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THE SINGULAR CAUCHY-NICOLETTI PROBLEM FOR THE SYSTEM OF TWO ORDINARY DIFFERENTIAL EQUATIONS

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Summary. In the paper the singular Cauchy-Nicoletti problem for the system of two ordinary differential equations is considered. New sufficient conditions for solvability of this problem are proved. In the proofs the topological method is applied. Some comparisons with known results are also given in the paper.

Keywords: Singular Cauchy-Nicoletti problem, retract

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1. INTRODUCTION

Consider the following singular Cauchy-Nicoletti problem for the system of two ordinary differential equations

$$\begin{aligned} y_1' &= f_1(x, y_1, y_2), \\ y_2' &= f_2(x, y_1, y_2) \end{aligned} \tag{1}$$

and

$$\begin{aligned} y_1(a^+) &= A, \\ y_2(b^-) &= B, \end{aligned} \tag{2}$$

where a, b, A, B are constants and $a < b$. Concerning the functions $f_i(x, y_1, y_2)$, $i = 1, 2$ we assume that they are continuous and satisfy a local Lipschitz condition in the variables y_1, y_2 in an open bounded region G such that the set $\{(x, y_1, y_2) \in G;\ x = x^*\} \neq \emptyset$ for each $x^* \in (a, b)$. Under these conditions the solutions of the system (1) are in G uniquely determined by their initial data but for $x = a$ or $x = b$ this need not be the case.

We define the solution of the problem (1), (2) as a vector-function $y(x) = (y_1(x), y_2(x)) \in C^1(a,b)$ which on (a,b) satisfies the system (1), $(x, y_1(x), y_2(x)) \subset G$ on (a,b) and $y_1(a^+) = A, y_2(b^-) = B$.

The Cauchy-Nicoletti problems, the generalized Cauchy problems or the boundary value problems for systems of ordinary differential equations have been considered by many authors (e.g. [6–8], [10–11], [13–15]). Singular problems of such types have been studied e.g. in works [1], [3–4], [7–9], [12] and [16–17].

In the present paper we obtain new sufficient conditions for solvability of the problem (1), (2) in the above mentioned sense. Moreover, some estimates for the components of solutions will be given. Certain results are formulated for more concrete right-hand sides of the system (1). Some comparisons with known results will be given in the paper.

2. Main results

We will consider real functions $\varphi_i(x)$, $\psi_i(x)$, $i = 1, 2$ which satisfy the following conditions (I):

$$\varphi_i(x), \quad \psi_i(x) \in C[a,b], \quad C^1(a,b), \quad i = 1,2;$$
$$\varphi_1(x) < \varphi_2(x) \text{ if } x \in (a,b], \quad \psi_1(x) < \psi_2(x) \text{ if } x \in [a,b);$$
$$\lim_{x\to a^+} \varphi_i(x) = A, \lim_{x\to b^-} \psi_i(x) = B, \quad i = 1,2.$$

The functions $\varphi_i(x)$, $\psi_i(x)$, $i = 1, 2$ are said to satisfy the conditions (II) if the conditions (I) hold and, moreover, $\varphi_i(x)$, $\psi_i(x) \in C^2(a,b)$.

Further we need the following auxiliary functions:

i) $W_1 \equiv W_1(x, y_1) \equiv (y_1 - \varphi_1(x))(y_1 - \varphi_2(x))$,
$W_2 \equiv W_2(x, y_2) \equiv (y_2 - \psi_1(x))(y_2 - \psi_2(x))$,
which are defined for $(x, y_1) \in [a,b] \times R$ or $(x, y_2) \in [a,b] \times R$, respectively;

ii) $\Phi_i(x, y_2) \equiv -\varphi_i'(x) + f_1(x, \varphi_i(x), y_2)$,
$\Psi_i(x, y_1) \equiv -\psi_i'(x) + f_2(x, y_1, \psi_i(x))$,
which are defined for $(x, y_2) \in G$ or $(x, y_1) \in G$, respectively, and where $i = 1$, 2 and $\varphi_i(x)$, $\psi_i(x)$ satisfy the conditions (I); if, moreover, $\varphi_i(x)$, $\psi_i(x)$ satisfy (II) and $f_i(x, y_1, y_2) \in C^1(G)$ for $i = 1$, 2, we write

iii) $D\Phi_i(x, y_2) \equiv -\varphi_i''(x) + f_{1x}'(x, \varphi_i(x), y_2) + f_{1y_1}'(x, \varphi_i(x), y_2) f_1(x, \varphi_i(x), y_2) +$
$+ f_{1y_2}'(x, \varphi_i(x), y_2) f_2(x, \varphi_i(x), y_2)$,
$D\Psi_i(x, y_1) \equiv -\psi_i''(x) + f_{2x}'(x, y_1, \psi_i(x)) + f_{2y_1}'(x, y_1, \psi_i(x)) f_1(x, y_1, \psi_i(x)) +$
$+ f_{2y_2}'(x, y_1, \psi_i(x)) f_2(x, y_1, \psi_i(x))$,
which are defined for $(x, y_2) \in G$ or $(x, y_1) \in G$, respectively.

