochastic process. Moreover, there exists a constant $K = K(m,d,r,T) < \infty$ such that

$$\sup_{0 \leqslant t \leqslant T} \boldsymbol{E}\big|J^Z(t)\big|^2_{L^2(\mathscr{O})} \leqslant K\|\!|Z\|\!|^2_{2,r,2}$$

for any $Z \in \mathscr{P}_{2,r,2}$.

Therefore, under the assumptions of Theorem 2.1 we can define the random field

$$J^Z(t,x) = \Big(\int_0^t S(t-s)Z(s)\,\mathrm{d}W(s)\Big)(x), \qquad 0 \leqslant t \leqslant T,\ x \in \mathscr{O}.$$

A priori, $J^Z(t,\cdot)$ is defined only as an element of $L^2(\mathscr{O})$, this means λ_d-almost everywhere. Our next theorem shows that, in fact, the space regularity of J^Z is much better. Prior to stating the result let us note that, obviously, $r > 4m/(2m-d)$ if $q > \theta$ and $r > \gamma(q)$.

Theorem 2.2. *Assume that $2m > d$, $q \in \,]\theta,\infty]$, $r \in \,]\gamma(q),\infty]$ and $\delta \in [2,\infty[$. Set*

$$b = \frac{m(1-\beta(q)\overline{r})}{\overline{r}} \wedge 1 = \Big(\frac{m(r-2)}{r} - \frac{d}{2} - \frac{d}{q}\Big) \wedge 1.$$

Then for each $s \in \,]0,b[$ there exists a constant $L = L(q,r,\delta,s) < \infty$ such that

$$\sup_{0 \leqslant t \leqslant T} \boldsymbol{E}|J^Z(t,\cdot)|^{\delta}_{W^{s,\delta}(\mathscr{O})} \leqslant L\|\!|Z\|\!|^{\delta}_{\delta,r,q} \tag{2.1}$$

for every $Z \in \mathscr{P}_{\delta,r,q}$. Hence $J^Z(t,\cdot) \in W^{s,\delta}(\mathscr{O})$ $\boldsymbol{P}$-almost surely, and if $\delta > d/b$ then $J^Z(t,\cdot) \in \mathscr{C}^{\lambda}(\overline{\mathscr{O}})$ $\boldsymbol{P}$-almost surely for each $\lambda \in [0, b - d/\delta[$.

As a simple consequence of the theorem we have

Corollary 2.1. (i) *If $2m - d = 1$ and $Z \in \bigcap_{r,q>2} \mathscr{P}_{\delta,r,q}$ for a certain $\delta > 2d$ then $J^Z(t,\cdot) \in \mathscr{C}^{\lambda}(\overline{\mathscr{O}})$ for $\lambda \in [0, 1/2 - d/\delta[$ and $t \in [0,T]$.*

(ii) *If $2m - d > 1$ and $Z \in \bigcap_{r,q>2} \mathscr{P}_{\delta,r,q}$ for a certain $\delta > d$ then $J^Z(t,\cdot) \in \mathscr{C}^{\lambda}(\overline{\mathscr{O}})$ for $\lambda \in [0, 1 - d/\delta[$ and $t \in [0,T]$.*

As usual in analogous situations, a local uniqueness argument makes it possible to weaken the integrability assumptions upon the process Z.

Corollary 2.2. *Assume that $2m > d$, let $q \in]\theta, \infty]$, $r \in]\gamma(q), \infty]$, and suppose that $Z \in \mathscr{P}_{0,r,q}$. Then for any $t \in [0,T]$ and $\lambda \in [0, b[$ one has $J^Z(t,\cdot) \in \mathscr{C}^\lambda(\overline{\mathscr{O}})$ $\boldsymbol{P}$-almost surely.*

Varying the proof of Theorem 2.2 we can investigate—under strengthened hypotheses—higher smoothness of the function $J^Z(t,\cdot)$, namely, we will prove the following assertion.

Theorem 2.3. *Assume that $2m > d+2k$ for an integer $k \geqslant 0$. Let $q \in]\theta(k), \infty]$, $r \in]\gamma(q,k), \infty]$ and $\delta \in [2, \infty[$. Set*

$$b(k) = \frac{m(1-\beta(q,k)\overline{r})}{\overline{r}} \wedge 1 = \left(\frac{m(r-2)}{r} - \frac{d}{2} - \frac{d}{q} - k\right) \wedge 1.$$

Then for each $s \in]0, b(k)[$ there exists a constant $\widehat{L} = \widehat{L}(q,r,\delta,s) < \infty$ such that

$$\sup_{0 \leqslant t \leqslant T} \boldsymbol{E}|D^\nu J^Z(t,\cdot)|^\delta_{W^{s,\delta}(\mathscr{O})} \leqslant \widehat{L}\|Z\|^\delta_{\delta,r,q}$$

for all processes $Z \in \mathscr{P}_{\delta,r,q}$ and any multi-index ν, $|\nu| \leqslant k$. Hence $J^Z(t,\cdot) \in W^{k+s,\delta}(\overline{\mathscr{O}})$ $\boldsymbol{P}$-almost surely, and if $\delta > d/b(k)$ then $J^Z(t,\cdot) \in \mathscr{C}^{k+\lambda}(\overline{\mathscr{O}})$ $\boldsymbol{P}$-almost surely for each $\lambda \in [0, b(k) - d/\delta[$.

Note that Theorem 2.2 is a particular case of Theorem 2.3, but we have treated the case $k = 0$ separately because of its special importance. As a consequence of the preceding theorem we can show that the stochastic convolution satisfies (in the classical sense) some of the boundary conditions that are fulfilled by the Green function G. Recall that $B_1, \dots, B_m$ are boundary operators of orders $r_1, \dots, r_m$, respectively, and $B_j G(\cdot,\cdot,y) = 0$ on $]0,T] \times \partial\mathscr{O}$ for any $y \in \mathscr{O}$.

Corollary 2.3. *Assume that $2m > d+2k$ for an integer $k \geqslant 0$. Let $q \in]\theta(k), \infty]$, $r \in]\gamma(q,k), \infty]$, $\delta \in]b(k), \infty[$, and $Z \in \mathscr{P}_{\delta,r,q}$. Then*

$$B_j J^Z(t,\cdot)_{|\partial\mathscr{O}} = 0 \qquad \boldsymbol{P}\text{-almost surely}$$

for any $t \in]0,T]$ and $j \in \{1, \dots, m\}$ such that $r_j \leqslant k$.

Our last theorem deals with the space-time regularity of J^Z. It is worth noticing that J^Z is less regular in time than in the space variables.

Theorem 2.4. *Assume that $2m > d$, $q \in]\theta, \infty]$, $r \in]\gamma(q), \infty]$ and $\delta \in [2, \infty[$. Then for each $s \in]0, (1-\beta(q)\overline{r})/(2\overline{r})[$ there exists a constant $M = M(q, r, \delta, s) < \infty$ such that*

$$\boldsymbol{E}|J^Z(\cdot,\cdot)|^\delta_{W^{s,\delta}(]0,T[\times\mathscr{O})} \leqslant M \|Z\|^\delta_{\delta,r,q} \tag{2.2}$$

for every $Z \in \mathscr{P}_{\delta,r,q}$. Consequently, $J^Z(\cdot,\cdot) \in W^{s,\delta}(]0,T[\times\mathscr{O})$ $\boldsymbol{P}$-almost surely. Moreover, if

$$\delta > \frac{2\overline{r}(d+1)}{1-\beta(q)\overline{r}} \tag{2.3}$$

then $J^Z(\cdot,\cdot) \in \mathscr{C}^\lambda([0,T]\times\overline{\mathscr{O}})$ $\boldsymbol{P}$-almost surely for each λ satisfying

$$0 \leqslant \lambda < \frac{1-\beta(q)\overline{r}}{2\overline{r}} - \frac{d+1}{\delta}. \tag{2.4}$$

After simple calculations we obtain

Corollary 2.4. *If $2m > d$ and $Z \in \bigcap_{r,q>2} \mathscr{P}_{\delta,r,q}$ for a certain $\delta > \frac{4m(d+1)}{2m-d}$ then $J^Z(\cdot,\cdot) \in \mathscr{C}^\lambda([0,T]\times\overline{\mathscr{O}})$ for any λ satisfying*

$$0 \leqslant \lambda < \frac{1}{2} - \frac{d}{4m} - \frac{d+1}{\delta}.$$

The estimate (2.2) can be viewed as a maximal inequality for stochastic convolutions. We state this result explicitly since maximal inequalities are very useful.

