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0–1 SEQUENCES HAVING THE SAME NUMBERS OF (1–1)-COUPLES OF GIVEN DISTANCES

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Summary. Let $\mathbf{a}$ be a 0–1 sequence with a finite number of terms equal to 1. The distance sequence $\delta^{(\mathbf{a})}$ of $\mathbf{a}$ is defined as a sequence of the numbers of (1–1)-couples of given distances. The paper investigates such pairs of 0–1 sequences $\mathbf{a}$, $\mathbf{b}$ that $\mathbf{a}$ is different from $\mathbf{b}$ and $\delta^{(\mathbf{a})} = \delta^{(\mathbf{b})}$.

Keywords: 0–1 sequence, distance sequence, uniform distribution, set covariance.

1. INTRODUCTION

Consider sets

$$\mathscr{A}_n = \{\mathbf{a};\, \mathbf{a} = \{a_i\}_{i=0}^{\infty},\ a_0 = 1, a_i \in \{0,1\} \text{ for } i \in \mathbb{N},\ \max\{i;\, i \in \mathbb{N}_0, a_i = 1\} = n\}$$

for each $n \in \mathbb{N}_0$, where $\mathbb{N}$ is the set of all positive integers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, and

$$\mathscr{A} = \bigcup_{n=0}^{\infty} \mathscr{A}_n.$$

For any $\mathbf{a} \in \mathscr{A}$ and $j \in \mathbb{N}_0$, put

$$n(\mathbf{a}) = \max\{i;\, i \in \mathbb{N}_0,\ a_i = 1\},$$
$$\delta_j^{(\mathbf{a})} = \sum_{i=0}^{\infty} a_i a_{i+j}$$

and

$$\delta^{(\mathbf{a})} = \{\delta_j^{(\mathbf{a})}\}_{j=0}^{\infty}.$$

The value of $\delta_j^{(\mathbf{a})}$ expresses the number of pairs of elements of the sequence $\mathbf{a}$ such that both are equal to 1 and that their distance is j. We shall call $\delta^{(\mathbf{a})}$ the *distance sequence* generated by the sequence $\mathbf{a}$.

It can be easily seen that the sets $\mathscr{A}_n$, $n \in \mathbb{N}_0$, are disjoint and that the following relations are true for any $\mathbf{a} \in \mathscr{A}$:

$$\mathbf{a} \in \mathscr{A}_{n(\mathbf{a})},$$

$$\delta_0^{(\mathbf{a})} = \operatorname{card}\{i;\, i \in \mathbb{N}_0,\ a_i = 1\}, \tag{1}$$

$$\delta_{n(\mathbf{a})}^{(\mathbf{a})} = 1, \tag{2}$$

$$\delta_j^{(\mathbf{a})} \in \{0, 1, \ldots, n(\mathbf{a}) - j + 1\} \quad \text{if} \quad j \in \{0, 1, \ldots, n(\mathbf{a})\}$$

and

$$\delta_j^{(\mathbf{a})} = 0 \quad \text{if} \quad j \in \mathbb{N}, \quad j > n(\mathbf{a}). \tag{3}$$

Let $\mathbf{a} \in \mathscr{A}$ and define a sequence $\mathbf{r}^{(\mathbf{a})} = \{r_i^{(\mathbf{a})}\}_{i=0}^{\infty}$ by

$$r_i^{(\mathbf{a})} = a_{n(\mathbf{a})-i} \quad \text{for} \quad i \leqslant n(\mathbf{a}),$$

and

$$r_i^{(\mathbf{a})} = 0 \quad \text{for} \quad i > n(\mathbf{a}).$$

We observe that $n(\mathbf{a}) = n(\mathbf{r}^{(\mathbf{a})})$ and that the finite subsequences

$$\{a_i\}_{i=0}^{n(\mathbf{a})} \quad \text{and} \quad \{r_i^{(\mathbf{a})}\}_{i=0}^{n(\mathbf{r}^{(\mathbf{a})})}$$

