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# ON PROJECTIVE INTERVALS IN A MODULAR LATTICE

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*Summary.* In this paper a combinatorial result concerning pairs of projective intervals of a modular lattice will be established.

*Keywords*: modular lattice, projective intervals, transposed intervals

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## 1. PRELIMINARIES

The recent papers dealing with combinatorial questions concerning partially ordered sets are rather frequent (cf., e.g., [2], [3], [4]).

Let $L$ be a modular lattice. We denote by $\mathscr{D}$ the collection of all systems $D = (a_1, a_2, a_3, u, v)$ of distinct elements of $L$ such that

$$u = a_1 \wedge a_2 = a_1 \wedge a_3 = a_2 \wedge a_3, \qquad v = a_1 \vee a_2 = a_1 \vee a_3 = a_2 \vee a_3.$$

An interval $[a_1, a_2]$ of $L$ will be said to be an $m$-interval if there is $D \in \mathscr{D}$ such that (under the above notation), $[a_1, a_2]$ is projective to $[u, a_1]$.

Let $\alpha = [b_1, b_2]$ and $\beta = [c_1, c_2]$ be distinct projective intervals of $L$. Assume that $\alpha$ is nontrivial (i.e. $b_1 \neq b_2$); then $\beta$ is nontrivial as well.

There exists a least positive integer $n$ such that for some $\alpha_0, \alpha_1, \ldots, \alpha_n$ in $L$ the following conditions are satisfied:

(i) $\alpha_0 = \alpha$ and $\alpha_n = \beta$;

(ii) for each $i \in \{1, 2, \ldots, n\}$, the interval $\alpha_i$ is transposed to the interval $\alpha_{i-1}$. We denote $\mu(\alpha, \beta) = n$.

Let $S(\alpha)$ be the collection of all systems $(y_0, y_1, y_2, \ldots, y_m)$ with $b_1 = y_0 < y_1 < y_2 < \ldots < y_m = b_2$. The collection $S(\beta)$ is defined analogously. For each $i \in \{1, 2, \ldots, m\}$ let $k(i)$ be a positive integer.

A system of distinct intervals

$$(1) \qquad (\beta_{ij})(i = 1,2,\ldots,m;\ j = 1,2,\ldots,k(i))$$

will be said to be a $p$-system for the intervals $\alpha$ and $\beta$ if the following conditions are satisfied:

(i) there are $Y = (y_0, y_1, \ldots, y_n) \in S(\alpha)$ and $Z = (z_1, z_2, \ldots, z_n) \in S(\beta)$ such that for each $i \in \{1,2,\ldots,m\}$ we have $\beta_{i1} = [y_{i-1}, y_1]$ and $\beta_{i,k_i} = [z_{i-1}, z_i]$;

(ii) for each $i \in \{1,2,\ldots,m\}$ and each $j \in \{1,2,\ldots,k(i)\}$ the interval $\beta_{i,j-1}$ is transposed to $\beta_{i,j}$. The collection of all $p$-systems for $\alpha$ and $\beta$ will be denoted by $p(\alpha,\beta)$. For $A \in P(\alpha,\beta)$ (where $A$ is as in (1)) let $A_0$ be the of all $\beta_{ij} \in A$ such that $\beta_{ij}$ fails to be an $m$-interval. We put

$$\nu(A) = \text{card } A_0,$$
$$\nu_0(\alpha,\beta) = \min\{\nu(A) : A \in P(\alpha,\beta)\}.$$

In this note it will be proved that we always have

$$(2) \qquad \nu_0(\alpha,\beta) \leqslant 3$$

and this estimate cannot be sharpened in general.