Corollary 2.5. *Let the assumptions of Theorem 2.4 be satisfied and let (2.3) hold. Then for any λ fulfilling (2.4) and any κ fulfilling*

$$0 < \kappa < \delta\left(\frac{1-\beta(q)\overline{r}}{2\overline{r}} - \frac{d+1}{\delta} - \lambda\right)$$

there exists a constant $\widetilde{M} < \infty$ such that

$$\boldsymbol{E} \sup_{0\leqslant t\leqslant T} \left| \int_0^t S(t-s)Z(s)\,\mathrm{d}W(s) \right|^\delta_{\mathscr{C}^\lambda(\overline{\mathscr{O}})} \leqslant \widetilde{M} T^\kappa \|Z\|^\delta_{\delta,r,q}$$

whenever $Z \in \mathscr{P}_{\delta,r,q}$.

The proofs of the above theorems are postponed to the final section.

Remark. Tracing the proofs one can see easily that the particular form of the operators $\mathscr{A}$ and $\{B_j\}$ has never been used, only the fact that the semigroup $S(t)$ is given by a kernel G satisfying the estimates of Theorem 1.2 is relevant. So we can treat the case of a (parabolic) system of operators with coefficients dependent on time as well, as just such systems are investigated in the papers [8], [23] we have relied on. We have contented ourselves, however, with the simplest case of a single operator having time independent coefficients not to obscure the basic idea. Moreover, all proofs virtually remain valid for a (cylindrical) Wiener process with an arbitrary covariance operator; we have chosen the standard cylindrical Wiener process with the covariance operator I as it is, in a sense, the worst possible case. On the other hand, the adopted method does not take into account the possible regularizing effect of nuclearity of the covariance operator.

3. Auxiliary estimates

Let $k \geqslant 0$ be a fixed integer, everywhere in this section we assume that $2m > d+2k$ and $q \in [2,\infty]$. Generic constants independent of x, t are denoted by C_i in each proof independently. Recall that $\theta(k)$, $\beta(q,k)$ and $\gamma(q,k)$ are defined at the end of Section 1.

Lemma 3.1. *For all $q \in]\theta(k),\infty]$ and $p \in [1, \beta(q,k)^{-1}[$ there exists a constant $L_1 = L_1(q,p) < \infty$ such that*

$$\int_0^t |D^\nu G(s,x,\cdot)|^{2p}_{L^{q'}(\mathscr{O})}\,\mathrm{d}s \leqslant L_1 t^{1-\beta(q,k)p}$$

for all $t \in [0,T]$, $x \in \mathscr{O}$, and any multi-index ν with $|\nu| \leqslant k$.

Proof. Let $q' < \infty$ (that is, $q > 2$). By Theorem 1.2 we have

$$\begin{aligned}\int_0^t |D^\nu G(s,x,\cdot)|^{2p}_{L^{q'}(\mathscr{O})}\,\mathrm{d}s &= \int_0^t \left(\int_{\mathscr{O}} |D^\nu G(s,x,y)|^{q'}\,\mathrm{d}y\right)^{2p/q'} \mathrm{d}s \\
&\leqslant C_1 \int_0^t s^{-(d+|\nu|)p/m} \left(\int_{\mathscr{O}} \exp\left(-cq'\left|\frac{x-y}{s^\mu}\right|^{1/(1-\mu)}\right)\mathrm{d}y\right)^{2p/q'} \mathrm{d}s \\
&\leqslant C_2 \int_0^t s^{-(d+k)p/m} \left[\int_{\mathbb{R}^d} \exp\left(-cq'\left|\frac{y}{s^\mu}\right|^{1/(1-\mu)}\right)\mathrm{d}y\right]^{2p/q'} \mathrm{d}s \\
&\leqslant C_2 \int_0^t s^{-(d+k)p/m+dp/(mq')} \left[\int_{\mathbb{R}^d} \exp\left(-cq'|z|^{1/(1-\mu)}\right)\mathrm{d}z\right]^{2p/q'} \mathrm{d}s \\
&\leqslant C_3 \int_0^t s^{-\beta(q,k)p}\,\mathrm{d}s \equiv L_1 t^{1-\beta(q,k)p}.\end{aligned}$$

The case $q' = \infty$ can be treated similarly. $\square$

Corollary 3.1. *Let $q \in]\theta(k), \infty]$ and $r \in]\gamma(q,k), \infty]$. Then we have*

$$\int_0^t |D^\nu G(t-s,x,\cdot)g(s)|^2_{L^2(\mathcal{O})} \, \mathrm{d}s \leqslant L_1(q,\overline{r})^{1/\overline{r}} t^{1/\overline{r}-\beta(q,k)} |g|^2_{L^r([0,T];L^q(\mathcal{O}))}$$

for all $t \in [0,T]$, $x \in \mathcal{O}$, $g \in L^r([0,T];L^q(\mathcal{O}))$, and any multi-index ν, $|\nu| \leqslant k$.

Proof. Note that $\overline{r} \in [1, \beta(q,k)^{-1}[$ if $r > \gamma(q,k)$. Applying the Hölder inequality we obtain

$$\begin{aligned}\int_0^t \int_{\mathcal{O}} |D^\nu G(t-s,x,y)g(s,y)|^2 \, \mathrm{d}y \, \mathrm{d}s & \\ &\leqslant \int_0^t |g(s)|^2_{L^q(\mathcal{O})} |D^\nu G(t-s,x,\cdot)|^2_{L^{q'}(\mathcal{O})} \, \mathrm{d}s \\ &\leqslant |g|^2_{L^r([0,T];L^q(\mathcal{O}))} |D^\nu G(\cdot,x,\cdot)|^2_{L^{2\overline{r}}([0,t];L^{q'}(\mathcal{O}))},\end{aligned}$$

and Lemma 3.1 yields the desired conclusion. $\square$

Lemma 3.2. *Assume that $q \in]\theta(k), \infty]$, $p \in [1, \beta(q,k)^{-1}[$ and $\alpha \in]0, 2m(1-\beta(q,k)p) \wedge 2p[$. Then there exists a constant $L_2 = L_2(q,p,\alpha)$ such that*

$$\int_0^t |D^\nu G(s,x_1,\cdot) - D^\nu G(s,x_2,\cdot)|^{2p}_{L^{q'}(\mathcal{O})} \, \mathrm{d}s \leqslant L_2 t^{1-\beta(q,k)p-\alpha/(2m)} |x_1 - x_2|^\alpha$$

for all $x_1, x_2 \in \mathcal{O}$, $t \in [0,T]$, and any multi-index ν, $|\nu| \leqslant k$.

