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TOLERANCES ON POWERS OF A FINITE ALGEBRA

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Summary. It is shown that any power A^n, $n \geqslant 2$, of a finite k-element algebra A, $k \geqslant 2$, has factorable tolerances whenever the power A^{4k^2-3k} has the same property.

Keywords: Finite algebra, power, factorable tolerance

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In [3] R. Willard proved that congruences on any power A^n, $n \geqslant 2$, of a finite k-element algebra A, $k \geqslant 2$, are factorable whenever the power $A^{k^3+k^2-k}$ has the same property. The aim of this paper is to find an adequate exponent for factorability of tolerances on powers of a finite algebra.

Definition 1. Let $C_1, \ldots, C_n$, $n \geqslant 2$, be algebras of the same type. We say that the product $B = C_1 \times \ldots \times C_n$ has *factorable tolerances* if for any tolerance T on B we have $T = T_1 \times \ldots \times T_n$ where T_i is a tolerance on C_i, $i \leqslant n$.

Notation 1. Let $C_1, \ldots, C_n$, $n \geqslant 2$, be algebras of the same type, $B = C_1 \times \ldots \times C_n$. The elements of B are denoted by x, u, v, …, i.e. $x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$, $u = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$, $v = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$, …, where $x_i, u_i, v_i \in C_i$, $i \leqslant n$. Let I, J be disjoint index sets such that $I \cup J = \{1, \ldots, n\}$. If

$$x_i = \begin{cases} u_i & \text{for } i \in I \\ v_i & \text{for } i \in J \end{cases}$$

then x can be expressed in the form $x = \begin{bmatrix} u_I \\ v_J \end{bmatrix}$.

Notation 2. Let x, y, u, v be elements of an algebra B. The symbol $T_B(\langle x,y\rangle, \langle u,v\rangle)$ denotes the least tolerance on B containing the pairs $\langle x,y\rangle$, $\langle u,v\rangle \in B^2$.

Notation 3. Let $C_1, \ldots, C_n$, $n \leqslant 2$, be algebras of the same type, $B = C_1 \times \ldots \times C_n$. Denote

$$\varrho(B) = \{ \langle a,b,c,d,e,f\rangle \in B^6 ; \forall i \leqslant n \quad \text{either } \langle a_i,b_i\rangle = \langle c_i,d_i\rangle \text{ or } \langle a_i,b_i\rangle = \langle e_i,f_i\rangle \}$$

and, further,

$$\begin{aligned} \tau(B) = \{ \langle a,b,c,d,e,f\rangle \in B^6 ; \forall i \leqslant n \quad & \text{either } \langle a_i,b_i\rangle = \langle c_i,d_i\rangle , d_i = e_i = f_i \\ & \text{or } a_i = b_i = d_i = e_i = f_i \\ & \text{or } a_i = b_i = e_i = f_i ,\ c_i = d_i \\ & \text{or } \langle a_i,b_i\rangle = \langle e_i,f_i\rangle ,\ b_i = c_i = d_i \}. \end{aligned}$$

Lemma 1. *Let $C_1, \ldots, C_n$, $n \geqslant 2$, be algebras of the same type, $B = C_1 \times \ldots \times C_n$. The following conditions are equivalent:*

(1) *B has factorable tolerances;*

(2) *$\langle c,d\rangle$, $\langle e,f\rangle \in T$ implies $\left\langle \begin{bmatrix} c_I \\ e_J \end{bmatrix}, \begin{bmatrix} d_I \\ f_J \end{bmatrix} \right\rangle \in T$ for any elements c, d, e, $f \in B$, an tolerance T on B and any disjoint index sets I, J, $I \cup J = \{1,\ldots,n\}$;*

(3) *$\left\langle \begin{bmatrix} c_I \\ e_J \end{bmatrix}, \begin{bmatrix} d_I \\ f_J \end{bmatrix} \right\rangle \in T_B(\langle c,d\rangle, \langle e,f\rangle)$ holds for any elements c, d, e, $f \in B$ and any disjoint index sets I, J, $I \cup J = \{1,\ldots,n\}$;*

(4) *$\langle a,b,c,d,e,f\rangle \in \varrho(B)$ implies $\langle a,b\rangle \in T_B(\langle c,d\rangle, \langle e,f\rangle)$ for any elements $a,b,c,d,e,f \in B$;*