The estimate (2) is a consequence of the following result:

(A) *Let $\alpha = [b_1, b_2]$ and $\beta = [c_1, c_2]$ be nontrivial intervals of a modular lattice $L$. Assume that $\alpha$ is projective to $\beta$. Then there exist elements $x_0, x_1, \ldots, x_m, y_0, y_1, \ldots, y_m$ in $L$ such that the following conditions are satisfied:*

(i) $b_1 = x_0 < x_1 < \ldots < x_m = b_2$, $c_1 = y_0 < y_1 < \ldots < y_m = c_2$ *and for each* $i \in \{1,2,\ldots,m\}$ *the interval* $[x_{i-1}, x_i]$ *is projective to* $[y_{i-1}, y_i]$;

(ii) *there is* $i(1) \in \{1,2,\ldots,m\}$ *such that* $[x_{i-1}, x_i]$ *is an m-interval for each* $i \in \{1,2,\ldots,m\} \setminus \{i(1)\}$, *and either* $[x_{i(1)-1}, x_{i(1)}]$ *is an m-interval, or there is an interval* $[t_1, t_2] \subseteq L$ *such that* $[x_{i(1)-1}, x_{i(1)}]$ *is transposed to* $[t_1, t_2]$ *and* $[t_1, t_2]$ *is transposed to* $[y_{i(1)-1}, y_{i(1)}]$.

## THE PROOF OF (A)

We will apply the notation from Section 1. Again, let $\alpha$ and $\beta$ be distinct nontrivial intervals of a modular lattice $L$. Assume that $\alpha$ and $\beta$ are projective. A $p$-system $A$ for $\alpha$ and $\beta$ will be said to be reduced if (under the notation as above), whenever $i \in \{1,2,\ldots,m\}$ and $j \in \{1,2,\ldots,k(i)-1\}$, then $\beta_{i,j-1}$ fails to be transposed to $\beta_{i,j+1}$.

The following lemma is easy to verify.

**2.1. Lemma.** *Let $A \in P(\alpha, \beta)$. Then there exists $A' \in P(\alpha, \beta)$ such that $A' \subseteq A$ and $A'$ is reduced.*

Let $[c_1, c_2]$ and $[d_1, d_2]$ be transposed intervals of $L$; then we have either

(i) $c_2 \wedge d_1 = c_1, \qquad c_2 \vee d_1 = d_2,$

or

(ii) $d_2 \wedge c_1 = d_1, \qquad d_2 \vee c_1 = c_2.$

If (i) is valid, then we write $[c_1, c_2] \nearrow [d_1, d_2]$; the validity of (ii) will be recorded by writing $[c_1, c_2] \searrow [d_1, d_2]$.

**2.2. Lemma.** *Let $A \in P(\alpha, \beta)$ and assume that $A$ is reduced. Let $A$ be as in (1). If $i \in \{1, 2, \ldots, m\}$, $j \in \{1, 2, \ldots, k(i) - 1\}$, $\alpha_{i,j-1} \nearrow \alpha_{i,j}$, then $\alpha_{i,j} \searrow \alpha_{i,j+1}$ (and dually).*

The proof is trivial.

Let $A \in P(\alpha, \beta)$ be as in (1). Let $i \in \{1, 2, \ldots, m\}$, $z_{i1} \in L$, $x_{i-1,1} < z_{i1} < x_{i1}$. We define elements $z_{i2}$, $z_{i3}$, ..., $z_{i,k(i)}$ by induction as follows: if $z_{i,j-1}$ ($j \in \{2, \ldots, k(i)-1\}$) is already defined and if $\alpha_{i,j-1} \nearrow \alpha_{i,j}$, then we put $z_{ij} = z_{i,j-1} \vee d_1$, where $d_1$ is the least element of $\alpha_{i,j}$; on the other hand, if $\alpha_{i,j-1} \searrow \alpha_{i,j}$, then we set $z_{ij} = z_{i,j-1} \wedge d_2$, where $d_2$ is the largest element of $\alpha_{ij}$.

Consider the system $A'$ which we obtain from the system $A$ if the $i$-th row $(\alpha_{i,1}, \alpha_{i,2}, \ldots, \alpha_{i,i(k)})$ of $A$ is replaced by the rows

$$\alpha'_{i,1},\ \alpha'_{i,2},\ \ldots,\ \alpha'_{i,i(k)}$$
$$\alpha''_{i,1},\ \alpha''_{i,2},\ \ldots,\ \alpha''_{i,n},$$

where

$$\alpha'_{i,j} = \{t \in \alpha_{ij} : t \leqslant z_{i,j}\}, \qquad \alpha''_{i,j} = \{t \in \alpha_{ij} : t \geqslant z_{i,j}\}.$$

Then we obviously have:

**2.3. Lemma.** *$A'$ is a $p$-system for the intervals $\alpha$ and $\beta$.*

The system $A'$ will be said to be generated by the system $A$ and by the element $z_{i1}$.