are mutually centrally symmetric. We write $\mathbf{a} \sim \mathbf{b}$ for $\mathbf{a}, \mathbf{b} \in \mathscr{A}$ if $\mathbf{b} = \mathbf{a}$ or if $\mathbf{b} = \mathbf{r}^{(\mathbf{a})}$ The relation $\sim$ is obviously an equivalence on each of the sets $\mathscr{A}$, $\mathscr{A}_0$, $\mathscr{A}_1$, …. Note that the set $\mathscr{B}_{\mathbf{a}}$ of all elements of $\mathscr{A}$ which are $\sim$-equivalent to $\mathbf{a}$ has either one or two elements for each $\mathbf{a} \in \mathscr{A}$. Denote by $\tilde{\mathscr{A}}$ ($\tilde{\mathscr{A}}_0$, $\tilde{\mathscr{A}}_1$, …) the factor-set $\mathscr{A}/_{\sim}$ ($\mathscr{A}_{0/\sim}$, $\mathscr{A}_{1/\sim}$, …), i.e. the set of $\sim$-equivalence classes of $\mathscr{A}$ ($\mathscr{A}_0$, $\mathscr{A}_1$, …). In the sequel, we shall treat any class from $\tilde{\mathscr{A}}$ as replaced by one of its elements, i.e. as a sequence from $\mathscr{A}$. Note that the mapping $\mathbf{a} \to \delta^{(\mathbf{a})}$ is $\sim$-invariant.

The aim of this paper is to characterize those pairs of sequences $\mathbf{a}, \mathbf{b} \in \tilde{\mathscr{A}}$ which satisfy

$$\delta(\mathbf{a}) = \delta(\mathbf{b}) \quad \text{and} \quad \mathbf{a} \neq \mathbf{b}. \tag{4}$$

The restriction to the factor-sets together with the assumption that $a_0 = 1$ for $\mathbf{a} \in \mathscr{A}$ makes it possible to formulate the assertions without the usual appendix "up to translation and central reflection".

H. Rost found an example (see [4]) of a pair $\mathbf{a}, \mathbf{b} \in \tilde{\mathscr{A}}_{15}$ such that (4) holds, i.e. the distance sequence $\delta^{(\mathbf{a})}$ does not determine in general the "parent" sequence $\mathbf{a}$ uniquely. We shall show how to construct all pairs $\mathbf{a}, \mathbf{b} \in \tilde{\mathscr{A}}$ satisfying (4) in Section 3. By (2) and (3), we find that for such a pair $n(\mathbf{a}) = n(\mathbf{b})$ is true, i.e. there exists an $n \in \mathbb{N}_0$ such that $\mathbf{a}, \mathbf{b} \in \tilde{\mathscr{A}}_n$. This n plays an important role in our investigation. Section 4 provides, for each $n \in \mathbb{N}_0$, some estimates of cardinality of the sets

$$\{\{\mathbf{a}, \mathbf{b}\}; \mathbf{a}, \mathbf{b} \in \tilde{\mathscr{A}}_n,\ \mathbf{a} \neq \mathbf{b}, \delta^{(\mathbf{a})} = \delta^{(\mathbf{b})}\}$$

and

$$\{\mathbf{b}; \mathbf{b} \in \tilde{\mathscr{A}}_n, \delta^{(\mathbf{a})} = \delta^{(\mathbf{b})}\} \quad \text{for} \quad \mathbf{a} \in \tilde{\mathscr{A}}_n,$$

respectively. Section 5 is devoted to the structure of those $\mathbf{a}, \mathbf{b} \in \tilde{\mathscr{A}}$ which satisfy (4).