Finally, by Ω we denote the domain

$$\Omega = \{(x, y_1, y_2) \colon a < x < b, W_1(x, y_1) < 0, W_2(x, y_2) < 0\}$$

and we wil suppose that $\Omega \subset G$.

The following three theorems represent the main results of the paper.

Theorem 1. *Let functions $\varphi_i(x)$, $\psi_i(x)$, $i = 1, 2$ satisfy the conditions (I), let $\Phi_1(x, y_2)\Phi_2(x, y_2) < 0$ for each $x \in (a, b)$ and $y_2 \in [\psi_1(x), \psi_2(x)]$ and $\Psi_1(x, y_1) \times \Psi_2(x, y_1) < 0$ for each $x \in (a, b)$ and $y_1 \in [\varphi_1(x), \varphi_2(x)]$. Then there is at least one solution $y(x) = (y_1(x), y_2(x))$ of the problem (1), (2) such that $(x, y_1(x), y_2(x)) \subset \Omega$ if $x \in (a, b)$.*

Theorem 2. *Let functions $\varphi_i(x)$, $\psi_i(x)$, $i = 1, 2$ satisfy the conditions (II), $f_i(x, y_1, y_2) \in C^1(G)$, $i = 1, 2$; for each fixed $x \in (a, b)$ let the functions $\Phi_i(x, y_2)$, $\Psi_i(x, y_1)$, $i = 1, 2$ be strong monotone with respect to y_2 on the interval $[\psi_1(x), \psi_2(x)]$ and with respect to y_1 on the interval $[\varphi_1(x), \varphi_2(x)]$. Further, on (a, b) let*

$$\Phi_i(x, \psi_1(x))\Phi_i(x, \psi_2(x)) < 0, \tag{3}$$

$$\Psi_i(x, \varphi_1(x))\Psi_i(x, \varphi_2(x)) < 0. \tag{4}$$

If, moreover,

$$D\Phi_1(x, y_2) < 0, \quad D\Phi_2(x, y_2) > 0 \tag{5}$$

on $(a, b) \times (\psi_1(x), \psi_2(x))$ and

$$D\Psi_1(x, y_1) < 0, \quad D\Psi_2(x, y_1) > 0 \tag{6}$$

on $(a, b) \times (\varphi_1(x), \varphi_2(x))$
then there is at least one solution $y(x) = (y_1(x), y_2(x))$ of the problem (1), (2) such that $(x, y_1(x), y_2(x)) \subset \Omega$ if $x \in (a, b)$.

Existence of a solution of the problem (1), (2) can be established even if some of the assumptions of Theorem 1 are combined with some of the assumptions of Theorem 2. One of the possible cases is presented in the following theorem.

Theorem 3. *Let functions $\varphi_i(x)$, $i = 1, 2$ satisfy the conditions (I), let functions $\psi_i(x)$, $i = 1, 2$ satisfy the conditions (II), $f_2(x, y_1, y_2) \in C^1(G)$, $\Phi_1(x, y_2)\Phi_2(x, y_2) < 0$ on $(a, b) \times [\psi_1(x), \psi_2(x)]$; for each fixed $x \in (a, b)$ let the functions $\Psi_i(x, y_1)$, $i = 1, 2$ be strong monotone with respect to y_1 on $[\varphi_1(x), \varphi_2(x)]$ and $\Psi_i(x, \varphi_1(x)) \times \Psi_i(x, \varphi_2(x)) < 0$ on (a, b). If, moreover,*

$$D\Psi_1(x, y_1) < 0, \quad D\Psi_2(x, y_1) > 0 \tag{7}$$

on $(a, b) \times (\varphi_1(x), \varphi_2(x))$ then there is at least one solution $y(x) = (y_1(x), y_2(x))$ of the problem (1), (2) such that $(x, y_1(x), y_2(x)) \subset \Omega$ if $x \in (a, b)$.

P r o o f of Theorem 1. As follows from (3), (4) the following cases are possible for the signs of Φ_1, Φ_2 and Ψ_1, Ψ_2 on the corresponding domains:
(i) $\Phi_1 < 0$, $\Phi_2 > 0$, $\Psi_1 < 0$ and $\Psi_2 > 0$;
(ii) $\Phi_1 > 0$, $\Phi_2 < 0$, $\Psi_1 > 0$ and $\Psi_2 < 0$;
(iii) $\Phi_1 < 0$, $\Phi_2 > 0$, $\Psi_1 > 0$ and $\Psi_2 < 0$;
(iv) $\Phi_1 > 0$, $\Phi_2 < 0$, $\Psi_1 < 0$ and $\Psi_2 > 0$.