Let $y, z \in L$, $b_1 < y < b_2$, $c_1 < z < c_2$. Suppose that $[b_1, y]$ is projective to $[c_1, z]$ and that $[y, b_2]$ is projective to $[z, c_2]$.

**2.4. Lemma.** *Let $A \in p([b_1, y], [c_1, z])$. (We apply the same notation as in (1) with the distinction that we now have $y$ and $z$ instead of $b_2$ and $c_2$.) Let $\beta_{m+1,i}$ ($i = 1, 2, \ldots, k(m+1)$) be intervals of $L$ such that $\beta_{m+1,1} = [y, b_2]$, $\beta_{m+1,k(m+1)} = [z, c_2]$*

*and for each $i \in \{2, 3, \ldots, k(m+1)\}$ the interval $\beta_{m+1,i-1}$ is transposed to $\beta_{m+1,i}$. Let $A'$ be the system*

$$(\beta_{ij}(i = 1, 2, \ldots, m+1;\ j = 1, 2, \ldots, k(i)).$$

*Then $A' \in p(\alpha, \beta)$.*

P r o o f. This is an immediate consequence of the definition of $p(\alpha, \beta)$.
The assertion dual to 2.4. is also valid. □

**2.5. Lemma.** *Let $x$, $y$ and $z$ be elements of a modular lattice $L$. Assume that the relations*

$$[x \wedge y, x] \nearrow [y, x \vee y] \quad \text{and} \quad [y, x \vee y] \searrow [y \wedge z, z]$$

*are valid. Then the sublattice $L_1$ of $L$ generated by the elements $x$, $y$ and $z$ is a homomorphic image of the lattice on Fig. 1.*

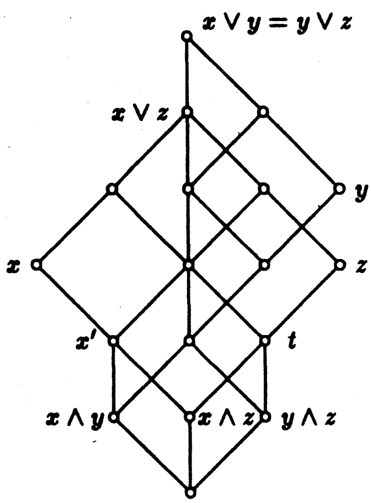


Fig. 1

P r o o f. If we consider the free modular lattice with three free generators (cf. e.g. [1], Chap. III, Theorem 8) $x$, $y$ and $z$, and if we take into account that in our case we have $x \vee y = y \vee z$, then we obtain the assertion of the lemma. □

**Theorem.** *Let $\alpha$ and $\beta$ be nontrivial distinct intervals of a modular lattice $L$. Assume that $\alpha$ is projective to $\beta$. Then there is $A \in P(\alpha, \beta)$ such that (under the notation as in (1) the following condition is satisfied: there is $i(1) \in \{1, 2, \ldots, m\}$*

*such that, whenever* $i \in \{1, 2, \ldots, m\} \setminus \{i(1)\}$ *and* $j \in \{1, 2, \ldots, k(i)\}$, *then* $\beta_{ij}$ *is an m-interval; next, either* $\beta_{i(1),1}$ *is an m-interval, or* $k(i(1)) \leqslant 3$.

Proof. Under the notation as in Section 1, let $\mu(\alpha, \beta) = n$. We have $n \geqslant 1$. If $n = 1$, then the assertion obviously holds (it suffices to consider the system $(\alpha_0, \alpha_1)$).

Suppose that $n \geqslant 2$ and let us apply induction with respect to $n$. First we consider the system

$$(\alpha_k) \qquad (k = 0, 1, 2, \ldots, n)$$

which obviously belongs to $p(\alpha, \beta)$. Without loss of generality we may assume that this system is reduced. Next, we can suppose that $\alpha_0 \nearrow \alpha_1 \searrow \alpha_2$ is valid (in the case $\alpha_0 \searrow \alpha_1 \nearrow \alpha_2$ we apply a dual procedure).