Proof. Note that $k+1 \leqslant 2m$, thus by Theorem 1.2 and the mean value theorem we have

$$\begin{aligned}|D^\nu G(s,x_1,y) - D^\nu G(s,x_2,y)| &= \left| \int_0^1 D_x D^\nu G(s, x_1 + \tau(x_2-x_1), y)(x_2-x_1) \, \mathrm{d}\tau \right| \\ &\leqslant C_1 |x_2 - x_1| \int_0^1 s^{-(d+|\nu|+1)\mu} \exp\left(-c \left| \frac{x_1 + \tau(x_2-x_1) - y}{s^\mu} \right|^{1/(1-\mu)} \right) \mathrm{d}\tau \\ &\leqslant C_2 s^{-(d+k+1)\mu} |x_2 - x_1|,\end{aligned}$$

where D_x stands for the Fréchet derivative with respect to the variable x. Take an $\alpha \in]0, 2m(1-\beta(q,k)p) \wedge 2p[$ and set $\varrho = \alpha/(2p)$. Then $\varrho \in]0,1[$ and

$$\begin{aligned}
&|D^\nu G(s,x_2,y) - D^\nu G(s,x_1,y)| \\
&\quad = |D^\nu G(s,x_2,y) - D^\nu G(s,x_1,y)|^\varrho \, |D^\nu G(s,x_2,y) - D^\nu G(s,x_1,y)|^{1-\varrho} \\
&\quad \leqslant C_2^\varrho |x_2 - x_1|^\varrho s^{-(d+k+1)\varrho\mu} \{ |D^\nu G(s,x_2,y)|^{1-\varrho} + |D^\nu G(s,x_1,y)|^{1-\varrho} \} \\
&\quad \leqslant C_3 |x_2 - x_1|^\varrho s^{-(d+k+\varrho)\mu} \Big\{ \exp\Big(c(\varrho - 1) \Big| \frac{x_2 - y}{s^\mu} \Big|^{1/(1-\mu)} \Big) + \\
&\qquad\qquad + \exp\Big(c(\varrho - 1) \Big| \frac{x_1 - y}{s^\mu} \Big|^{1/(1-\mu)} \Big) \Big\}.
\end{aligned}$$

Assuming for simplicity that $q > 2$ we obtain

$$\begin{aligned}
&\int_0^t |D^\nu G(s,x_2,\cdot) - D^\nu G(s,x_1,\cdot)|^{2p}_{L^{q'}(\mathscr{O})} \, \mathrm{d}s \\
&\leqslant C_4 |x_2 - x_1|^{2p\varrho} \int_0^t s^{-(d+k+\varrho)p/m} \Big[\int_{\mathbb{R}^d} \exp\Big(cq'(\varrho - 1) \Big| \frac{y}{s^\mu} \Big|^{1/(1-\mu)} \Big) \, \mathrm{d}y \Big]^{2p/q'} \mathrm{d}s \\
&\leqslant C_5 |x_2 - x_1|^\alpha \int_0^t s^{-\beta(q,k)p - \varrho p/m} \, \mathrm{d}s,
\end{aligned}$$

and it remains to observe that $\beta(q,k)p + \alpha/(2m) < 1$ due to the choice of α. □

Corollary 3.2. *Let $q \in]\theta(k), \infty]$, $r \in]\gamma(q,k), \infty]$, and $\alpha \in]0, 2m(1-\beta(q,k)\overline{r}) \wedge 2\overline{r}[$. Then we have*

$$\begin{aligned}
&\int_0^t |[D^\nu G(t-s,x_1,\cdot) - D^\nu G(t-s,x_2,\cdot)] g(s)|^2_{L^2(\mathscr{O})} \, \mathrm{d}s \\
&\qquad \leqslant L_2(q,\overline{r},\alpha)^{1/\overline{r}} t^{1/\overline{r} - \beta(q,k) - \alpha/(2m\overline{r})} |x_1 - x_2|^{\alpha/\overline{r}} |g|^2_{L^r([0,T];L^q(\mathscr{O}))}
\end{aligned}$$

for all $x_1, x_2 \in \mathscr{O}$, $t \in [0,T]$, $g \in L^r([0,T]; L^q(\mathscr{O}))$, and any multi-index ν with $|\nu| \leqslant k$.

Lemma 3.3. *Let $q \in]\theta, \infty]$ and $p \in [1, \beta(q)^{-1}[$. Then there exists a constant $L_3 = L_3(q,p) < \infty$ such that*

$$\int_0^h |G(s,x,\cdot)|^{2p}_{L^{q'}(\mathscr{O})} \, \mathrm{d}s \leqslant L_3 h^{1-\beta(q)p}$$

for all $x \in \mathscr{O}$ and $h \in [0,T]$.

Proof. From Theorem 1.2 we obtain easily

$$\int_0^h |G(s,x,\cdot)|^{2p}_{L^{q'}(\mathscr{O})}\,\mathrm{d}s \leqslant C_1 \int_0^h s^{-\beta(q)p}\,\mathrm{d}s,$$

and the desired estimate follows. □

Corollary 3.3. *Let $q \in]\theta,\infty]$, $r \in]\gamma(q),\infty]$. Then one has*

$$\int_t^{t+h} \bigl|G(t+h-s,x,\cdot)g(s)\bigr|^2_{L^2(\mathscr{O})}\,\mathrm{d}s \leqslant L_3(q,\overline{r})^{1/\overline{r}}|g|^2_{L^r([0,T];L^q(\mathscr{O}))}h^{(1-\beta(q)\overline{r})/\overline{r}}$$

for all $x \in \mathscr{O}$, $t \in [0,T[$, $h \in [0,T-t]$ and for any $g \in L^r([0,T];L^q(\mathscr{O}))$.

Lemma 3.4. *For arbitrary $q \in]\theta,\infty]$, $p \in \bigl[1,\beta(q)^{-1}\bigl[$ and $\alpha \in]0,1-\beta(q)p[$ there exists a constant $L_4 = L_4(q,p,\alpha) < \infty$ such that*

$$\int_0^t |G(s+h,x,\cdot)-G(s,x,\cdot)|^{2p}_{L^{q'}(\mathscr{O})}\,\mathrm{d}s \leqslant L_4 t^{1-\alpha-\beta(q)p}\,h^\alpha$$

for all $x \in \mathscr{O}$, $t \in [0,T[$ and $h \in [0,T-t[$.

Proof. From (1.1) and Theorem 1.2 we have

$$\begin{aligned}\left|\frac{\partial}{\partial v}G(v,x,y)\right| &= |\mathscr{A}G(v,x,y)| \\ &\leqslant C_1 v^{-1-d\mu}\exp\left(-c\left|\frac{x-y}{v^\mu}\right|^{1/(1-\mu)}\right)\end{aligned}$$

for all $x,y \in \mathscr{O}$, $v \in [0,T]$. Hence

$$\begin{aligned}|G(s+h,x,y)-G(s,x,y)| &= \left|\int_0^h \frac{\partial}{\partial r}G(s+r,x,y)\,\mathrm{d}r\right| \\ &\leqslant C_2\int_0^h (s+r)^{-1-d\mu}\,\mathrm{d}r \leqslant C_2 h s^{-1-d\mu}.\end{aligned}$$

