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SEMIDOMATIC AND TOTAL SEMIDOMATIC NUMBERS OF DIRECTED GRAPHS

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Summary. Certain numerical invariants of directed graphs, analogous to the domatic number and to the total domatic number of an undirected graph, are introduced and studied.

Keywords: outside-semidomatic number, inside-semidomatic number, total outside-semidomatic number, total inside-semidomatic number.

AMS classification: 05C20, 05C35

The domatic number of an undirected graph was introduced by E. J. Cockayne and S. T. Hedetniemi in [1] and [2], the total domatic number by the same authors and R. Dawes in [3]. The concept of domatic number was transferred to directed by introducing semidomatic numbers in [6]. Here we will continue the study of semidomatic numbers. Further, we shall transfer the concept of total domatic number to directed graphs, analogously to [6].

All graphs considered will be finite directed graphs without loops, except the case when we explicitly state the contrary. Two vertices may be joined by at most two edges; if there are two edges joining the same pair of vertices, then they must be directed oppositely. The symbol xy, where x and y are vertices, always denotes the directed edge from x to y (we omit arrows).

A subset D of the vertex set $V(G)$ of a graph G is called outside-semidominating (or inside-semidominating) in G, if for each vertex $x \in V(G) - D$ there exists a vertex $y \in D$ such that yx (or xy, respectively) is an edge of G. A partition of $V(G)$, all of whose classes are outside-semidominating (or inside-semidominating) in G, is called outside-semidomatic (or inside-semidomatic, respectively). The maximum number of classes of an outside-semidomatic (or inside-semidomatic) partition of G is called the outside-semidomatic (or inside-semidomatic, respectively) number of G. The outside-semidomatic number of G is denoted by $d^+(G)$, the inside-semidomatic number of G is denoted by $d^-(G)$.

A subset D of $V(G)$ is called total outside-semidominating (or total inside-semidominating) in G, if for each vertex $x \in V(G)$ there exists a vertex $y \in D$ such that yx (or xy, respectively) is an edge of G. There exists at least one total outside-semidominating (or total inside-semidominating) set in G if and only if G has no source (or no sink, respectively). Namely, the whole set $V(G)$ has this property. If G contains a source x, then there is no vertex $y \in V(G)$ such that yx is an edge of G and thus no total outside-semidominating set in G exists; analogously for a sink. Note that isolated vertices are simultaneously sources and sinks.

Let G be without sources. A partition of $V(G)$, all of whose classes are total outside-semidominating sets in G, is called total outside-semidomatic. The maximum number of classes of a total outside-semidomatic partition of G is called the total outside-semidomatic number of G and denoted by $d_t^+(G)$. Analogously for a graph G without sinks we define its total inside-domatic number $d_t^-(G)$.

The expressions "outside" and "inside" will be shortened to the letters O and I. Most of the assertions will concern the O-semidomatic number and the total O-semidomatic number. For these assertions it is possible to formulate dual assertions concerning the I-semidomatic number and the total I-semidomatic number; this is left to the reader.

2. Semidomatic numbers

We shall treat quasicomponents of graphs. A quasicomponent Q of G is called initial, if no edge comes into Q from another quasicomponent. It is called terminal, if no edge goes from Q to another quasicomponent.

Theorem 1. *Let G be a directed graph, let d_1^+ be the minimum of O-semidomatic numbers of its quasicomponents, let d_2^+ be the minimum of O-semidomatic numbers of its initial quasicomponents. Then*

$$d_1^+ \leqslant d^+(G) \leqslant d_2^+.$$

Proof. As the union of two O-semidominating sets is evidently again an O-semidominating set, in every graph there exist O-semidomatic partitions of all cardinalities not exceeding the O-semidomatic number. Therefore in each quasicomponent Q we may choose an O-semidomatic partition $\{D_1(Q), \dots, D_{d_1^+}(Q)\}$. Now for $i = 1, \dots, d_1^+$ let D_i be the union of all sets $D_i(Q)$ for all quasicomponents Q of G. Each D_i is an O-semidominating set in G; namely if x belongs to $V(G) - D_i$, then x belongs to some quasicomponent Q of G and there exists and edge into x from a vertex of $D_i(Q) \subseteq D_i$. This implies $d_1^+ \leqslant d^+(G)$.