(5) *$\langle a,b,c,d,e,f\rangle \in \tau(B)$ implies $\langle a,b\rangle \in T_B(\langle c,d\rangle, \langle e,f\rangle)$ for any elements $a,b,c,d,e,f \in B$.*

Proof. (1)$\Rightarrow$ (2): Suppose that $\langle c,d\rangle$, $\langle e,f\rangle \in T$ for a tolerance T on B. By hypothesis $T = T_1 \times \ldots \times T_n$ for some tolerances T_i on C_i, $i \leqslant n$. Then $\langle c_i,d_i\rangle$, $\langle e_i,f_i\rangle \in T_i$, $i \leqslant n$, and so $\langle c_i,d_i\rangle \in T_i$, $i \in I$, $\langle e_i,f_i\rangle \in T_i$, $i \in J$, for any disjoint index sets I, J, $I \cup J = \{1,\ldots,n\}$. In other words, we have $\left\langle \begin{bmatrix} c_I \\ e_J \end{bmatrix}, \begin{bmatrix} d_I \\ f_J \end{bmatrix} \right\rangle \in T_1 \times \ldots \times T_n = T$.

(2) $\Rightarrow$ (3) is trivial.

(3) $\Rightarrow$ (4) follows from the definition of $\varrho(B)$.

(4) $\Rightarrow$ (5) is evident since $\tau(B) \subseteq \varrho(B)$.

(5) ⇒ (4): Let $\langle a,b,c,d,e,f\rangle \in \varrho(B)$. Then

$$\langle a,b,c,d,e,f\rangle = \left\langle \begin{bmatrix} c_I \\ e_J \end{bmatrix}, \begin{bmatrix} d_I \\ f_J \end{bmatrix}, \begin{bmatrix} c_I \\ c_J \end{bmatrix}, \begin{bmatrix} d_I \\ d_J \end{bmatrix}, \begin{bmatrix} e_I \\ e_J \end{bmatrix}, \begin{bmatrix} f_I \\ f_J \end{bmatrix} \right\rangle$$

for some disjoint index sets I, J, $I \cup J = \{1,\dots,n\}$. If $I = \emptyset$ or $J = \emptyset$ then the conclusion of (4) holds trivially. In the opposite case we proceed as follows:

(i)

$$\left\langle \begin{bmatrix} c_I \\ d_J \end{bmatrix}, \begin{bmatrix} d_I \\ d_J \end{bmatrix}, \begin{bmatrix} c_I \\ c_J \end{bmatrix}, \begin{bmatrix} d_I \\ d_J \end{bmatrix}, \begin{bmatrix} d_I \\ d_J \end{bmatrix}, \begin{bmatrix} d_I \\ d_J \end{bmatrix} \right\rangle \in \tau(B)$$

yields

$$\left\langle \begin{bmatrix} c_I \\ d_J \end{bmatrix}, \begin{bmatrix} d_I \\ d_J \end{bmatrix} \right\rangle \in T_B\left(\left\langle \begin{bmatrix} c_I \\ c_J \end{bmatrix}, \begin{bmatrix} d_I \\ d_J \end{bmatrix} \right\rangle, \left\langle \begin{bmatrix} d_I \\ d_J \end{bmatrix}, \begin{bmatrix} d_I \\ d_J \end{bmatrix} \right\rangle\right) =$$

$$= T_B\left(\left\langle \begin{bmatrix} c_I \\ d_J \end{bmatrix}, \begin{bmatrix} d_I \\ d_J \end{bmatrix} \right\rangle\right) \subseteq T_B(\langle c,d\rangle);$$