I. consider the case (i). It is easy to verify that each point $M^* = (x^*, y_1^*, y_2^*)$ of the set

$$\Omega_1 = \{(x, y_1, y_2) \in \partial\Omega, x \in (a, b)\}$$

is a point of strict egres of Ω, that is $\Omega_1 = \Omega_{se}$ with respect to the corresponding solution of (1), because the derivatives of the functions $W_i(x, y_i)$, $i = 1, 2$ along solutions of the system (1) are positive. (The definitions of points of strict egress and of the other notions which we will use can be found e.g. in [5].) Indeed, if $M^* \in \Omega_1$ and

α) $W_1(x^*, y_1^*) = 0$ then

$$\frac{dW_1(x, y_1)}{dx}\Big|_{M^*} = [(f_1(x, y_1, y_2) - \varphi_1'(x))(y_1 - \varphi_2(x)) + \\ + (y_1 - \varphi_1(x))(f_1(x, y_1, y_2) - \varphi_2'(x))]\Big|_{M^*}.$$

As $y_1^* = \varphi_1(x^*)$ or $y_1^* = \varphi_2(x^*)$, we have

$$\frac{dW_1(x, y_1)}{dx}\Big|_{M^*} = \Phi_1(x^*, y_2)(\varphi_1(x^*) - \varphi_2(x^*)) > 0$$

or

$$\frac{dW_1(x, y_1)}{dx}\Big|_{M^*} = (\varphi_2(x^*) - \varphi_1(x^*))\Phi_2(x^*, y_2) > 0;$$

β) $W_2(x^* y_2^*) = 0$ then, as above, we obtain

$$\frac{dW_2(x, y_2)}{dx}\Big|_{M^*} > 0.$$

Obviously, the set of the egress points Ω_e of Ω coincides with the set Ω_{se}. Let $\{x_i\}$ be a decreasing sequence of numbers such that $x_i \in (a, b)$ and $\lim_{i\to\infty} x_i = a$. Let the index i be fixed. Denote $S_i = \{(x, y_1, y_2) \in \bar{\Omega}, x = x_i\}$. Then the set $S_i \cap \Omega_e$ is a retract of the set Ω_e because the mapping

$$\pi_I : (x, y_1, y_2) \in \Omega_e \to (x_i, y_1^0, y_2^0) \in S_i \cap \Omega_e$$

where

$$y_1^0 = \varphi_1(x_i) + (y_1 - \varphi_1(x))\frac{\varphi_2(x_i) - \varphi_1(x_i)}{\varphi_2(x) - \varphi_1(x)},$$
$$y_2^0 = \psi_1(x_i) + (y_2 - \psi_1(x))\frac{\psi_2(x_i) - \psi_1(x_i)}{\psi_2(x) - \psi_1(x)}$$

is continuous, identical on $S_i \cap \Omega_e$; however, it is not a retract of S_i since the boundary of a sphere is not its retract ([2]). As follows from the topological method of T. Ważewski (e.g. [5], [18]), there is a nonempty set $M_{iI} \subset S_i \cap \Omega$ such that if $(x_i, y_{1i}^0, y_{2i}^0) \in M_{iI}$ then for the corresponding solution of the system (1) $y_i(x) = (y_{1i}(x), y_{2i}(x))$ starting from the point $(x_i, y_{1i}^0, y_{2i}^0)$ the relation $(x, y_{1i}(x), y_{2i}(x)) \subset \Omega$ holds on its right-hand maximal interval of existence. This interval is equal to $[x_i, b)$ because the set Ω is bounded. The set M_{iI} is closed (including the case when M_{iI} consists of one point only) since in the opposite case we get a contradiction with the continuous dependence of solutions of the system (1) upon their initial data. Let $\chi\{M_{iI}, [x_i, b)\}$ be the set of all solutions of (1) with the initial data from the set M_{iI} on the interval $[x_i, b)$. Obviously $M_{iI}' \subset M_{1I}$ where

$$M_{iI}' = \{(x, y_1, y_2) \colon x = x_1, y_1 = y_1(x_1), y_2 = y_2(x_2),\\ (y_1(x), y_2(x)) \in \chi\{M_{iI}, [x_i, b)\}\}$$