Now let $\mathcal{D}$ be an O-semidomatic partition of G with $d^+(G)$ classes. Let Q_0 be an initial quasicomponent of G with $d^+(Q_0) = d_2^+$. Let $\mathcal{D}_0 = \{D \cap V(Q_0) \mid D \in \mathcal{D}\}$. Let

$D \in \mathcal{D}$, $x \in V(Q_0) - D$, $D_0 = D \cap V(Q_0)$. Then there exists $y \in D$ such that an edge of G goes from y to x. As $x \in V(Q_0)$ and Q_0 is an initial quasicomponent of G, the vertex y must belong to Q_0. Therefore $y \in D \cap V(Q_0) = D_0$. We have proved that each set from $\mathcal{D}_0$ is O-semidominating in Q_0 and $\mathcal{D}_0$ is an O-semidomatic partition of Q_0. This implies that $d^+(G) \leqslant d^+(Q_0) = d_2^+$. □

Theorem 2. *Let d_1^+, d_2^+, d^+ be three positive integers, let $d_1^+ \leqslant d^+ \leqslant d_2^+$. Then there exists a directed graph G with the property that $d^+(G) = d^+$, the minimum of O-semidomatic numbers of quasicomponents of G is d_1^+ and the minimum of O-semidomatic numbers of initial quasicomponents of G is d_2^+.*

Proof. First, let $d_1^+ \geqslant 2$. Let Q_1, Q_2 be two vertex-disjoint complete directed graphs, let $V(Q_1) = \{u(1), \ldots, u(d_1^+)\}$, $V(Q_2) = \{v(1), \ldots, v(d_2^+)\}$. Evidently $d^+(Q_1) = d_1^+$, $d^+(Q_2) = d_2^+$. If $d^+ = d_1^+$, then let G be the graph obtained from Q_1 and Q_2 by adding the edge from $v(1)$ to $u(1)$. Then Q_1, Q_2 are quasicomponents of G and Q_2 is its unique initial quasicomponent. According to Theorem 1 we have $d^+(G) \geqslant d^+(Q_1) = d_1^+ = d^+$. Now suppose that $d^+(G) > d^+$ and let $\mathcal{D}$ be an O-semidomatic partition of G with $d^+(G)$ classes. According to the Pigeon Hole Principle there exists a class $D \in \mathcal{D}$ which contains no vertex of Q_1. As $d_1^+ \geqslant 2$, there exists a vertex $u(2)$ into which no edge from Q_2 comes, hence also no edge from D, which is a contradiction. Therefore $d^+(G) = d_1^+ d^+$. Now let $d^+ > d_1^+$. Then let G be the graph obtained from Q_1 and Q_2 by adding all edges from the vertices $v(d_1^+ + 1)$, …, $v(d^+)$ into all vertices of Q_1. Let $D_1 = \{u(1), v(1)\} \cup \{v(j) \mid d_1^+ + 1 \leqslant j \leqslant d_2^+\}$, $D_i = \{u(i), v(i)\}$ for $i = 2, \ldots, d_1^+$, $D_i = \{v(i)\}$ for $i = d_1^+ + 1, \ldots, d^+$. Evidently, the classes D_i for $i = 1, \ldots, d^+$ form an O-semidomatic partition of G and thus $d^+(G) \geqslant d^+$. Suppose that $d^+(G) > d^+$ and let $\mathcal{D}^*$ be an O-semidomatic partition of G with $d^+(G)$ classes. Then, by the Pigeon Hole Principle, there are at least $d^+ - d_1^+ + 1$ classes of $\mathcal{D}^*$ containing no vertex of Q_1. By the same principle, among these classes there is at least one class D which contains none of the vertices $v(d_1^+ + 1)$, …, $v(d^+)$. Each edge with a terminal vertex in Q_1 has its initial vertex either in Q_1, or among the vertices $v(d_1^+ + 1)$, …, $v(d^+)$; this is a contradiction with the assumption that D is an O-semidominating set. Therefore $d^+(G) = d^+$. Thus the proof is complete for $d_1^+ \geqslant 2$. If $d_1^+ = 1$, we modify the construction of G in such a way that Q_1 is a cycle of lenght 3 with the vertices $u(1)$, $u(2)$, $u(3)$. □