Take an $\alpha \in]0,1-\beta(q)p[$ and set $\varrho = \alpha/(2p)$. Then

$$\begin{aligned}&|G(s+h,x,y)-G(s,x,y)| \\ &\qquad\leqslant C_2^\varrho h^\varrho s^{-\varrho(1+d\mu)}\bigl\{|G(s,x,y)|^{1-\varrho}+|G(s+h,x,y)|^{1-\varrho}\bigr\} \\ &\qquad\leqslant C_3 h^\varrho s^{-\varrho-d\mu}\exp\left(c(\varrho-1)\left|\frac{x-y}{s^\mu}\right|^{1/(1-\mu)}\right) \\ &\qquad\qquad + C_3 h^\varrho s^{-\varrho(1+d\mu)}(s+h)^{-(1-\varrho)d\mu}\exp\left(c(\varrho-1)\left|\frac{x-y}{(s+h)^\mu}\right|^{1/(1-\mu)}\right) \\ &\qquad\equiv I_1(s,x,y)+I_2(s,x,y).\end{aligned}$$

First,

$$\int_0^t |I_1(s,x,\cdot)|^{2p}_{L^{q'}(\mathscr{O})}\,\mathrm{d}s$$
$$\leqslant C_3^{2p}h^{2p\varrho}\int_0^t s^{-2p\varrho-2pd\mu}\left|\exp\left(c(\varrho-1)\left|\frac{x-\cdot}{s^\mu}\right|^{1/(1-\mu)}\right)\right|^{2p}_{L^{q'}(\mathscr{O})}\mathrm{d}s$$
$$\leqslant C_4h^\alpha\int_0^t s^{-\alpha-\beta(q)p}\,\mathrm{d}s$$

and the last integral is convergent by the choice of α. Further,

$$\int_0^t |I_2(s,x,\cdot)|^{2p}_{L^{q'}(\mathscr{O})}\,\mathrm{d}s \leqslant C_5h^\alpha\int_0^t s^{-2p\varrho(1+\mu d)}(s+h)^{2p\varrho\mu d-2p\mu d+2p\mu d/q'}\,\mathrm{d}s$$
$$= C_5h^\alpha\int_0^t s^{-\alpha(1+d\mu)}(s+h)^{-\beta(q)p+\alpha d\mu}\,\mathrm{d}s \equiv I_3.$$

Note that $\alpha\mu d-\beta(q)p=\mu d\{\alpha-(1+2/q)p\}<0$, thus we have

$$s^{-\alpha(1+d\mu)}(s+h)^{-\beta(q)p+\alpha d\mu} \leqslant s^{-\alpha(1+d\mu)}s^{-\beta(q)p+\alpha d\mu} = s^{-\alpha-\beta(q)p}.$$

Hence

$$I_3 \leqslant C_5h^\alpha\int_0^t s^{-\alpha-\beta(q)p}\,\mathrm{d}s$$

and, since $\alpha+\beta(q)p<1$, the integral is finite, which completes the proof. □

Corollary 3.4. *Let $q\in\,]\theta,\infty]$, $r\in\,]\gamma(q),\infty]$ and $\alpha\in\,]0,1-\beta(q)\overline{r}[$. Then we have*

$$\int_0^t \left|\big(G(t+h-s,x,\cdot)-G(t-s,x,\cdot)\big)g(s)\right|^2_{L^2(\mathscr{O})}\mathrm{d}s$$
$$\leqslant L_4(q,\overline{r},\alpha)^{1/\overline{r}}t^{(1-\alpha)/\overline{r}-\beta(q)}\,|g|^2_{L^r([0,T];L^q(\mathscr{O}))}\,h^{\alpha/\overline{r}}$$

for all $t\in[0,T[$, $h\in[0,T-t]$, $x\in\mathscr{O}$ and for any $g\in L^r([0,T];L^q(\mathscr{O}))$.

4. Proofs

Proof of Theorem 2.1. For every $u \in L^2(\mathscr{O})$ one has

$$[S(t-s)Z(s)u](\cdot) = \int_{\mathscr{O}} G(t-s,\cdot,y)Z(s,y)u(y)\,\mathrm{d}y.$$

Hence (see e.g. [19], Theorem VI.23)

$$\|S(t-s)Z(s)\|^2_{(\mathrm{HS})} = |G(t-s,\cdot,\cdot)Z(s,\cdot)|^2_{L^2(\mathscr{O}\times\mathscr{O})}.$$

This yields

$$\begin{aligned}
\int_0^t \|S(t-s)Z(s)\|^2_{(\mathrm{HS})}\,\mathrm{d}s &= \int_0^t\int_{\mathscr{O}}\Big(\int_{\mathscr{O}}|G(t-s,x,y)|^2\,\mathrm{d}x\Big)|Z(s,y)|^2\,\mathrm{d}y\,\mathrm{d}s \\
&\leqslant C\int_0^t\int_{\mathscr{O}}\Big(\int_{\mathscr{O}}(t-s)^{-d/m}\exp\Big(-2c\Big|\frac{x-y}{(t-s)^{\mu}}\Big|^{1/(1-\mu)}\Big)\,\mathrm{d}x\Big)|Z(s,y)|^2\,\mathrm{d}y\,\mathrm{d}s \\
&\leqslant C_1\int_0^t (t-s)^{-d/(2m)}|Z(s,\cdot)|^2_{L^2(\mathscr{O})}\,\mathrm{d}s \\
&\leqslant C_1\Big(\int_0^T |Z(s,\cdot)|^r_{L^2(\mathscr{O})}\,\mathrm{d}s\Big)^{2/r}\Big(\int_0^T s^{-d\overline{r}/(2m)}\,\mathrm{d}s\Big)^{1/\overline{r}}
\end{aligned}$$

and it remains to note that $d\overline{r}/(2m) < 1$ provided $r > 4m/(2m-d)$. If $Z \in \mathscr{P}_{2,r,2}$ then

$$\boldsymbol{E}|J^Z(t)|^2_{L^2(\mathscr{O})} = \boldsymbol{E}\int_0^t \|S(t-s)Z(s)\|^2_{(\mathrm{HS})}\,\mathrm{d}s \leqslant C_2\boldsymbol{E}|Z|^2_{L^r([0,T];L^2(\mathscr{O}))}$$

by the preceding estimate. □

The proofs of Theorems 2.2 and 2.3 are based on the possibility to switch between the Hilbert space approach to stochastic evolution equations and the random fields setting, namely on the following lemma.

Lemma 4.1. *Let $B(t)$ be an $(\mathscr{F}_t)$-adapted one-dimensional Wiener process on Ω. Let $\varphi\colon [0,T]\times\Omega \longrightarrow L^2(\mathscr{O})$ be an $(\mathscr{F}_t)$-adapted measurable stochastic process satisfying*

$$\boldsymbol{E}\int_0^T |\varphi(s)|^2_{L^2(\mathscr{O})}\,\mathrm{d}s < \infty, \tag{4.1}$$

and such that for λ_d-almost every $x \in \mathscr{O}$, $(\varphi(s,\omega)(x),\ 0 \leqslant s \leqslant T)$ is a well-defined real valued $(\mathscr{F}_t)$-adapted measurable stochastic process. Define

$$K_t\colon \Omega \times \mathscr{O} \longrightarrow \mathbb{R}, \quad (\omega, x) \longmapsto \int_0^t \varphi(s)(x)\,\mathrm{d}B(s), \qquad 0 \leqslant t \leqslant T.$$

Then

$$\int_0^t \varphi(s)\,\mathrm{d}B(s) = K_t \qquad \text{in } L^2(\Omega \times \mathscr{O}) \tag{4.2}$$

for every $t \in [0,T]$.