and, if $i > 2$ then $M_{iI}' \subset M_{i-1,I}'$. Since the sets M_{iI}', $i = 1, 2, \ldots$ are compact there is a nonempty set $M_{\infty I} = \bigcap_{i=1}^{\infty} M_{iI}'$. If $(x_1, y_{1\infty}, y_{2\infty}) \in M_{\infty I}$ then for the corresponding solution $y_\infty(x) = (y_{1\infty}(x), y_{2\infty}(x))$ we have $(x, y_{1\infty}(x), y_{2\infty}(x)) \subset \Omega$ if $x \in (a, b)$ and this yields the conclusion of Theorem 1 as $\lim_{x \to a^+} y_{1\infty}(x) = A$ and $\lim_{x \to b^-} y_{2\infty}(x) = B$. In the case (ii) we have

$$\frac{dW_1(x, y_1)}{dx}\Big|_{M^*} < 0, \quad \frac{dW_2(x, y_2)}{dx}\Big|_{M^*} < 0$$

and the proof can proceed by analogy if, applying the topological method, we reverse the orientation of the axis x.

II. Let the conditions (iii) hold. Then the set Ω_e of all points of agress of Ω with respect to the system (1) is equal to the set Ω_{se} of all points of strict egress, that is

$$\Omega_e = \Omega_{se} = \{(x, y_1, y_2) \colon a < x < b, W_1(x, y_1) = 0, y_2 \in [\psi_1(x), \psi_2(x)]\} \subset \Omega_1.$$

Indeed, if $M^* = (x^*, y_1^*, y_2^*) \in \Omega_e$, then as above we obtain $\frac{dW_1(x, y_1)}{dx}\big|_{M^*} > 0$ and in the case when $M^* \in \Omega_1$ and $W_2(x^*, y_2^*) = 0$ we have

$$\frac{dW_2(x, y_2)}{dx}\Big|_{M^*} < 0.$$

In the next part of the proof we will proceed by analogy with part I. Let $\{x_i\}$ be a decreasing sequence of numbers with the above mentioned properties. We denote

$$S_i(C_2) = \{(x, y_1, y_2) \colon x = x_i, y_1 \in [\varphi_1(x), \varphi_2(x)], y_2 = C_2\},$$

$i = 1, 2, \ldots$ where $C_2 = \text{const}$ and $C_2 \in [\psi_1(x_i), \psi_2(x_i)]$. The set $S_i(C_2) \cap \Omega_e$ is a retract of the set Ω_e as the mapping

$$\pi_{II} \colon (x, y_1, y_2) \in \Omega_e \to (x_i, y_1^0, C_2) \in S_i(C_2) \cap \Omega_e$$

where

$$y_1^0 = \varphi_1(x_i) + (y_1 - \varphi_1(x))\frac{\varphi_2(x_i) - \varphi_1(x_i)}{\varphi_2(x) - \varphi_1(x)}$$

is continuous, identical on $S_i(C_2) \cap \Omega_e$; however it is not a retract of the set $S_i(C_2)$. Then there is a nonempty set $M_i(C_2) \subset S_i(C_2)$ such that if $(x_i, y_{1i}, y_{2i}) \in M_i(C_2)$ then, for the corresponding solution $y_i(x) = (y_{1i}(x), y_{2i}(x))$ of the system (1), the relation $(x, y_{1i}(x), y_{2i}(x)) \in \Omega$ holds if $x \in [x_i, b)$. Varying C_2 on the interval $[\psi_1(x_i), \psi_2(x_i)]$, we construct the closed set $M_{i,II} = \bigcup M_i(C_2)$. Now we consider the set $\chi\{M_{i,II}, [x_i, b)\}$. We have $M'_{i,II} \subset M_{1,II}$ where $M'_{i,II} \equiv \{(x, y_1, y_2)\colon x = x_1,\ y_1 = y_1(x_1),\ y_2 = y_2(x_1),\ (y_1(x), y_2(x)) \in \chi\{M_{i,II}, [x_i, b)\}\}$, and if $i > 2$ then $M'_{i,II} \subset M'_{i-1,II}$ and $M'_{i-1,II}$ is compact. Consequently, the set $M_{\infty,II} = \bigcap_{i=1}^{\infty} M'_{i,II}$ contains at least one point and as above the conclusion of the theorem holds.