The concept of the O-semidominating set is related to the concepts described already in the classical books of D. König [4] ("Punktbasis") and O. Ore [5], and also to the problem of C. F. Gauss concerning eight queens on the chessboard.

3. TOTAL SEMIDOMATIC NUMBERS

Proposition 1. *Let G be a directed graph without sources, let $\delta^-(G)$ be the minimum indegree of a vertex of G. Then*

$$d_t^+(G) \leqslant \delta^-(G).$$

Proof. Let $d_t^+(G) = d$ and let $\mathcal{D} = \{D_1, \ldots, D_d\}$ be a total O-semidomatic partition of G with d classes. Let $x \in V(G)$; then in each D_i for $i = 1, \ldots, d$ there exists a vertex y_i such that $y_i x$ is an edge of G. The vertices $y_1, \ldots, y_d$ are pairwise distinct, therefore the indegree of x is at least d. As x was chosen arbitrarily, we have $d_t^+ \leqslant \delta^-(G)$. □

Proposition 2. *Let G be a directed graph without sources. Then*

$$d_t^+(G) \geqslant \left\lfloor \frac{1}{2} d^+(G) \right\rfloor .$$

Proof. If $d^+(G) = 1$, the assertion is evident. Thus suppose $d^+(G) \geqslant 2$. It is easy to prove that the union of at least two disjoint O-semidominating sets is a total O-semidominating set. If an O-semidomatic partition $\mathcal{D}$ with $d^+(G)$ classes is given, then we can construct a total O-semidomatic partition of G in such a way that at most one of its classes is the union of three classes of $\mathcal{D}$ and each other class is the union of two classes of $\mathcal{D}$. The partition thus obtained has $\lfloor \frac{1}{2} d^+(G) \rfloor$ classes, which implies the assertion. □

Proposition 3. *Let G be a directed graph with n vertices without sources. Then*

$$d_t^+(G) \leqslant \left\lfloor \frac{1}{2} n \right\rfloor .$$

If, moreover, in G any pair of vertices is joined by at most one edge, then

$$d_t^+(G) \leqslant \left\lfloor \frac{1}{3} n \right\rfloor .$$

Proof. Let D be a total O-semidominating set in G. As G has no loops, for each vertex $x \in D$ there exists another vertex $y \in D$ such that yx is an edge of G. Hence $|D| \geqslant 2$. If $|D| = 2$, then $|D| = \{x, y\}$ and both xy and yx are edges of G. Thus if in G any pair of vertices is joined by at most one edge, then any total O-semidominating set in G must have at least three vertices. This implies the assertions. □

Now we will state a theorem analogous to Theorem 1.

Theorem 3. *Let G be a directed graph without sources, let d_1^+ be the minimum of total O-semidomatic numbers of its quasicomponents, let d_2^+ be the minimum of total O-semidomatic numbers of its initial quasicomponents. Then*

$$d_1^+ \leqslant d_t^+(G) \leqslant d_2^+.$$

Proof is analogous to the proof of Theorem 1. □

The following theorem is analogous to Theorem 2.