Proof. Fix $t \in [0,T]$. First, note that the definition of K_t is correct, since (4.1) implies

$$\lambda_d\Big\{x \in \mathscr{O};\ \boldsymbol{E}\int_0^t |\varphi(s)(x)|^2\,\mathrm{d}s = \infty\Big\} = 0.$$

Hence $K_t(\cdot,x)$ is well-defined for λ_d-almost every $x \in \mathscr{O}$, and $K_t \in L^2(\Omega \times \mathscr{O})$. The assertion (4.2) is obvious if φ is a step function. The general case may be proved by a standard approximation argument. □

Let Z satisfy the assumptions of Theorem 2.2. By the definition of the stochastic integral we have

$$J^Z(t,\cdot) = \sum_{k=1}^{\infty} \int_0^t S(t-s)Z(s)e_k\,\mathrm{d}W_k(s) \qquad \text{in } L^2(\Omega; L^2(\mathscr{O})).$$

Hence there exists a subsequence $\{l_n\}$ such that for λ_d-almost every $x \in \mathscr{O}$,

$$J^Z(t,x) = \lim_{n\to\infty} \sum_{k=1}^{l_n} \Big(\int_0^t S(t-s)Z(s)e_k\,\mathrm{d}W_k(s)\Big)(x) \qquad \boldsymbol{P}\text{-almost surely.}$$

Further, we claim that the series

$$\sum_{k=1}^{\infty} \int_0^t [S(t-s)Z(s)e_k](x)\,\mathrm{d}W_k(s)$$

converges in $L^2(\Omega)$ for λ_d-almost every $x \in \mathscr{O}$. Indeed,

$$\sum_{k=1}^{\infty} \boldsymbol{E}\int_0^t \big|\langle G(t-s,x,\cdot)Z(s,\cdot), e_k(\cdot)\rangle\big|^2\,\mathrm{d}s = \boldsymbol{E}\int_0^t |G(t-s,x,\cdot)Z(s,\cdot)|^2_{L^2(\mathscr{O})}\,\mathrm{d}s < \infty$$

by Corollary 3.1, so the independence of $\{W_k\}$ yields

$$\begin{aligned}
\boldsymbol{E}\left|\sum_{k=n}^{l}\int_0^t [S(t-s)Z(s)e_k](x)\,\mathrm{d}W_k(s)\right|^2 &= \sum_{k=n}^{l}\boldsymbol{E}\int_0^t |[S(t-s)Z(s)e_k](x)|^2\,\mathrm{d}s\\
&= \sum_{k=n}^{l}\boldsymbol{E}\int_0^t \left|\int_{\mathscr{O}} G(t-s,x,y)Z(s,y)e_k(y)\,\mathrm{d}y\right|^2\mathrm{d}s\\
&= \sum_{k=n}^{l}\boldsymbol{E}\int_0^t \left|\langle G(t-s,x,\cdot)Z(s,\cdot),e_k(\cdot)\rangle\right|^2\mathrm{d}s \xrightarrow[l,n\to\infty]{} 0.
\end{aligned}$$

Consequently, by Lemma 4.1 we obtain

Lemma 4.2. *Let $2m > d$, $q \in \,]\theta,\infty]$, $r \in \,]\gamma(q),\infty]$ and $Z \in \mathscr{P}_{2,r,q}$. Then for each $t \in [0,T]$ there exists a measurable set $\mathscr{N}(t) \subseteq \mathscr{O}$ such that $\lambda_d(\mathscr{N}(t)) = 0$ and*

$$\boldsymbol{P}\Big\{J^Z(t,x) = \sum_{k=1}^{\infty}\int_0^t [S(t-s)Z(s)e_k](x)\,\mathrm{d}W_k(s)\Big\} = 1$$

for every $x \in \mathscr{O}\setminus\mathscr{N}(t)$.

Lemma 4.3. *Suppose that $2m > d$, let $q \in \,]\theta,\infty]$, $r \in \,]\gamma(q),\infty]$, and $\alpha \in \,]0, 2m(1-\beta(q)\overline{r}) \wedge 2\overline{r}[$. Then there exists a constant $L_5 = L_5(q,r,\delta,\alpha) < \infty$ such that*

$$\boldsymbol{E}|J^Z(t,x_1) - J^Z(t,x_2)|^\delta \leqslant L_5 T^{\delta(1/\overline{r}-\beta(q)-\alpha/(2m\overline{r}))/2}\, |\!|\!| Z |\!|\!|_{\delta,r,q}^{\delta}\, |x_1-x_2|^{\alpha\delta/(2\overline{r})}$$

for any $Z \in \mathscr{P}_{\delta,r,q}$ and for all $t \in [0,T]$, $x_1, x_2 \in \mathscr{O}\setminus\mathscr{N}(t)$.

Proof. Proceeding as in the proof of Lemma 4.2, using the Burkholder-Gundy inequality and Corollary 3.2 we obtain

$$\begin{aligned}
&\boldsymbol{E}|J^Z(t,x_1) - J^Z(t,x_2)|^\delta\\
&\quad= \boldsymbol{E}\left|\sum_{k=1}^{\infty}\int_0^t \{[S(t-s)Z(s)e_k](x_1) - [S(t-s)Z(s)e_k](x_2)\}\,\mathrm{d}W_k(s)\right|^\delta\\
&\quad\leqslant C_1\boldsymbol{E}\left(\int_0^t\sum_{k=1}^{\infty}\left|[S(t-s)Z(s)e_k](x_1) - [S(t-s)Z(s)e_k](x_2)\right|^2\mathrm{d}s\right)^{\delta/2}\\
&\quad= C_1\boldsymbol{E}\left(\int_0^t \left|[G(t-s,x_1,\cdot) - G(t-s,x_2,\cdot)]Z(s,\cdot)\right|_{L^2(\mathscr{O})}^2\mathrm{d}s\right)^{\delta/2}\\
&\quad\leqslant C_1\boldsymbol{E}\left(L_2(q,\overline{r},\alpha)^{1/\overline{r}}t^{1/\overline{r}-\beta(q)-\alpha/(2m\overline{r})}\,|Z|_{L^r([0,T];L^q(\mathscr{O}))}^2|x_1-x_2|^{\alpha/\overline{r}}\right)^{\delta/2}\\
&\quad\leqslant L_5 T^{\delta(1/\overline{r}-\beta(q)-\alpha/(2m\overline{r}))/2}\,|\!|\!| Z |\!|\!|_{\delta,r,q}^{\delta}|x_1-x_2|^{\alpha\delta/(2\overline{r})},
\end{aligned}$$

and Lemma 4.3 follows. □

Proof of Theorem 2.2. First, using the above procedure and Corollary 3.1 we get

$$\begin{aligned}
\boldsymbol{E}|J^Z(t,\cdot)|^\delta_{L^\delta(\mathscr{O})} &= \int_{\mathscr{O}} \boldsymbol{E}|J^Z(t,x)|^\delta \,\mathrm{d}x \\
&\leqslant C_1 \int_{\mathscr{O}} \boldsymbol{E}\Big(\int_0^t |G(t-s,x,\cdot)Z(s,\cdot)|^2_{L^2(\mathscr{O})}\,\mathrm{d}s\Big)^{\delta/2}\,\mathrm{d}x \\
&\leqslant C_2 \|Z\|^\delta_{\delta,r,q}.
\end{aligned}$$

Now, let $s \in \,]0,b[$, where b is defined in Theorem 2.2. Then one can choose an $\alpha \in \,]2\overline{r}s, 2m(1-\beta(q)\overline{r}) \wedge 2\overline{r}[$. Lemma 4.3 and the Fubini theorem yield

$$\begin{aligned}
\boldsymbol{E}\int_{\mathscr{O}}\int_{\mathscr{O}} &\frac{|J^Z(t,x)-J^Z(t,y)|^\delta}{|x-y|^{d+s\delta}}\,\mathrm{d}x\,\mathrm{d}y \\
&\leqslant C_3 \|Z\|^\delta_{\delta,r,q} \int_{\mathscr{O}}\int_{\mathscr{O}} |x-y|^{-d+\delta(-s+\alpha/(2\overline{r}))}\,\mathrm{d}x\,\mathrm{d}y.
\end{aligned}$$