In the case (iv) the proof is analogous. □

P r o o f of Theorem 2. We consider only one case as the remaining cases may be proved analogously. Let e.g. the functions $\Phi_i(x, y_2)$, $i = 1, 2$ be decreasing with respect to y_2 on the interval $[\psi_1(x), \psi_2(x)]$, and let the functions $\Psi_i(x, y_1)$, $i = 1, 2$ be increasing with respect to y_1 on the interval $[\varphi_1(x), \varphi_2(x)]$. Then (3), (4) imply

$$\begin{aligned} &\Phi_i(x, \psi_1(x)) > 0, \Phi_i(x, \psi_2(x)) < 0, \\ &\Psi_i(x, \varphi_1(x)) < 0, \Psi_i(x, \varphi_2(x)) > 0 \end{aligned} \tag{8}$$

on (a, b) for $i = 1, 2$. From the conditions of the theorem and from (8) we conclude that the equations

$$\Phi_i(x, Y_{2i}) = 0, \quad \Psi_i(x, Y_{1i}) = 0, \quad i = 1, 2 \tag{9}$$

have unique solutions

$$Y_{2i} = Y_{2i}(x) \in C^1(a, b), \quad Y_{1i} = Y_{1i}(x) \in C^1(a, b), \quad i = 1, 2$$

such that on (a, b)

$$\psi_1(x) < Y_{2i}(x) < \psi_2(x), \quad \varphi_1(x) < Y_{1i}(x) < \varphi_2(x).$$

For $(x, y_1, y_2) \in G$ we define functions

$$\begin{aligned} W_3(x, y_1, y_2) \equiv\ &(y_1 - \varphi_2(x))(y_2 - Y_{21}(x)) + \\ &+ (y_1 - \varphi_1(x))(y_2 - Y_{22}(x)), \end{aligned}$$

$$\begin{aligned} W_4(x, y_1, y_2) \equiv\ &(y_2 - \psi_2(x))(y_1 - Y_{11}(x)) + \\ &+ (y_2 - \psi_1(x))(y_1 - Y_{12}(x)) \end{aligned}$$

and introduce sets

$$
\begin{aligned}
W_1^+ &= \{(x, y_1, y_2) \in \bar{\Omega} \colon a < x < b, W_1(x, y_1) = 0, W_3(x, y_1, y_2) < 0\},\\
W_1^- &= \{(x, y_1, y_2) \in \bar{\Omega} \colon a < x < b, W_1(x, y_1) = 0, W_3(x, y_1, y_2) > 0\},\\
W_1^0 &= \{(x, y_1, y_2) \in \bar{\Omega} \colon a < x < b, W_1(x, y_1) = 0, W_3(x, y_1, y_2) = 0\}
\end{aligned}
$$

and

$$
\begin{aligned}
W_2^+ &= \{(x, y_1, y_2) \in \bar{\Omega} \colon a < x < b, W_2(x, y_2) = 0, W_4(x, y_1, y_2) > 0\},\\
W_2^- &= \{(x, y_1, y_2) \in \bar{\Omega} \colon a < x < b, W_2(x, y_2) = 0, W_4(x, y_1, y_2) < 0\},\\
W_2^0 &= \{(x, y_1, y_2) \in \bar{\Omega} \colon a < x < b, W_2(x, y_2) = 0, W_4(x, y_1, y_2) = 0\}.
\end{aligned}
$$

As above we may verify that, along a solution of (1),

$$
\operatorname{sign} \frac{dW_1(x, y_1)}{dx}\Big|_{M_1^*} = \begin{cases} +1 & \text{if } M_1^* \in W_1^+, \\ -1 & \text{if } M_1^* \in W_1^-, \\ 0 & \text{if } M_1^* \in W_1^0 \end{cases}
$$

and

$$
\operatorname{sign} \frac{dW_2(x, y_2)}{dx}\Big|_{M_2^*} = \begin{cases} +1 & \text{if } M_2^* \in W_2^+, \\ -1 & \text{if } M_2^* \in W_2^-, \\ 0 & \text{if } M_2^* \in W_2^0. \end{cases}
$$

The points of the sets W_i^0, $i = 1, 2$ are the points of exterior tangency of the corresponding integral curves. Indeed, we verify that $\frac{d^2 W_i(x, y_i)}{dx^2}\big|_{M_i^*} > 0$, where $M_i^* \in W_i^0$ and $i = 1, 2$. Direct computations yield

$$
\begin{aligned}
\frac{d^2 W_1(x, y_1)}{dx^2} = {} & 2[f_1(x, y_1, y_2) - \varphi_1'(x)][f_1(x, y_1, y_2) - \varphi_2'(x)] + \\
& + [f_{1x}'(x, y_1, y_2) + f_{1y_1}'(x, y_1, y_2) f_1(x, y_1, y_2) + \\
& + f_{1y_2}'(x, y_1, y_2) f_2(x, y_1, y_2) - \varphi_1''(x)](y_1 - \varphi_2(x)) + \\
& + [f_{1x}'(x, y_1, y_2) + f_{1y_1}'(x, y_1, y_2) f_1(x, y_1, y_2) + \\
& + f_{1y_2}'(x, y_1, y_2) f_2(x, y_1, y_2) - \varphi_1''(x)](y_1 - \varphi_1(x))
\end{aligned}
$$