(ii) further, from

$$\left\langle \begin{bmatrix} c_I \\ f_J \end{bmatrix}, \begin{bmatrix} d_I \\ f_J \end{bmatrix}, \begin{bmatrix} c_I \\ d_J \end{bmatrix}, \begin{bmatrix} d_I \\ d_J \end{bmatrix}, \begin{bmatrix} d_I \\ f_J \end{bmatrix}, \begin{bmatrix} d_I \\ f_J \end{bmatrix} \right\rangle \in \tau(B)$$

we get

$$\left\langle \begin{bmatrix} c_I \\ f_J \end{bmatrix}, \begin{bmatrix} d_I \\ f_J \end{bmatrix} \right\rangle \in T_B\left(\left\langle \begin{bmatrix} c_I \\ d_J \end{bmatrix}, \begin{bmatrix} d_I \\ d_J \end{bmatrix} \right\rangle, \left\langle \begin{bmatrix} d_I \\ f_J \end{bmatrix}, \begin{bmatrix} d_I \\ f_J \end{bmatrix} \right\rangle\right) =$$

$$= T_B\left(\left\langle \begin{bmatrix} c_I \\ d_J \end{bmatrix}, \begin{bmatrix} d_I \\ d_J \end{bmatrix} \right\rangle\right) \subseteq T_B(\langle c,d\rangle),$$

by (i);

(iii)

$$\left\langle \begin{bmatrix} f_I \\ e_J \end{bmatrix}, \begin{bmatrix} f_I \\ f_J \end{bmatrix}, \begin{bmatrix} e_I \\ e_J \end{bmatrix}, \begin{bmatrix} f_I \\ f_J \end{bmatrix}, \begin{bmatrix} f_I \\ f_J \end{bmatrix}, \begin{bmatrix} f_I \\ f_J \end{bmatrix} \right\rangle \in \tau(B)$$

implies

$$\left\langle \begin{bmatrix} f_I \\ e_J \end{bmatrix}, \begin{bmatrix} f_I \\ f_J \end{bmatrix} \right\rangle \in T_B\left(\left\langle \begin{bmatrix} e_I \\ e_J \end{bmatrix}, \begin{bmatrix} f_I \\ f_J \end{bmatrix} \right\rangle, \left\langle \begin{bmatrix} f_I \\ f_J \end{bmatrix}, \begin{bmatrix} f_I \\ f_J \end{bmatrix} \right\rangle\right) =$$

$$= T_B\left(\left\langle \begin{bmatrix} e_I \\ e_J \end{bmatrix}, \begin{bmatrix} f_I \\ f_J \end{bmatrix} \right\rangle\right) = T_B(\langle e,f\rangle);$$

(iv) from

$$\left\langle \begin{bmatrix} d_I \\ e_J \end{bmatrix}, \begin{bmatrix} d_I \\ f_J \end{bmatrix}, \begin{bmatrix} f_I \\ e_J \end{bmatrix}, \begin{bmatrix} f_I \\ f_J \end{bmatrix}, \begin{bmatrix} d_I \\ f_J \end{bmatrix}, \begin{bmatrix} d_I \\ f_J \end{bmatrix} \right\rangle \in \tau(B)$$

we obtain

$$\left\langle \begin{bmatrix} d_I \\ e_J \end{bmatrix}, \begin{bmatrix} d_I \\ f_J \end{bmatrix} \right\rangle \in T_B\left(\left\langle \begin{bmatrix} f_I \\ e_J \end{bmatrix}, \begin{bmatrix} f_I \\ f_J \end{bmatrix} \right\rangle, \left\langle \begin{bmatrix} d_I \\ f_J \end{bmatrix}, \begin{bmatrix} d_I \\ f_J \end{bmatrix} \right\rangle \right) =$$
$$= T_B\left(\left\langle \begin{bmatrix} f_I \\ e_J \end{bmatrix}, \begin{bmatrix} f_I \\ f_J \end{bmatrix} \right\rangle \right) \subseteq T_B(\langle e, f\rangle),$$

by (iii);

(v)

$$\left\langle \begin{bmatrix} c_I \\ e_J \end{bmatrix}, \begin{bmatrix} d_I \\ f_J \end{bmatrix}, \begin{bmatrix} c_I \\ f_J \end{bmatrix}, \begin{bmatrix} d_I \\ f_J \end{bmatrix}, \begin{bmatrix} d_I \\ e_J \end{bmatrix}, \begin{bmatrix} d_I \\ f_J \end{bmatrix} \right\rangle \in \tau(B)$$