Theorem 4. *Let d_1^+, d_2^+, d^+ be three positive integers, let $d_1^+ \leqslant d^+ \leqslant d_2^+$. Then there exists a directed graph G with the property that $d_t^+(G) = d^+$, the minimum of total O-semidomatic numbers of quasicomponents of G is d_1^+ and the minimum of total O-semidomatic numbers of initial quasicomponents of G id d_2^+.*

Proof. Let Q_1, Q_2 be two vertex-disjoint complete directed graphs, let

$$V(Q_1) = \{u(1), \ldots, u(d_1^+), u'(1), \ldots, u'(d_1^+)\},$$
$$V(Q_2) = \{v(1), \ldots, v(d_2^+), v'(1), \ldots, v'(d_2^+)\}.$$

Evidently $d_t^+(Q_1) = d_1^+$, $d_t^+(Q_2) = d_2^+$. If $d^+ = d_1^+$, then let G be the graph obtained from Q_1 and Q_2 by adding an edge from $v(1)$ to $u(1)$. Then Q_1, Q_2 are quasicomponents of G and Q_2 is its unique initial quasicomponent. According to Theorem 3 we have $d_t^+(G) \geqslant d_t^+(Q_1) = d_1^+ = d^+$. Now suppose that $d_t^+(G) > d^+$ and let $\mathcal{D}$ be a total O-semidomatic partition of G with $d_t^+(G)$ classes. According to the Pigeon Hole Principle either there exists a class $D \in \mathcal{D}$ which contains no vertex of Q_1, or there exist classes $D_1 \in \mathcal{D}$, $D_2 \in \mathcal{D}$, each of which contains exactly one vertex of Q_1. In the first case no edge from D comes into $u'(1)$, which is a contradiction. In the second case at least one of the sets D_1, D_2 does not contain $u(1)$; then no edge from this class comes into its vertex being in Q_1, which is again a contradiction. Therefore $d_t^+(G) = d_1^+ = d^+$. Now let $d^+ > d_1^+$. Then let G be the graph obtained from Q_1 and Q_2 by adding all edges from the vertices $v(d_1^+ + 1)$, …, $v(d^+)$ into all vertices of Q_1. Let $D_1 = \{u(1), u'(1), v(1), v'(1)\} \cup \{u(j) \mid d_1^+ + 1 \leqslant j \leqslant d_2^+\} \cup \{v(j) \mid d_1^+ + 1 \leqslant j \leqslant d_2^+\}$, $D_i = \{u(i), u'(i), v(i), v'(i)\}$ for $i = 2$, …, d_1^+, $D_i = \{v(i), v'(i)\}$ for $i = d_1^+ + 1$, …, d^+. Evidently, the classes D_i for $i = 1$, …, d^+ form a total O-semidomatic partition of G and thus $d_t^+(G) \geqslant d^+$. Suppose that $d_t^+(G) > d^+$ and let $\mathcal{D}^*$ be a total O-semidomatic partition of G with $d_t^+(G)$ classes. Then, by the Pigeon Hole Principle, there are at least $d^+ - d_1^+ + 1$ classes of $\mathcal{D}^*$ containing at most one vertex of Q_1. By the same principle, among these classes there is at least one class D which contains none of the vertices $v(d_1^+ + 1)$, …, $v(d^+)$. Each edge with a terminal vertex on Q_1 has its initial vertex either in Q_1, or among the vertices $v(d_1^+ + 1)$, …, $v(d^+)$; this is a contradiction with the assumption that D is a total O-semidominating set. Therefore $d_t^+(G) = d^+$. □

Now we shall prove a theorem concerning tournaments.