Since $\delta(-s+\alpha/(2\overline{r})) > 0$ the double integral is finite, therefore there exists a constant $L < \infty$ such that

$$\boldsymbol{E}|J^Z(t,\cdot)|^\delta_{W^{s,\delta}(\mathscr{O})} \leqslant L\|Z\|^\delta_{\delta,r,q},$$

which proves (2.1). Now note that if $\delta > d/b$ and $\lambda \in [0, b-d/\delta[$ then one can choose an $s \in \,]d/\delta+\lambda, b[$. Consequently, $W^{s,\delta}(\mathscr{O}) \hookrightarrow \mathscr{C}^\lambda(\overline{\mathscr{O}})$ by Theorem 1.1. □

Proof of Corollary 2.2. Fix $\lambda \in [0,b[$ and find $\delta \in [2,\infty[$ such that $\lambda + d/\delta < b$. Define stopping times τ_N, $N \in \mathbb{N}$, by

$$\tau_N(\omega) = \inf\Big\{t \in [0,T];\ \int_0^t |Z(s,\cdot,\omega)|^r_{L^q(\mathscr{O})}\,\mathrm{d}s \geqslant N\Big\},$$

with the convention $\inf\emptyset = T$. Setting

$$Z_N(s,\cdot,\omega) = \chi_{\{\tau_N(\omega)>s\}} Z(s,\cdot,\omega), \qquad N \in \mathbb{N},$$

we can easily see that $Z_N \in \mathscr{P}_{\delta,r,q}$, thus $J^{Z_N}(t,\cdot) \in \mathscr{C}^\lambda(\overline{\mathscr{O}})$ $\boldsymbol{P}$-almost surely for any fixed $t \in [0,T]$ due to Theorem 2.2. Moreover, let $\Omega_N = \{\omega;\ \tau_N(\omega) = T\}$, then for any $t \in [0,T]$ one has

$$J^{Z_N}(t,\cdot) = J^Z(t,\cdot) \qquad \boldsymbol{P}\text{-almost surely on } \Omega_N$$

according to a well known property of stochastic integrals. Obviously,

$$\boldsymbol{P}\Big(\Omega \setminus \bigcup_{N\geqslant 1}\Omega_N\Big) = 0$$

by the choice of the process Z, and Corollary 2.2 follows. □

Proof of Theorem 2.3. As usual, we denote by $\mathscr{D}(\mathscr{O})$ the space of all $\mathscr{C}^\infty$-functions with compact supports in $\mathscr{O}$, and by $\mathscr{D}'(\mathscr{O})$ its dual space (i.e., the Schwartz distributions on $\mathscr{O}$). As was already mentioned, the proof proceeds along the lines of the proof of Theorem 2.2, so we will discuss only the differences.

Fix $t \in]0,T]$, a multi-index ν with $|\nu| \leqslant k$, and a process $Z \in \mathscr{P}_{\delta,r,q}$. Let us define operators $m_s \in L(L^2(\mathscr{O}))$ by

$$m_s f = \int_{\mathscr{O}} D^\nu G(s,\cdot,y) f(y)\,\mathrm{d}y, \qquad f \in L^2(\mathscr{O}).$$

Then

$$\boldsymbol{E} \int_0^t \|m_{t-s} Z(s)\|^2_{(\mathrm{HS})}\,\mathrm{d}s < \infty$$

by the same argument as in the proof of Theorem 2.1, so we can set

$$M(t,\cdot) = \int_0^t m_{t-s} Z(s)\,\mathrm{d}W(s).$$

Repeating the proof of Lemma 4.2 we find a set $\mathscr{E} \subseteq \mathscr{O}$ such that $\lambda_d(\mathscr{E}) = 0$ and

$$\boldsymbol{P}\Big\{M(t,x) = \sum_{k=1}^\infty \int_0^t [m_{t-s} Z(s) e_k](x)\,\mathrm{d}W_k(s)\Big\} = 1$$

for any $x \in \mathscr{O} \setminus \mathscr{E}$. We aim at proving

$$\boldsymbol{P}\big\{D^\nu J^Z(t,\cdot) = M(t,\cdot) \text{ in } \mathscr{D}'(\mathscr{O})\big\} = 1. \tag{4.3}$$

To avoid clumsy notation, we verify (4.3), with no essential loss of generality, only in the case $\nu = (1,0,\dots,0)$. Let $\varphi \in \mathscr{D}(\mathscr{O})$ be arbitrary but fixed, set $\mathbf{e} = (1,0,\dots,0) \in \mathbb{R}^d$ and find $h_0 > 0$ such that

$$\operatorname{supp}(\varphi) \subseteq \bigcap_{|h| \leqslant h_0} (\mathscr{O} + h\mathbf{e}).$$

First, note that

$$\lim_{h\to 0} \int_{\mathscr{O}} J^Z(t,x) \frac{\varphi(x+h\mathbf{e}) - \varphi(x)}{h}\,\mathrm{d}x = \int_{\mathscr{O}} J^Z(t,x) D^\nu \varphi(x)\,\mathrm{d}x \quad \text{in } L^2(\Omega) \tag{4.4}$$

by the dominated convergence theorem. Moreover, for $h \in [-h_0, h_0]$ one has

$$\begin{aligned}
&\int_{\mathscr{O}} J^Z(t,x) \frac{\varphi(x+h\mathbf{e}) - \varphi(x)}{h}\,\mathrm{d}x \\
&\quad = \frac{1}{h}\Big\{\int_{\mathscr{O}+h\mathbf{e}} J^Z(t,x-h\mathbf{e})\varphi(x)\,\mathrm{d}x - \int_{\mathscr{O}} J^Z(t,x)\varphi(x)\,\mathrm{d}x\Big\} \\
&\quad = \int_{\mathscr{O}} \frac{J^Z(t,x-h\mathbf{e}) - J^Z(t,x)}{h}\varphi(x)\,\mathrm{d}x.
\end{aligned}$$

Further, we proceed as in the proofs of Lemma 4.3 to obtain the following estimates:

$$
\begin{aligned}
&\boldsymbol{E}\left|\int_{\mathscr{O}}\left\{\frac{J^Z(t,x-h\mathbf{e})-J^Z(t,x)}{h}+M(t,x)\right\}\varphi(x)\,\mathrm{d}x\right|^2\\
&\leqslant |\varphi|^2_{L^2(\mathscr{O})}\int_{\mathscr{O}}\boldsymbol{E}\left|\frac{J^Z(t,x-h\mathbf{e})-J^Z(t,x)}{h}+M(t,x)\right|^2\mathrm{d}x\\
&\leqslant |\varphi|^2_{L^2(\mathscr{O})}\int_{\mathscr{O}}\boldsymbol{E}\left|\sum_{k=1}^{\infty}\int_0^t\int_{\mathscr{O}}\left\{\frac{G(t-s,x-h\mathbf{e},y)-G(t-s,x,y)}{h}\right.\right.\\
&\qquad\qquad\left.\left.+D^\nu G(t-s,x,y)\right\}Z(s,y)e_k(y)\,\mathrm{d}y\,\mathrm{d}W_k(s)\right|^2\mathrm{d}x\\
&\leqslant |\varphi|^2_{L^2(\mathscr{O})}\int_{\mathscr{O}}\boldsymbol{E}\int_0^t\left|\left\{\frac{G(t-s,x-h\mathbf{e},\cdot)-G(t-s,x,\cdot)}{h}\right.\right.\\
&\qquad\qquad\left.\left.+D^\nu G(t-s,x,\cdot)\right\}Z(s,\cdot)\right|^2_{L^2(\mathscr{O})}\mathrm{d}s\,\mathrm{d}x\\
&\leqslant |\varphi|^2_{L^2(\mathscr{O})}\|Z\|^2_{2,r,q}\int_{\mathscr{O}}\int_0^t\left|\frac{G(t-s,x-h\mathbf{e},\cdot)-G(t-s,x,\cdot)}{h}\right.\\
&\qquad\qquad\left.+D^\nu G(t-s,x,\cdot)\right|^{2\bar{r}}_{L^{q'}(\mathscr{O})}\mathrm{d}s\,\mathrm{d}x.
\end{aligned}
$$