2. Two equivalent formulations of the problem in question

Let $\mathbf{a} \in \mathscr{A}$, put

$$A = \{i; i \in \mathbb{N}_0, a_i = 1\}$$

and consider two independent random variables X and Y having the uniform distribution concentrated on the set A. Then the distribution of $Z = X - Y$ is by (1)

$$(5) \qquad P(Z = j) = \sum_{i \in A} P(X = i + j) P(Y = i) = \sum_{i=0}^{\infty} a_i a_{i+j} \left(\frac{1}{\delta_0^{(\mathbf{a})}}\right)^2 = \delta_j^{(\mathbf{a})} \left(\frac{1}{\delta_0^{(\mathbf{a})}}\right)^2$$

for any $j \in \mathbb{N}_0$ and

$$P(Z = j) = P(Z = -j) \quad \text{for any integer } j.$$

R. Pyke posed in [3] the following question:

Let X and Y be independently distributed uniform random variables over the same closed subset of the real line. Given the distribution of $Z = X - Y$, can one determine B (up to translation and reflection)?

We observe that if only the sets $B \subset \mathbb{N}_0$ were considered (as a matter of fact, the Rost's example in [4] has this property) then it would require by (5) to decide whether the distance sequence $\delta^{(\mathbf{a})}$ determines $\mathbf{a} \in \tilde{\mathscr{A}}$ uniquely or not.

Another situation in which this problem appeared concerns stochastic geometry. A compact set $K \subset \mathbb{R}^d$ is characterized in [1] by the volumes of its dilations by compact sets C, i.e. the values of

$$\Phi_C(K) = \mu(K \oplus \check{C})$$

are considered, where μ is a translation invariant measure on $\mathbf{R}^d$ with $\mu(K) < \infty$ (usually the Lebesgue measure or the counting measure), $\oplus$ denotes the Minkowski addition of sets and

$$\check{C} = \{-x\,;\, x \in C\}.$$

Assume that $\Phi_C(K)$ is known for each set $C \subset \mathbf{R}^d$ containing at most two elements. Thus, the function

$$\Psi_1^K(y) = \sum_{C \subset \{0,y\}} (-1)^{\operatorname{card} C+1} \Phi_C(K) \quad \text{for } y \in \mathbf{R}^d$$

is known as well and, moreover,

$$\Psi_1^K(y) = \mu\big(K \cap (K \oplus \{-y\})\big).$$

Note that the function Ψ_1^K is called the *set covariance* of K and is widely used in mathematical morphology and automatic image analysis—see [5]. It is proved in [2] that the values $\{\Psi_1^K(y);\, y \in \mathbf{R}^2\}$ determine a planar convex polygon up to translation and central reflection. On the other hand, the paper [1] shows that their knowledge is not sufficient to determine each compact subset of $\mathbf{R}^d$ (up to translation, central reflection, and symmetric difference of μ-measure zero) even for $d = 1$. In [1], μ is the Lebesgue measure on $\mathbf{R}^1$ but it can be easily seen that the essence of the example given there is to consider the above mentioned problem for sets $K \subset \mathbf{N}_0$ with μ being the counting measure on $\mathbf{R}^1$. To observe the connection with the 0–1 sequences discussed in the introduction, let μ be the counting measure on $\mathbf{R}^1$, let $\mathbf{a} \in \tilde{\mathscr{A}}$ and let

$$K = \{i\,;\, i \in \mathbf{N}_0,\ a_i = 1\}.$$

Then

$$\Psi_1^K(j) = \delta_j^{(\mathbf{a})} \quad \text{for } j \in \mathbf{N}_0.$$

Note that the two examples given in [1] and [4] are not identical. Moreover, the corresponding pairs of sets (or, equivalently, sequences) are elements of different sets $\tilde{\mathscr{A}}_n$ because $n = 11$ in the former case and $n = 15$ in the latter one. Their structure is, however, analogous—cf. Section 5.