and so

$$\langle a, b\rangle = \left\langle \begin{bmatrix} c_I \\ e_J \end{bmatrix}, \begin{bmatrix} d_I \\ f_J \end{bmatrix} \right\rangle \in T_B\left(\left\langle \begin{bmatrix} c_I \\ f_J \end{bmatrix}, \begin{bmatrix} d_I \\ f_J \end{bmatrix} \right\rangle, \left\langle \begin{bmatrix} d_I \\ e_J \end{bmatrix}, \begin{bmatrix} d_I \\ f_J \end{bmatrix} \right\rangle \right) =$$
$$= T_B\left(\left\langle \begin{bmatrix} c_I \\ f_J \end{bmatrix}, \begin{bmatrix} d_I \\ f_J \end{bmatrix} \right\rangle \right) \vee T_B\left(\left\langle \begin{bmatrix} d_I \\ e_J \end{bmatrix}, \begin{bmatrix} d_I \\ f_J \end{bmatrix} \right\rangle \right) \subseteq$$
$$\subseteq T_B(\langle c, d\rangle) \vee T_B(\langle e, f\rangle) = T_B(\langle c, d\rangle, \langle e, f\rangle),$$

by (ii) and (iv).

(4) $\Rightarrow$ (3): See again the definition of $\varrho(B)$.

(3) $\Rightarrow$ (2): Let T be a tolerance on B and let $\langle c, d\rangle$, $\langle e, f\rangle \in T$. Then evidently $T_B(\langle c, d\rangle, \langle e, f\rangle) \subseteq T$ and further $\left\langle \begin{bmatrix} c_I \\ e_J \end{bmatrix}, \begin{bmatrix} d_I \\ f_J \end{bmatrix} \right\rangle \in T_B(\langle c, d\rangle, \langle e, f\rangle)$ for any disjoint index sets I, J, $I \cup J = \{1, \ldots, n\}$, by hypothesis (3). Altogether, $\left\langle \begin{bmatrix} c_I \\ e_J \end{bmatrix}, \begin{bmatrix} d_I \\ f_J \end{bmatrix} \right\rangle \in T$ as claimed.

(2) $\Rightarrow$ (1): Let T be a tolerance on $B = C_1 \times \ldots \times C_n$. Denote by T_i the projection of T on C_i, i.e. $T_i = \{\langle x_i, y_i\rangle \in C_i^2;\ \langle x, y\rangle \in T \text{ for some } x, y \in B\}$, $i \leqslant n$. The inclusion $T \subseteq T_1 \times \ldots \times T_n$ is trivial. Conversely, let $\langle u, v\rangle \in T_1 \times \ldots \times T_n$. Then there are pairs $\langle c, d\rangle$, $\langle e, f\rangle \in T$ such that $\langle u_1, v_1\rangle = \langle c_1, d_1\rangle$ and $\langle u_2, v_2\rangle = \langle e_2, f_2\rangle$. Choose index sets $I = \{1\}$, $J = \{2, \ldots, n\}$ and apply the hypothesis (2) to the assumption $\langle c, d\rangle$, $\langle e, f\rangle \in T$. Then we have $\left\langle \begin{bmatrix} c_I \\ e_J \end{bmatrix}, \begin{bmatrix} d_I \\ f_J \end{bmatrix} \right\rangle = \left\langle \begin{bmatrix} c_1 \\ e_2 \\ \vdots \\ e_n \end{bmatrix}, \begin{bmatrix} d_1 \\ f_2 \\ \vdots \\ f_n \end{bmatrix} \right\rangle =$
$\left\langle \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ e_n \end{bmatrix}, \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ f_n \end{bmatrix} \right\rangle \in T$. Repeating this process we find that $\langle u, v\rangle \in T$, as required. The proof is complete. □

Lemma 2. *Let B, C be algebras of the same type, φ a homomorphism from B to C. Then $\langle a,b\rangle \in T_B(\langle c,d\rangle, \langle e,f\rangle)$ implies*

$$\langle \varphi(a), \varphi(b)\rangle \in T_c(\langle \varphi(c), \varphi(d)\rangle, \langle \varphi(e), \varphi(f)\rangle)$$

for any elements a, b, c, d, e, $f \in B$.