and in view of the conditions (5) we have

$$
\frac{d^2 W_1(x, y_1)}{dx^2}\Big|_{M_1^*} = \begin{cases} D\Phi_2(x, Y_{22}(x))(\varphi_2(x) - \varphi_1(x)) \\ \qquad \text{if } y_1 = \varphi_2(x), y_2 = Y_{22}(x) \\ \\ D\Phi_1(x, Y_{21}(x))(\varphi_1(x) - \varphi_2(x)) \\ \qquad \text{if } y_1 = \varphi_1(x), y_2 = Y_{21}(x) \end{cases}
$$

and, consequently, $\frac{d^2W_1(x,y_1)}{dx^2}|_{M_1^*} > 0$. By analogy with the previous computations (in view of the conditions (6)) we obtain

$$\frac{d^2W_2(x,y_2)}{dx^2}|_{M_2^*} > 0.$$

Then the set Ω_e is equal to the set Ω_{se} and $\Omega_e = \Omega_{se} = W_1^+ \cup W_2^+$. Let $\{x_i\}$ be the sequence as above. Let an index i be fixed. Denote $S_i = \{(x, y_1, y_2) \in \Omega \cup \Omega_e, x = x_i\}$. Then the set $S_i \cap \Omega_e$ is a retract of the set Ω_e because the mapping

$$\pi_{III}: (x, y_1, y_2) \in \Omega_e \to (x_i, y_1^0, y_2^0) \in S_i \cap \Omega_e$$

where

$$y_1^0 = \varphi_1(x_i), \quad y_2^0 = Y_{21}(x_i) + (y_2 - Y_{21}(x))\frac{\psi_2(x_i) - Y_{21}(x_i)}{\psi_2(x) - Y_{21}(x)} \text{ if } y_1 = \varphi_1(x);$$

$$y_1^0 = \varphi_2(x_i), \quad y_2^0 = \psi_1(x_i) + (y_2 - \psi_1(x))\frac{Y_{22}(x_i) - \psi_1(x_i)}{Y_{22}(x) - \psi_1(x)} \text{ if } y_1 = \varphi_2(x);$$

$$y_1^0 = \varphi_1(x_i) + (y_1 - \varphi_1(x))\frac{Y_{11}(x_i) - \varphi_1(x_i)}{Y_{11}(x) - \varphi_1(x)}, \quad y_2^0 = \psi_1(x_i) \text{ if } y_2 = \psi_1(x)$$

and

$$y_1^0 = Y_{12}(x_i) + (y_1 - Y_{12}(x))\frac{\varphi_2(x_i) - Y_{12}(x_i)}{\varphi_2(x) - Y_{12}(x)}, \quad y_2^0 = \psi_2(x_i) \text{ if } y_2 = \psi_2(x)$$

is continuous and identical on $S_i \cap \Omega_e$. On the other hand, the set $S_i \cap \Omega_e$ is not a retract of the set S_i because, by the above mentioned argument, it is not a retract of a connected set $\tilde{S}_i \subset S_i$ which consists of two points $\alpha \in W_1^+$, $\beta \in W_2^+$ and a continuous curve without self-intersection $l \subset \operatorname{int} S_i$ such that $\bar{l} \cap W_1^+ = \alpha$, $\bar{l} \cap W_2^+ = \beta$. Topological method yields that there is a nonempty set $M_i \subset S_i$ such that if $(x_i, y_{1i}, y_{2i}) \in M_i$ then as in the proof of Theorem 1 the corresponding solution $y_i(x) = (y_{1i}(x), Y_{2i}(x))$ of the system (1) satisfies the relation $(x, y_{1i}(x), y_{2i}(x)) \in \Omega$ on $[x_i, b)$. The remaining part of the proof is analogous to the proof of Theorem 1. □

P r o o f of Theorem 3. The proof is an analogue of the proofs of Theorems 1, 2 and therefore is omitted. □

R e m a r k 1. I. Let all conditions of Theorem 2 be fulfilled except the conditions (5), (6) which are replaced by

$$D\Phi_1(x, Y_{21}(x)) < 0, D\Phi_2(x, Y_{22}(x)) > 0, \quad x \in (a, b)$$

and

$$D\Psi_1(x, Y_{11}(x)) < 0, D\Psi_2(x, Y_{12}(x)) > 0, \quad x \in (a, b), \tag{6'}$$

respectively, and let $Y_{2i}(x)$, $Y_{1i}(x)$, $i = 1, 2$ be solutions of the equations (9). Then the conclusion of Theorem 2 remains valid.

The proof of this remark follow immediately from the proof of Theorem 2.