3. The polynomial approach

This section shows how to find all pairs $\mathbf{a}, \mathbf{b} \in \tilde{\mathscr{A}}$ satisfying (4). Let $n \in \mathbb{N}_0$ and let $\mathbf{a} \in \mathscr{A}_n$. The sequence $\mathbf{a}$ determines a polynomial

$$p^{(\mathbf{a})}(x) = \sum_{i=0}^{\infty} a_i x^i = \sum_{i=0}^{n} a_i x^i.$$

For each polynomial h of a degree $k \in \mathbb{N}_0$ we put

$$\hat{h}(x) = x^k \cdot h(x^{-1}).$$

The values of $\delta_j^{(\mathbf{a})}$ appear in the product

$$q^{(\mathbf{a})}(x) = p^{(\mathbf{a})}(x) \cdot \hat{p}^{(\mathbf{a})}(x) = \sum_{j=-n}^{n} \delta_{|j|}^{(\mathbf{a})} x^{j+n}.$$

Thus, the relation (4) is equivalent to

$$q^{(\mathbf{a})} = q^{(\mathbf{b})}, \tag{6}$$

$$p^{(\mathbf{a})} \neq p^{(\mathbf{b})} \tag{7}$$

and

$$p^{(\mathbf{a})} \neq \hat{p}^{(\mathbf{b})}. \tag{8}$$

The polynomial $p^{(\mathbf{a})}$ can be written as a product

$$p^{(\mathbf{a})} = s \cdot u$$

of two polynomials. Further, put

$$p^{(\mathbf{b})} = s \cdot \hat{u},$$

so that

$$\hat{p}^{(\mathbf{a})} = \hat{s} \cdot \hat{u}$$

and

$$\hat{p}^{(\mathbf{b})} = \hat{s} \cdot u.$$

Thus, the relation (6) is fulfilled and the conditions (7) and (8) are equivalent to

$$u \neq \hat{u} \tag{9}$$

and

$$s \neq \hat{s}. \tag{10}$$

Conversely, if two polynomials s and u are taken in such a way that (9) and (10) hold and that the products $s \cdot u$ and $s \cdot \hat{u}$ are polynomials all coefficients of which belong to $\{0, 1\}$ then we get $\mathbf{a}, \mathbf{b} \in \tilde{\mathscr{A}}$ satisfying (4) by puting $p^{(\mathbf{a})} = s \cdot u$ and $p^{(\mathbf{b})} = s \cdot \hat{u}$.

Since each of the polynomials $p^{(\mathbf{a})}$, $\hat{p}^{(\mathbf{a})}$, $p^{(\mathbf{b})}$ and $\hat{p}^{(\mathbf{b})}$, for any $\mathbf{a}, \mathbf{b} \in \mathscr{A}$, contains obviously the absolute term 1, the polynomials s and u have the same property as well. Many pairs $\mathbf{a}, \mathbf{b} \in \tilde{\mathscr{A}}$ satisfying (4) can be obtained by using polynomials s and u such that all their coefficients belong to $\{0, 1\}$. The two examples presented in [1] and [4]—see also Tables 1 and 2 below—use the polynomials

$$s(x) = 1 + x + x^4,$$
$$u(x) = 1 + x^2 + x^7$$

and

$$s(x) = 1 + x^4 + x^{12},$$
$$u(x) = 1 + x + x^3,$$

respectively.

i	0	1	2	3	4	5	6	7	8	9	10	11	$\geqslant 12$
a_i	1	1	1	1	1	0	1	1	1	0	0	1	0
b_i	1	1	0	0	1	1	1	1	1	1	0	1	0
$\delta_i^{(\mathbf{a})} = \delta_i^{(\mathbf{b})}$	9	6	5	5	5	4	3	3	2	1	1	1	0

Table 1

i	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	$\geqslant 16$
a_i	1	1	0	1	1	1	0	1	0	0	0	0	1	1	0	1	0
b_i	1	0	1	1	1	0	1	1	0	0	0	0	1	0	1	1	0
$\delta_i^{(\mathbf{a})} = \delta_i^{(\mathbf{b})}$	9	4	4	4	3	2	2	2	3	2	2	2	3	1	1	1	0