Theorem 5. *Let d^+, d^-, n be three positive integers, let $d^+ \leqslant \lfloor \frac{1}{3}n \rfloor$, $d^- \leqslant \lfloor \frac{1}{3}n \rfloor$. Then there exists a tournament T with n vertices such that $d_t^+(T) = d^+$, $d_t^-(T) = d^-$.*

Proof. Let V be a set of n vertices. If $3d^+ + 3d^- \leqslant n$, then we choose a subset C of V of the cardinality $n - 3d^+ - 3d^-$. Then we decompose $V - C$ into two disjoint subsets A, B such that $|A| = 3d^+$, $|B| = 3d^-$. If $3d^+ + 3d^- > n$, then we choose a subset C of V such that $|C| \leqslant 2$, $|C| \equiv n (\bmod 3)$. Then we choose two sets A, B such that $A \cup B = V - C$, $|A| = 3d^+$, $|B| = 3d^-$. (In both cases C may be empty.) In both cases the number of vertices of $A \cup B$ is divisible by 3 and (by the Inclusion-Exclusion Principle) so are the numbers of vertices of $A \cap B$, $A - B$ and $B - A$. Thus we choose a partition $\mathcal{D}$ of $A \cup B$ into three-element sets with the property that each class of $\mathcal{D}$ is a subset of one of the sets $A \cap B$, $A - B$, $B - A$. If $D \in \mathcal{D}$, then we denote the vertices of D by $x(D)$, $y(D)$, $z(D)$ and lead edges from $x(D)$ to $y(D)$, from $y(D)$ to $z(D)$ and form $z(D)$ to $x(D)$. Now we choose a linear ordering $\prec$ of $\mathcal{D}$ in such a way that if either $D_1 \subseteq A-B$, $D_2 \subseteq A \cap B$, or $D_1 \subseteq A \cap B$, $D_2 \subseteq B - A$, then always $D_1 \prec D_2$. Now we lead edges between vertices of different classes of D. If $D_1 \subseteq A - B$, $D_2 \subseteq A$, $D_1 \prec D_2$, then we lead edges as in Fig. 1. If $D_1 \subseteq A \cap B$, $D_2 \subseteq B - A$, $D_1 \prec D_2$, we lead them as in Fig. 3. If $D_1 \subseteq A \cap B$, $D_2 \subseteq A \cap B$, $D_1 \prec D_2$, we lead them as in Fig. 2.

Further, we draw edges from all vertices of $A-B$ into all vertices of $(B-A) \cup C$ and (if $C \neq \emptyset$) from all vertices of C into all vertices of $B - A$. If $A \cap B \neq \emptyset$, $C \neq \emptyset$, then we draw edges from all vertices of C into all vertices $x(D)$ for $D \subseteq A \cap B$ and from all vertices $y(D)$, $z(D)$ for $D \subseteq A \cap B$ into all vertices of C. In the graph in Fig. 2 the sets D_1, D_2 are both O-semidominating and I-semidominating. In the graph in Fig. 1 they are both O-semidominating, but only D_2 is I semidominating. In the graph in Fig. 3 they are both I-semidominating, but only D_1 is O-semidominating. Let $\mathcal{D}_A$ (or $\mathcal{D}_B$) be the partition of A (or B, respectively) induced by $\mathcal{D}$. From the construction of T it follows that $\mathcal{D}_A$ is total O-semidomatic partition of T and $\mathcal{D}_B$ is a total I-semidomatic partition of T; this implies $d_t^+(T) \geqslant d^+$, $d_t^-(T) \geqslant d^-$. Let $D_{\min}$ be the first element in the ordering $\prec$. Consider $z(D_{\min})$. Into $z(D_{\min})$ edges go only from the vertex $y(D_{\min})$ and from the vertices $z(D)$ for all $D \in \mathcal{D}_A - \{D_{\min}\}$. Hence the indegree if $z(D_{\min})$ is d^+, which implies $d_t^+(T) \leqslant \delta^-(T) \leqslant d^+$ and thus $d_t^+(T) = d^+$. Similarly, if $D_{\max}$ is the least element in $\prec$, then from $x(D_{\max})$ edges go only to all vertices $z(D)$ for $D \in \mathcal{D}_B$. the outdegree of $x(D_{\max})$ id d^-, which implies $d_t^-(T) \leqslant \delta^+(T) \leqslant d^-$ and thus $d_t^-(T) = d^-$. □