Obviously,

$$
\lim_{h\to 0}\left\{\frac{G(s,x-h\mathbf{e},y)-G(s,x,y)}{h}+D^\nu G(s,x,y)\right\}=0
$$

for any $x,y\in\mathscr{O}$, $s\in\,]0,T]$ and by the mean value theorem we have

$$
\begin{aligned}
&\left|\frac{G(s,x-h\mathbf{e},y)-G(s,x,y)}{h}+D^\nu G(s,x,y)\right|\\
&\leqslant Cs^{-(d+1)/(2m)}\left\{\exp\left(-c\left|\frac{x-h\varrho\mathbf{e}-y}{s^\mu}\right|^{1/(1-\mu)}\right)+\exp\left(-c\left|\frac{x-y}{s^\mu}\right|^{1/(1-\mu)}\right)\right\}
\end{aligned}
$$

for a $\varrho\in\,]0,1[$. So applying the dominated convergence theorem twice we obtain

$$
\lim_{h\to 0}\int_{\mathscr{O}}\int_0^t\left|\frac{G(s,x-h\mathbf{e},\cdot)-G(s,x,\cdot)}{h}+D^\nu G(s,x,\cdot)\right|^{2\bar{r}}_{L^{q'}(\mathscr{O})}\mathrm{d}s\,\mathrm{d}x=0.
$$

This yields

$$
\lim_{h\to 0}\int_{\mathscr{O}}J^Z(t,x)\frac{\varphi(x+h\mathbf{e})-\varphi(x)}{h}\,\mathrm{d}x=-\int_{\mathscr{O}}M(t,x)\varphi(x)\,\mathrm{d}x\quad\text{in } L^2(\Omega). \tag{4.5}
$$

Comparing (4.4) and (4.5) and taking into account that φ was arbitrary we see that

$$\text{(4.6)} \qquad \boldsymbol{P}\Big\{\int_{\mathscr{O}} J^Z(t,x)D^\nu\varphi(x)\,\mathrm{d}x = -\int_{\mathscr{O}} M(t,x)\varphi(x)\,\mathrm{d}x\Big\} = 1$$

holds for any $\varphi \in \mathscr{D}(\mathscr{O})$. The space $\mathscr{D}(\mathscr{O})$ is separable and first countable, hence

$$\boldsymbol{P}\Big\{\int_{\mathscr{O}} J^Z(t,x)D^\nu\psi(x)\,\mathrm{d}x = -\int_{\mathscr{O}} M(t,x)\psi(x)\,\mathrm{d}x \quad \text{for all } \psi \in \mathscr{D}(\mathscr{O})\Big\} = 1,$$

which is equivalent to our claim (4.3).

Finally, the required estimate

$$\boldsymbol{E}|M(t,\cdot)|^\delta_{W^{s,\delta}(\mathscr{O})} \leqslant \widehat{L}\,|\!|\!|Z|\!|\!|^\delta_{\delta,r,q}$$

can be derived by exactly the same procedure as the related estimate of J^Z in the proof of Theorem 2.2. □

Proof of Corollary 2.3. Fix $t \in]0,T]$, $j \in \{1,\dots,m\}$ with $r_j \leqslant k$ and $Z \in \mathscr{P}_{\delta,r,q}$; take $h \in \partial\mathscr{O}$ arbitrary. We may assume that the coefficients $b_{j\nu}$ of the operator B_j are defined on the whole $\overline{\mathscr{O}}$ and $b_{j\nu} \in \mathscr{C}^{2m-r_j+\eta}(\overline{\mathscr{O}})$. We know that there exists a measurable set $\mathscr{E} \subseteq \mathscr{O}$ such that $\lambda_d(\mathscr{E}) = 0$ and

$$\boldsymbol{P}\{B_jJ^Z(t,x) = \sum_{l=1}^\infty \int_0^t\Big(\int_{\mathscr{O}} B_jG(t-s,x,v)Z(s,v)e_l(v)\,\mathrm{d}v\Big)\,\mathrm{d}W_l(s)\} = 1$$

for any $x \in \mathscr{O}\setminus\mathscr{E}$. Let us find $y_l \in \mathscr{O}\setminus\mathscr{E}$ satisfying $y_l \longrightarrow h$ as $l \to \infty$; we aim at proving that

$$\text{(4.7)} \qquad \lim_{l\to\infty} \boldsymbol{E}|B_jJ^Z(t,y_l)|^\delta = 0.$$

Since $y_l \notin \mathscr{E}$ we obtain

$$\begin{aligned}\boldsymbol{E}|B_jJ^Z(t,y_l)|^\delta &\leqslant C_1\boldsymbol{E}\Big(\int_0^t |B_jG(t-s,y_l,\cdot)Z(s,\cdot)|^2_{L^2(\mathscr{O})}\,\mathrm{d}s\Big)^{\delta/2}\\ &\leqslant C_1|\!|\!|Z|\!|\!|^\delta_{\delta,r,q}\Big(\int_0^t |B_jG(s,y_l,\cdot)|^{2\overline{r}}_{L^{q'}(\mathscr{O})}\,\mathrm{d}s\Big)^{\delta/(2\overline{r})}.\end{aligned}$$

By the properties of the Green function,

$$\lim_{l\to\infty} B_jG(s,y_l,z) = 0$$

for any $s \in]0,T]$ and $z \in \mathscr{O}$. Furthermore, as has been already established,

$$\sup_{l\in\mathbb{N}} |B_jG(s,y_l,\cdot)|^{2\overline{r}}_{L^{q'}(\mathscr{O})} \leqslant C_2 s^{-\beta(q,k)\overline{r}},$$

hence

$$\lim_{l\to\infty} \int_0^t |B_jG(s,y_l,\cdot)|^{2\overline{r}}_{L^{q'}(\mathscr{O})} \, \mathrm{d}s = 0$$

by the dominated convergence theorem, and (4.7) holds true. At the same time $B_jJ^Z(t,\cdot) \in \mathscr{C}(\overline{\mathscr{O}})$ $\boldsymbol{P}$-almost surely, so (4.7) yields that $B_jJ^Z(t,h) = 0$ almost surely. Therefore, Corollary 2.3 follows by continuity of $B_jJ^Z(t,\cdot)$ on $\partial\mathscr{O}$. □

To prove Theorem 2.4 we need the following lemma.