Let $x$, $y$ and $z$ be the greatest element of $\alpha_0$, the least element of $\alpha_1$ and the greatest element of $\alpha_2$, respectively. (Cf. Fig. 1.) Then

$$\alpha_0 = [x \wedge y, x], \quad \alpha_1 = [y, x \vee y], \quad \alpha_2 = [x \wedge z, z].$$

At the same time, $x \vee y = y \vee z$. Put $x' = (x \wedge y) \vee (x \wedge z)$. We have obviously

$$x \wedge y \leqslant x' \leqslant x.$$

From $x \wedge y < x$ we infer that either $x \wedge y < x'$ or $x' < x$.

Let us distinguish the following cases.

(a) Let $x \wedge y = x'$. Then $\alpha = \alpha_0 = [x', x]$. In view of Fig. 1, $\alpha_0$ is an $m$-interval; therefore $\alpha_1$, $\alpha_2$, …, $\alpha_n$ are $m$-intervals as well. Now it suffices to put $a = (\alpha_i)$ $(i = 0, 1, 2, \ldots, n)$.

(b) Let $x' = x$. Then $\alpha = \alpha_0 = [x \wedge y, x]$. Next, $\alpha_2 = [y \wedge z, t]$, where $t = (x \wedge z) \vee (y \wedge z)$. Denote $\alpha_1' = [x \wedge y \wedge z, x \wedge z]$. We have (cf. Fig. 1)

$$\alpha_0 \searrow \alpha_1' \nearrow \alpha_2.$$

Thus the system $A'$ consisting of the intervals

$$\alpha_0,\ \alpha_1',\ \alpha_2,\ \alpha_3,\ \ldots,\ \alpha_n$$

belongs to $P(\alpha, \beta)$. Since $\alpha_2 \nearrow d_3$, according to 2.2 the system $A'$ fails to be reduced. Thus in view of 2.1 there exists a system

$$\beta_0,\ \beta_1,\ \ldots,\ \beta_l$$

which belongs to $P(\alpha, \beta)$ such that $l < n$. Therefore by the induction hypothesis, the assertion of the theorem is valid for $\alpha$ and $\beta$.

(c) Let $x \wedge y < x' < x$. Let $A_1$ be the system

$$(\alpha_i) \qquad (i = 0, 1, 2, \ldots, n)$$

and let $A_2$ be the system generated by $A_1$ and the element $x'$. Then (under the notation as in Lemma 2.3) the system $A_2$ consists of intervals

$$\alpha'_0,\ \alpha'_1,\ \ldots,\ \alpha'_n,$$
$$\alpha''_o,\ \alpha''_1,\ \ldots,\ \alpha''_n,$$

where

$$\alpha'_0 = [x \wedge y, x'], \quad \alpha'_n = [y \wedge z, t],$$
$$\alpha''_0 = [x', x], \qquad \alpha''_n = [t, z].$$

Since $\alpha''_0$ is an $m$-interval, all $\alpha'_i$ ($i$ = 1, 2, ..., $n$) must be $m$-intervals. Next, by the same argument as in (b) we can verify that there exists a system $A_3$ consisting of intervals

$$\beta_0,\ \beta_1,\ \ldots,\ \beta_l$$

with $1 < n$ such that $A_3 \in p([x', x], [t, z])$. Hence by the induction hypothesis, the assertion of the theorem is valid for the intervals $[x', x]$ and $[t, z]$. Now it suffices to apply Lemma 2.3. □

Theorem (A) in Section 1 is obviously a consequence of (in fact, equivalent to) Theorem 2.6.

**2.7. Example.** Let $L$ be as in Fig. 1 Consider the intervals $\alpha = [x \wedge y, x']$ and $\beta = [y \wedge z, t]$. It is easy to verify that $\mu_0(\alpha, \beta) = 3$. Hence the estimate (2) cannot be sharpened in general.

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