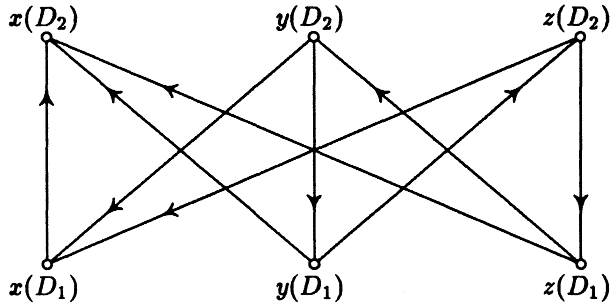

Fig. 1

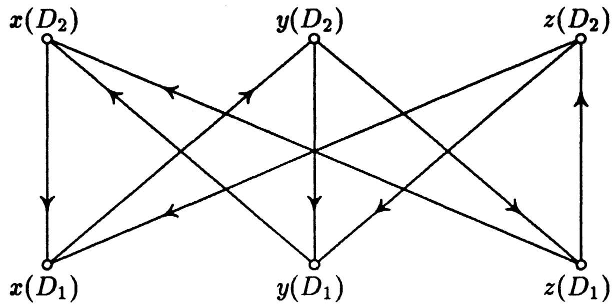

Fig. 2

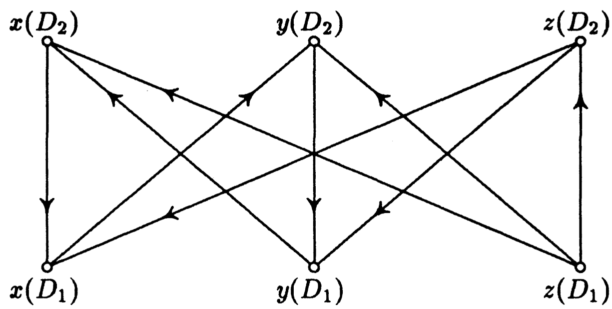

Fig. 3

4. Remarks on infinite graphs

The semidomatic number and the total semidomatic numbers can be extended also to infinite graphs in such a way that instead of minima we consider suprema.

Theorem 6. *Let G be a directed graph. If one of the numbers $d^+(G)$, $d_t^+(G)$ is infinite, then*

$$d^+(G) = d_t^+(G).$$

Proof. This assertion can be proved analogously as Proposition 2. □

Theorem 7. *There exists a directed graph G such that*

$$d^+(G) = d_t^+(G) = \aleph_0,$$

while all quasicomponents of G are finite.

Proof. The vertex set $V(G)$ is the set of all ordered pairs (i,j) of positive integers such that $i \geqslant j$. If (i_1, j_1), (i_2, j_2) are two vertices of G, then an edge goes from (i_1, j_1) into (i_2, j_2) if and only if $i_1 \geqslant i_1$. (In particular, if $i_1 = i_2$, then these vertices are jointed by a pair of oppositely directed edges.) The graph G has quasicomponents Q_i for all positive integers i. For each i the vertex set $V(Q_1)$ of Q_i is the set of all pairs (i,j) for $j \leqslant i$. Each quasicomponent Q_i is a complete directed graph with i vertices. Now for each positive integer j let D_j be the set of all pairs (i,j) for $i \geqslant j$. Let (i_0, j_0) be a vertex of G, let j be a positive integer, let $k = \max(i_0 + 1, j)$. Then $(k,j) \in D_j$ and, as $k > i$, there exists an edge from (k,j) into (i_0, j_0). As (i_0, j_0) was chosen arbitrarily, D_j is a total O-semidomatic set in G. As j was chosen arbitrarily, the sets D_j for all positive integers j form a total O-domatic partition of G and thus $d_t^+(G) = \aleph_0$. (Evidently it cannot be greater.) □

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