Lemma 4.4. *Assume that $2m > d$, $q \in]\theta,\infty]$, $r \in]\gamma(q),\infty]$, and $\delta \in [2,\infty[$. Then for any $\alpha \in]0, 1-\beta(q)\overline{r}[$ there exists a constant $L_6 = L_6(q,r,\delta,\alpha) < \infty$ such that*

$$\boldsymbol{E}|J^Z(t_1,x) - J^Z(t_2,x)|^\delta \leqslant L_6 T^{\delta(1/\overline{r}-\beta(q)-\alpha/\overline{r})/2} \|Z\|^\delta_{\delta,r,q} |t_1-t_2|^{\alpha\delta/(2\overline{r})} \tag{4.8}$$

holds for all $Z \in \mathscr{P}_{\delta,r,q}$, $t_1,t_2 \in [0,T]$ and any $x \in \mathscr{O} \setminus (\mathscr{N}(t_1) \cup \mathscr{N}(t_2))$.

Proof. For definiteness, assume that $t_1 < t_2$. If $x \notin \mathscr{N}(t_1) \cup \mathscr{N}(t_2)$ then by Lemma 4.2, the Burkholder-Gundy inequality, and by Corollaries 3.3 and 3.4 we have

$$\begin{aligned}
&\boldsymbol{E}|J^Z(t_2,x) - J^Z(t_1,x)|^\delta \\
&= \boldsymbol{E}\left|\sum_{k=1}^\infty \left\{\int_0^{t_2} [S(t_2-s)Z(s)e_k](x)\,\mathrm{d}W_k(s) - \int_0^{t_1} [S(t_1-s)Z(s)e_k](x)\,\mathrm{d}W_k(s)\right\}\right|^\delta \\
&\leqslant C_1\boldsymbol{E}\left|\sum_{k=1}^\infty \int_{t_1}^{t_2} [S(t_2-s)Z(s)e_k](x)\,\mathrm{d}W_k(s)\right|^\delta \\
&\qquad + C_1\boldsymbol{E}\left|\sum_{k=1}^\infty \int_0^{t_1} \Big\{[S(t_2-s)Z(s)e_k](x) - [S(t_1-s)Z(s)e_k](x)\Big\}\,\mathrm{d}W_k(s)\right|^\delta \\
&\leqslant C_2\boldsymbol{E}\left(\int_{t_1}^{t_2} |G(t_2-s,x,\cdot)Z(s,\cdot)|^2_{L^2(\mathscr{O})}\,\mathrm{d}s\right)^{\delta/2} \\
&\qquad + C_2\boldsymbol{E}\left(\int_0^{t_1} \left|[G(t_2-s,x,\cdot) - G(t_1-s,x,\cdot)]Z(s,\cdot)\right|^2_{L^2(\mathscr{O})}\,\mathrm{d}s\right)^{\delta/2} \\
&\leqslant C_3\|Z\|^\delta_{\delta,r,q}\left\{(t_2-t_1)^{(1-\beta(q)\overline{r})\delta/(2\overline{r})} + T^{\delta(1-\beta(q)\overline{r}-\alpha)/(2\overline{r})}(t_2-t_1)^{\alpha\delta/(2\overline{r})}\right\},
\end{aligned}$$

which proves (4.8). □

Set

$$\mathscr{N} = \{((t_1,x_1),(t_2,x_2)) \in ([0,T]\times\mathscr{O})^2 : x_1, x_2 \notin \mathscr{N}(t_1)\cup\mathscr{N}(t_2)\}.$$

Obviously, $\lambda_{2d+2}(\mathscr{N})=0$. Combining Lemmas 4.3 and 4.4 we obtain

Corollary 4.1. *Under the assumptions of Theorem 2.4, for any $\alpha \in]0, 1-\beta(q)\overline{r}[$ there exists a constant $L_7 = L_7(q,r,\delta,\alpha) < \infty$ such that*

$$\boldsymbol{E}|J^Z(t_1,x_1)-J^Z(t_2,x_2)|^\delta \leqslant L_7 T^{\delta(1/\overline{r}-\beta(q)-\alpha/\overline{r})/2} |\!|\!| Z |\!|\!|_{\delta,r,q}^{\delta} \big|(t_1,x_1)-(t_2,x_2)\big|^{\alpha\delta/(2\overline{r})}$$

for any $Z\in\mathscr{P}_{\delta,r,q}$ and all $((t_1,x_1),(t_2,x_2)) \in ([0,T]\times\mathscr{O})^2\setminus\mathscr{N}$.

P r o o f o f T h e o r e m 2.4 a n d C o r o l l a r y 2.5. We can complete the proof of Theorem 2.4 proceeding in the same way as in the proof of Theorem 2.2. We repeat the main steps as now we are interested in the dependence of constants on T. First,

$$\begin{aligned}\boldsymbol{E}\big|J^Z(\cdot,\cdot)\big|^\delta_{L^\delta(]0,T[\times\mathscr{O})} &\leqslant C_1\int_{\mathscr{O}}\int_0^T \boldsymbol{E}\Big(\int_0^t \big|G(t-s,x,\cdot)Z(s,\cdot)\big|^2_{L^2(\mathscr{O})}\,\mathrm{d}s\Big)^{\delta/2}\,\mathrm{d}t\,\mathrm{d}x\\ &\leqslant C_2 |\!|\!| Z |\!|\!|^\delta_{\delta,r,q}\int_0^T t^{\delta(1-\beta(q)\overline{r})/(2\overline{r})}\,\mathrm{d}t\\ &\leqslant C_3 T^{\delta(1-\beta(q)\overline{r})/(2\overline{r})} |\!|\!| Z |\!|\!|^\delta_{\delta,r,q}.\end{aligned}$$

Further, for any $\alpha\in]2\overline{r}s, 1-\beta(q)\overline{r}[$ we have

$$\boldsymbol{E}\int_{]0,T[\times\mathscr{O}}\int_{]0,T[\times\mathscr{O}} \frac{|J^Z(t,y)-J^Z(\tau,x)|^\delta}{|(t,y)-(\tau,x)|^{d+1+s\delta}}\,\mathrm{d}\tau\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}y$$
$$\leqslant C_4 T^{\delta(1-\alpha-\beta(q)\overline{r})/(2\overline{r})} |\!|\!| Z |\!|\!|^\delta_{\delta,r,q}$$

by Corollary 4.1. Hence for any ϱ fulfilling

$$0\leqslant\varrho<\delta\left(\frac{1}{2\overline{r}}-\frac{\beta(q)}{2}-s\right) \tag{4.9}$$

there is a constant C_5 (dependent on ϱ) such that the estimate

$$\boldsymbol{E}\big|J^Z(\cdot,\cdot)\big|^\delta_{W^{s,\delta}(]0,T[\times\mathscr{O})} \leqslant C_5 T^\varrho |\!|\!| Z |\!|\!|^\delta_{\delta,r,q}$$

holds.

To prove Corollary 2.5, assume that (2.3) is fulfilled and take $s > \lambda + (d+1)/\delta$, then

$$\begin{aligned}\boldsymbol{E} \sup_{0\leqslant t\leqslant T} \left|J^Z(t,\cdot)\right|^{\delta}_{\mathscr{C}^{\lambda}(\overline{\mathcal{O}})} &\leqslant \boldsymbol{E}\left|J^Z(\cdot,\cdot)\right|^{\delta}_{\mathscr{C}^{\lambda}([0,T]\times\overline{\mathcal{O}})} \leqslant C_6\boldsymbol{E}\left|J^Z(\cdot,\cdot)\right|^{\delta}_{W^{s,\delta}(]0,T[\times\mathcal{O})}\\ &\leqslant C_5C_6T^{\varrho}\|Z\|^{\delta}_{\delta,r,q}\end{aligned}$$

for any ϱ satisfying (4.9). Since we can take s arbitrarily close to $\lambda + (d+1)/\delta$ the proof is completed. □

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