Table 2

4. The quantitative results

All pairs $\mathbf{a}, \mathbf{b} \in \tilde{\mathscr{A}}_n$ were investigated for $n = 1, 2, \ldots, 14, 15$ with the use of PC Olivetti M 28. There are no pairs $\mathbf{a}, \mathbf{b} \in \tilde{\mathscr{A}}_n$ satisfying (4) for $n \leqslant 10$. Table 3 contains the list of the pairs $\mathbf{a}, \mathbf{b} \in \tilde{\mathscr{A}}_n$ satisfying (4), the corresponding sequences

n	$\{a_i\}_{i=0}^n$ $\{b_i\}_{i=0}^n$	$\{\delta_i^{(\mathbf{a})}\}_{i=0}^n = \{\delta_i^{(\mathbf{b})}\}_{i=0}^n$	$s(x)$	$u(x)$
11	11100 01010 01 11000 01101 01	62211 22111 11	$1 + x + x^2 + x^3 + x^4 + x^6$	$1 - x^3 + x^5$
	11111 01110 01 11001 11111 01	96555 43321 11	$1 + x + x^4$	$1 + x^2 + x^7$
12	11010 01110 001 11000 11101 001	73222 33211 011	$1 - x^2 + x^3$	$1 + x + x^2 + x^3 + x^7 + x^8 + x^9$
	11100 10101 001 11000 10110 101	72322 31311 111	$1 + x + x^2 + x^3 + x^4 + x^5 + x^7$	$1 - x^3 + x^5$
	11110 11101 001 11011 11100 101	95554 44312 111	$1 + x + x^7$	$1 + x^2 + x^5$
	11111 01011 001 11001 01111 101	95554 34232 111	$1 + x + x^4$	$1 + x^2 + x^8$
	11011 10101 101 11010 11011 101	94554 43323 111	$1 + x + x^3$	$1 + x^5 + x^9$
	11110 10110 011 11101 11001 011	95444 43322 221	$1 - x^3 + x^5$	$1 + x + x^2 + x^3 + 2x^4 + x^5 + x^6 + x^7$

Table 3

$\delta^{(\mathbf{a})} = \delta^{(\mathbf{b})}$ and the polynomials s and u for $n = 11$ and $n = 12$. The number of such pairs for $n = 11, 12, 13, 14, 15$ is given in Table 4. (There is no group of more than two elements of $\tilde{\mathscr{A}}_n$ with the same distance sequence for those n's—cf. Proposition 2). Since the cardinality of the set $\tilde{\mathscr{A}}_n$ grows exponentially in n there is a litle chance to get such complete information for larger n. A lower bound for the number of pairs $\mathbf{a}, \mathbf{b} \in \tilde{\mathscr{A}}_n$ satisfying (4) can be obtained for any $n \geqslant 11$ as follows: Put

$$s(x) = 1 + x + x^m,$$
$$u(x) = 1 + x^2 + x^{n-m}$$

for any $m \in \mathbb{N}$ such that $4 \leqslant m \leqslant \frac{n-3}{2}$ if $n \geqslant 11$ and

$$s(x) = 1 + x^2 + x^m,$$
$$u(x) = 1 + x + x^{n-m}$$

for any $m \in \mathbb{N}$ such that $5 \leqslant m \leqslant \frac{n-2}{2}$ if $n \geqslant 12$. Since these polynomials s and u generate in total $[\frac{n-9}{2}] + [\frac{n-10}{2}] = n - 10$ disjoint pairs $\{\mathbf{a}, \mathbf{b}\} \subseteq \tilde{\mathscr{A}}_n$ with $\mathbf{a}$, $\mathbf{b}$ satisfying (4), where $[y]$ means the integer part of y, we have the following estimate.