II. The conclusion of Theorem 3 also remains valid if in its formulation the condition (7) is replaced by (6′).

3. Some applications and comparisons

I. Let the system (1) have the form

$$\begin{aligned} y_1' &= y_2\omega_1(x, y_1), \\ y_2' &= y_1\omega_2(x, y_2). \end{aligned} \tag{10}$$

Then Theorem 2 may be formulated more precisely as

Theorem 4. *Let functions $\varphi_i(x), \psi_i(x)$, $i = 1, 2$ satisfy the conditions (II), $\omega_i(x, y_i) \in C^1(G)$, $i = 1, 2$ and $\omega_1(x, \varphi_i(x)) \neq 0$, $\omega_2(x, \psi_i(x)) \neq 0$, $i = 1, 2$ on (a, b). Let*

$$\varphi_1(x) < \frac{\psi_i'(x)}{\omega_2(x, \psi_i(x))} < \varphi_2(x), \quad i = 1, 2$$

and

$$\psi_1(x) < \frac{\varphi_i'(x)}{\omega_1(x, \varphi_i(x))} < \psi_2(x), \quad i = 1, 2$$

on (a, b). If, moreover,

$$\begin{aligned} (-1)^i \Big[\varphi_i'(x)\frac{\omega_{1x}'(x, \varphi_i(x))}{\omega_1(x, \varphi_i(x))} &+ (\varphi_i'(x))^2\frac{\omega_{1y_1}'(x, \varphi_i(x))}{\omega_1(x, \varphi_i(x))} \\ &+ \varphi_i(x)\omega_1(x, \varphi_i(x))\omega_2\Big(x, \frac{\varphi_i'(x)}{\omega_1(x, \varphi_i(x))}\Big)\Big] > (-1)^i\varphi_i''(x) \end{aligned}$$

and

$$\begin{aligned} (-1)^i \Big[\psi_i'(x)\frac{\omega_{2x}'(x, \psi_i(x))}{\omega_2(x, \psi_i(x))} &+ (\psi_i'(x))^2\frac{\omega_{2y_2}'(x, \psi_i(x))}{\omega_2(x, \psi_i(x))} \\ &+ \psi_i(x)\omega_2(x, \psi_i(x))\omega_1\Big(x, \frac{\psi_i'(x)}{\omega_2(x, \psi_i(x)}\Big)\Big] > (-1)^i\psi_i''(x) \end{aligned}$$

for $i = 1, 2$ on (a, b) then there is at least one solution $y(x) = (y_1(x), y_2(x))$ of the problem (10), (2) such that $(x, y_1(x), y_2(x)) \subset \Omega$ if $x \in (a, b)$.

Proof. It is enough to put $f_1(x, y_1, y_2) \equiv y_2\omega_1(x, y_1)$, $f_2(x, y_1, y_2) \equiv y_1\omega_2(x, y_2)$ and verify all assumptions of the first part of Remark 1. In this case

$$Y_{1i} \equiv \frac{\psi_i'(x)}{\omega_2(x, \psi_i(x))}, \quad i = 1, 2$$

and

$$Y_{2i} \equiv \frac{\varphi_i'(x)}{\omega_1(x, \varphi_i(x))}, \quad i = 1, 2.$$

□

II. Linear case. Let the system (1) have the form

$$
\begin{aligned}
y_1' &= a_{11}(x)y_1 + a_{12}(x)y_2 + \omega_1(x),\\
y_2' &= a_{21}(x)y_1 + a_{22}(x)y_2 + \omega_2(x).
\end{aligned} \tag{11}
$$

Then Theorems 1, 2 may be formulated as follows:

Theorem 5. *Let functions $\varphi_i(x)$, $\psi_i(x)$, $i = 1, 2$ satisfy the conditions (I), $a_{ij}(x) \in C(a,b)$, $i,j = 1, 2$ and $\omega_i(x) \in C(a,b)$, $i = 1, 2$. If*

$$
\begin{aligned}
&\big(a_{11}(x)\varphi_1(x) + a_{12}(x)y_2 + \omega_1(x) - \varphi_1'(x)\big)\times\\
&\qquad\big(a_{11}(x)\varphi_2(x) + a_{12}(x)y_2 + \omega_1(x) - \varphi_2'(x)\big) < 0
\end{aligned}
$$

for each $x \in (a,b)$, $y_2 \in [\psi_1(x), \psi_2(x)]$, and

$$
\begin{aligned}
&\big(a_{21}(x)y_1 + a_{22}(x)\psi_1(x) + \omega_2(x) - \psi_1'(x)\big)\times\\
&\qquad\big(a_{21}(x)y_1 + a_{22}(x)\psi_2(x) + \omega_2(x) - \psi_2'(x)\big) < 0
\end{aligned}
$$

for each $x \in (a,b)$, $y_1 \in [\varphi_1(x), \varphi_2(x)]$ then there is at least one solution $y(x) = (y_1(x), y_2(x))$ of the problem (11), (2) such that $(x, y_1(x), y_2(x)) \subset \Omega$ on (a,b).