P r o o f. The assumption $\langle a,b\rangle \in T_B(\langle c,d\rangle, \langle e,f\rangle)$ can be rewritten to

$$(*) \qquad \begin{aligned} a &= t(c,d,e,f,b_1,\dots,b_m),\\ b &= t(d,c,f,e,b_1,\dots,b_m) \end{aligned}$$

for some elements $b_1, \dots, b_m \in B$ and a $(4+m)$-ary term t, see e.g. [2]. Applying φ to the above equations $(*)$ we immediately get

$$\begin{aligned} \varphi(a) &= t\big(\varphi(c), \varphi(d), \varphi(e), \varphi(f), \varphi(b_1), \dots, \varphi(b_m)\big),\\ \varphi(b) &= t\big(\varphi(d), \varphi(c), \varphi(f), \varphi(e), \varphi(b_1), \dots, \varphi(b_m)\big), \end{aligned}$$

which means that $\langle \varphi(a), \varphi(b)\rangle \in T_C\big(\langle \varphi(c), \varphi(d)\rangle, \langle \varphi(e), \varphi(f)\rangle\big)$, see [2] again. □

N o t a t i o n 4. Let A be an algebra, $n \geqslant 2$, $p_1, \dots, p_n \colon A^n \to A$ canonical projections, and S a subset of A^n. Then $p_1^S, \dots, p_n^S$ denote the restrictions of $p_1, \dots, p_n$, respectively, to S.

Theorem. *Let A be a finite algebra. The following conditions are equivalent:*
(1) A^n has factorable tolerances for any $n \geqslant 2$;
(2) $A^{\tau(A)}$ has factorable tolerances.

P r o o f. (1) $\Rightarrow$ (2) is trivial.
(2) $\Rightarrow$ (1): Take $\langle a,b,c,d,e,f\rangle \in \tau(A^n)$. It is a routine to verify that
(i) $\langle a_i, b_i, c_i, d_i, e_i, f_i\rangle \in \tau(A)$, $\quad i \leqslant n$;
(ii) $\Big\langle p_1^{\tau(A)}, p_2^{\tau(A)}, p_2^{\tau(A)}, p_4^{\tau(A)}, p_5^{\tau(A)}, p_6^{\tau(A)}\Big\rangle \in \tau(A^{\tau(A)})$;
(iii) the correspondence $\varphi\colon g \mapsto \begin{bmatrix} g(a_1,b_1,c_1,d_1,e_1,f_1)\\ \dots\\ g(a_n,b_n,c_n,d_n,e_n,f_n)\end{bmatrix}$ is a homomorphism from $A^{\tau(A)}$ to A^n which sends $p_1^{\tau(A)}, p_2^{\tau(A)}, p_3^{\tau(A)}, p_4^{\tau(A)}, p_5^{\tau(A)}, p_6^{\tau(A)}$ to a,b,c,d,e,f, respectively.

By hypothesis $A^{\tau(A)}$ has factorable tolerances and so (ii) implies

$$(*) \qquad \Big\langle p_1^{\tau(A)}, p_2^{\tau(A)}\Big\rangle \in T_{A^{\tau(A)}}\Big(\Big\langle p_3^{\tau(A)}, p_4^{\tau(A)}\Big\rangle, \Big\langle p_5^{\tau(A)}, p_6^{\tau(A)}\Big\rangle\Big),$$

by Lemma 1(5). Applying the homomorphism φ to the relation formula $(*)$ we obtain

$$\langle a,b\rangle \in T_{A^n}(\langle c,d\rangle, \langle e,f\rangle),$$

see Lemma 2. In this way we get that $\langle a,b,c,d,e,f\rangle \in \tau(A^n)$ implies $\langle a,b\rangle \in T_{A^n}(\langle c,d\rangle, \langle e,f\rangle)$, which establishes the factorability of tolerances on algebra A^n, by Lemma 1(5) again. The proof is complete. □

Corollary. *Let A be a finite k-element algebra, $k \geqslant 2$. The following conditions are equivalent:*

(1) *A^n has factorable tolerances for any $n \geqslant 2$;*

(2) *A^{4k^2-3k} has factorable tolerances.*

Proof. Evidently card $\tau(A) = 4k^2 - 3k$ whenever card $A = k$. □

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Souhrn

TOLERANCE NA MOCNINÁCH KONEČNÉ ALGEBRY

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V článku je ukázáno, že libovolná mocnina A^n, $n \geqslant 2$, konečné k-prvkové algebry A, $k \geqslant 2$, má rozložitelné tolerance, jestliže tuto vlastnost má již mocnina A^{4k^2-3k}.

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