Proposition 1. *There exist at least $n - 10$ disjoint pairs $\{\mathbf{a}, \mathbf{b}\} \subseteq \tilde{\mathscr{A}}_n$ with $\mathbf{a}$, $\mathbf{b}$ satisfying (4) for any $n \in \mathbb{N}$, $n \geqslant 11$.*

We know that the distance sequence $\delta^{(\mathbf{a})}$ defined in the introduction does not determine in general the "parent" sequence $\mathbf{a} \in \tilde{\mathscr{A}}$ uniquely. All the examples given in Table 3 have a common feature that just two elements of $\tilde{\mathscr{A}}$ correspond to the same distance sequence. It seams to be useful to demonstrate that more than two elements of $\tilde{\mathscr{A}}$ can have the same distance sequence. Put

$$s(x) = 1 + x + x^3,$$
$$u(x) = 1 + x^4 + x^9,$$
$$w(x) = 1 + x^{13} + x^{27}$$

and

$$p^{(\mathbf{a})} = s \cdot u \cdot w,$$
$$p^{(\mathbf{b})} = s \cdot u \cdot \hat{w},$$
$$p^{(\mathbf{c})} = s \cdot \hat{u} \cdot w,$$
$$p^{(\mathbf{d})} = s \cdot \hat{u} \cdot \hat{w}.$$

The reader can easily verify that all the coefficients of the polynomials $p^{(\mathbf{a})}$, $p^{(\mathbf{b})}$, $p^{(\mathbf{c})}$ and $p^{(\mathbf{d})}$ belong to $\{0, 1\}$, that $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$ are different elements of $\tilde{\mathscr{A}}_{39}$ and that

$$\delta^{(\mathbf{a})} = \delta^{(\mathbf{b})} = \delta^{(\mathbf{c})} = \delta^{(\mathbf{d})}.$$

Proposition 2. *Let $n \in \mathbb{N}$ and $z \in \mathbb{N}$ be such that*

$$n \geqslant \sum_{i=1}^{z+1} 3^i.$$

Then there exist at least 2^z different elements of $\tilde{\mathscr{A}}_n$ having the same distance sequence.

Proof. In a similar way as above, we put

$$m = n - \sum_{i=1}^{z} 3^i$$
$$\alpha_i = \frac{1}{2}(3^i - 1) \quad \text{for} \quad i \in \mathbb{N},$$
$$\beta_i = 3^i \quad \text{for} \quad i \in \mathbb{N}$$

and form the polynomials

$$(1+x+x^3)\prod_{i=2}^{z}(1+x^{\alpha_i}+x^{\beta_i})^{y_i}(1+x^{\alpha_i+1}+x^{\beta_i})^{1-y_i}$$
$$\times(1+x^{\alpha_z+1}+x^m)^{y_{z+1}}(1+x^{m-\alpha_z+1}+x^m)^{1-y_{z+1}}$$

for all $(y_2,\dots,y_{z+1})\in\{0,1\}^z$. These polynomials generate 2^z different elements of $\tilde{\mathscr{A}}_n$ having the required properties. □

n	11	12	13	14	15
number of unordered pairs $\mathbf{a},\mathbf{b}\in\tilde{\mathscr{A}}_n$ which satisfy (4)	2	6	12	16	37

Table 4

5. The structure of the pairs $\mathbf{a},\mathbf{b}\in\tilde{\mathscr{A}}$ satisfying (4)

To any finite subset $A\subseteq\mathbf{R}$ we attach the uniform probability measure P_A over A with respect to the counting measure. We shall say that such a *subset A has a property $\mathscr{P}$* if there exist two non-empty finite subsets $S,U\subseteq\mathbf{R}$ such that

(11) $$S,\ U\neq\{0\}$$

and

(12) $$P_A=P_S*P_U,$$

where $*$ denotes the operation of convolution. Further, we shall say that a *sequence* $\mathbf{a}\in\mathscr{A}$ *has the property $\mathscr{P}$* if its *support set*

(13) $$A=\{i;\ a_i=1\}$$

has the property $\mathscr{P}$. It can be easily seen that the property $\mathscr{P}$ is $\sim$-invariant so that we can deal with the elements of the factor-set $\tilde{\mathscr{A}}$.

R. Pyke posed in [4] a question which can be re-formulated as follows: Consider a pair $\mathbf{a},\mathbf{b}\in\tilde{\mathscr{A}}$ satisfying (4). Have $\mathbf{a}$ and $\mathbf{b}$ necessarily the property $\mathscr{P}$? The answer is negative, as can be found considering the first example given in Table 3 for $n=11$. In this case, the support sets—cf. (13)—corresponding to those $\mathbf{a},\mathbf{b}\in\mathscr{A}_{11}$ are (up to central reflection)

(14) $$A=\{0,1,2,6,8,11\}$$

and

(15) $$B = \{0, 1, 6, 7, 9, 11\}.$$

Suppose that the set A given by (14) has the property $\mathscr{P}$ and let S, U be the finite nonempty subsets of $\mathbf{R}$ such that (11) and (12) hold. Since $\min S + \min U = \min A = 0$ and

(16) $$\max S + \max U = \max A = 11$$

we can assume without loss of generality that $\min S = \min U = 0$ and that $\max S \leqslant \max U$, i.e.

(17) $$\max U \geqslant 6.$$

This implies $S \cup U \subset A$. Since all the three measures in question are uniform, $S \cap U = \{0\}$ holds. Finally, there are three possibilities by (17):

1) if $\max U = 6$ then $\max S = 5$ by (16) but $5 \notin A$;

2) if $\max U = 8$ then $\max S = 3$ by (16) but $3 \notin A$;

3) if $\max U = 11$ then $\max S = 0$ by (16), i.e. $S = \{0\}$, which contradicts to (11).

Thus, the set A given by (14) cannot have the property $\mathscr{P}$. Similarly, the same result is obtained for the set B given by (15). We conclude that none of the elements $\mathbf{a}$, $\mathbf{b}$ of the set $\tilde{\mathscr{A}}_{11}$ given in the first row of Table 3 has the property $\mathscr{P}$.

The example just discussed concerned the particular case of $n = 11$. General n's are considered in

Proposition 3. *For any integer $n \geqslant 11$, there exists a pair $\mathbf{a}, \mathbf{b} \in \tilde{\mathscr{A}}_n$ satisfying* (4) *and such that neither* $\mathbf{a}$ *nor* $\mathbf{b}$ *has the property* $\mathscr{P}$.

Proof. It remains to deal with $n \geqslant 12$ only. Use the polynomials

$$s(x) = 1 + x + x^2 + x^3 + x^{n-5} + x^{n-4} + x^{n-3}$$

and

$$u(x) = 1 - x^2 + x^3.$$

The sequences $\mathbf{a}, \mathbf{b} \in \mathscr{A}_n$ corresponding to $p^{(\mathbf{a})} = s \cdot u$ and $p^{(\mathbf{b})} = s \cdot \hat{u}$ have the support sets

(18) $$A = \{i;\, a_i = 1\} = \{0, 1, 3, 6, n-5, n-4, n\}$$

and

(19) $$B = \{i;\, b_i = 1\} = \{0, 3, 5, 6, n-5, n-1, n\}.$$

Following the ideas applied in the example above, we find that the sets A, B given by (18) and (19) cannot have the property $\mathscr{P}$. (In the cases of

$$\max S = \max U = 6 \quad \text{if} \quad n = 12,$$

or

$$\max U = n - 5 \quad \text{and} \quad \max S = 5$$

if the set B is considered, the second greatest elements of S, U should be discussed to find the contradiction to (12).) □

6. Open problems

1) In spite of the fact that the number of elements of $\mathscr{A}$ having the same distance sequence is not bounded—cf. Proposition 2, no example that this number equals 3 is known to the authors of this paper. This problem seems to be associated with the question whether there exist $\mathbf{a}, \mathbf{b} \in \tilde{\mathscr{A}}$ satisfying (4) such that $\operatorname{card} \mathscr{B}_{\mathbf{a}} = 1$ and $\operatorname{card} \mathscr{B}_{\mathbf{b}} = 2$ (the set $\mathscr{B}_{\mathbf{a}}$ has been introduced in Section 1). In the words of polynomials, these conditions can be expressed by $p^{(\mathbf{a})} = \hat{p}^{(\mathbf{a})}$ and $p^{(\mathbf{b})} \neq \hat{p}^{(\mathbf{b})}$.

2) A question whether there exist 5 different elements of $\tilde{\mathscr{A}}$ having the same distance sequence seems to be much harder than the problem 1.

3) Proposition 2 provides an upper bound for the minimum of those $n \in \mathbb{N}$ that there exist at least $2^z = 2, 4, 8, \dots$ different elements of $\tilde{\mathscr{A}}_n$ having the same distance sequence. What is the minimal n with this property for each $z \in \mathbb{N}$? Note that the upper bound is equal to 12 and 39 for $z = 1$ and $z = 2$, respectively, but it is possible to take $n = 11$ for $z = 1$ and $n = 35$ for $z = 2$—see Tables 3 and 5, respectively.

$\{a_i\}_{i=0}^{35}$	11010 01110 11110 01110 00111 01001 110001
$\{b_i\}_{i=0}^{35}$	10001 11001 11101 11001 01110 00111 001011
$\{c_i\}_{i=0}^{35}$	10010 11100 11110 11100 01110 01011 100011
$\{d_i\}_{i=0}^{35}$	11000 11101 11100 11101 00111 00011 101001
$\{\delta_i^{(\mathbf{e})}\}_{i=0}^{35}$ for $\mathbf{e} = \mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$	21,12,9,8,10,13,13,11,8,8,9,9,9,9,6,6,7,7,7,6,4,4, 5,7,4,3,2,2,3,3,2,1,1,0,1,1

Table 5

4) The computer search study shows that the number $\delta_0^{(\mathbf{a})}$ of 1's in the sequences $\mathbf{a} \in \tilde{\mathscr{A}}_n$ for which there exists $\mathbf{b} \in \tilde{\mathscr{A}}_n$ such that the pair $\mathbf{a}$, $\mathbf{b}$ satisfies (4) is as

follows:

$$\delta_0^{(a)} \in \{6,9\} \quad \text{for} \quad n = 11,$$
$$\delta_0^{(a)} \in \{7,9\} \quad \text{for} \quad n = 12,$$
$$\delta_0^{(a)} \in \{6,7,8,9\} \quad \text{for} \quad n = 13,$$
$$\delta_0^{(a)} \in \{6,7,8,9,10\} \quad \text{for} \quad n = 14$$

and

$$\delta^{(a)} \in \{7,8,9,10,11,12\} \quad \text{for} \quad n = 15.$$

Is it possible to state that the distance sequence determines the "parent" element of $\tilde{\mathscr{A}}$ uniquely if $\delta_0^{(a)}$ is small enough or large enough (compared to n)? And if it is so what are the limits for a given $n \in \mathbb{N}$?

5) When looking for the polynomials s and u (cf. Section 3) such that the pair $\mathbf{a}, \mathbf{b} \in \tilde{\mathscr{A}}$ corresponding to $p^{(a)} = s \cdot u$ and $p^{(b)} = s \cdot \hat{u}$ satisfies (4), the following basic problem appears to be of interest: For which polynomials s exists there such a polynomial u (both s and u having arbitrary coeficients) that all the coefficients of the product polynomial $s \cdot u$ are from $\{0,1\}$?

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