Theorem 6. *Let functions $\varphi_i(x)$, $\psi_i(x)$, $i = 1, 2$ satisfy the conditions (II), $a_{ij}(x) \in C^1(a,b)$, $i,j = 1, 2$ and $\omega_i(x) \in C^1(a,b)$, $i = 1, 2$. Let $a_{12}(x)a_{21}(x) \neq 0$,*

$$
\begin{aligned}
&\psi_1(x) < [a_{12}(x)]^{-1}(\varphi_i'(x) - a_{11}(x)\varphi_i(x) - \omega_1(x) < \psi_2(x), \quad i = 1,2,\\
&\varphi_1(x) < [a_{21}(x)]^{-1}(\psi_i'(x) - a_{22}(x)\psi_i(x) - \omega_2(x)) < \varphi_2(x), \quad i = 1,2
\end{aligned}
$$

on (a,b). If, moreover,

$$
\begin{aligned}
(-1)^i &\left[\left(\frac{a_{12}'(x)}{a_{12}(x)} + a_{11}(x) + a_{22}(x)\right)\varphi_i'(x) + \right.\\
&+ \left(a_{11}'(x) - a_{11}(x)\frac{a_{12}'(x)}{a_{12}(x)} + a_{12}(x)a_{21}(x) - a_{11}(x)a_{22}(x)\right)\varphi_i(x) + \\
&\left. + \omega_1'(x) - \left(\frac{a_{12}'(x)}{a_{12}(x)} + a_{22}(x)\right)\omega_1(x) + a_{12}(x)\omega_2(x)\right] > (-1)^i\varphi_i''(x)
\end{aligned}
$$

and

$$
\begin{aligned}
(-1)^i &\left[\left(\frac{a_{21}'(x)}{a_{21}(x)} + a_{11}(x) + a_{22}(x)\right)\psi_i'(x)\right.\\
&+ \left(a_{22}'(x) - a_{22}(x)\frac{a_{21}'(x)}{a_{21}(x)} + a_{12}(x)a_{21}(x) - a_{11}(x)a_{22}(x)\right)\psi_i(x)\\
&\left. + \omega_2'(x) - \left(\frac{a_{21}'(x)}{a_{21}(x)} + a_{11}(x)\right)\omega_2(x) + a_{21}(x)\omega_1(x)\right] > (-1)^i\psi_i''(x)
\end{aligned}
$$

for $i = 1, 2$ on (a, b), then there is at least one solution $y(x = (y_1(x)), y_2(x))$ of the problem (11), (2) such that $(x, y_1(x), y_2(x)) \subset \Omega$ if $x \in (a, b)$.

Theorem 5 follows from Theorem 1 if we put $f_i(x, y_1, y_2) \equiv a_{i1}(x)y_1 + a_{i2}(x)y_2 + \omega_i(x)$, $i = 1, 2$. Theorem 6 follows from the first part of Remark 1 if we put f_i, $i = 1$, 2 as above and note that in this case

$$\begin{aligned} Y_{1i}(x) &\equiv [a_{21}(x)]^{-1}(\psi_i'(x) - a_{22}(x)\psi_i(x) - \omega_2(x)), \\ Y_{2i}(x) &\equiv [a_{12}(x)]^{-1}(\varphi_i'(x) - a_{11}(x)\varphi_i(x) - \omega_1(x)). \end{aligned}$$

Example 1. Let the system (11) have the form

$$\begin{aligned} y_1' &= 10(x-1)^{-2}y_2 + 1, \\ y_2' &= 10x^{-2}y_1 + 1. \end{aligned} \tag{12}$$

Let $a = 0$, $b = 1$ and $A = B = 0$. Then all assumptions of Theorem 6 are fulfilled if we put $\varphi_1(x) = -x$, $\varphi_2(x) = x$, $\psi_1(x) = x - 1$ and $\psi_2(x) = -x + 1$. Consequently, there is at least one solution $y(x) = (y_1(x), y_2(x))$ of the problem (12), (2) such that $|y_1(x)| < x$, $|y_2(x)| < 1 - x$ if $x \in (0, 1)$.

Remark 2. The book [7] contains some theorems on existence and uniqueness of solutions of singular Cauchy-Nicoletti problems for systems of ordinary differential equations. We note that our results are independent of the above mentioned theorems. For example, if we apply Theorem 4.1 from [7, Chapter II, §4, pp. 37–38] to the problem (12), (2), then, moreover, the inequality

$$(10(x-1)^{-2}y_2 + 1) \text{ sign } y_1 \leqslant -a(x)|y_1| + g(x, |y_1|, |y_2|)$$

must be valid on the set

$$\{(x, y_1, y_2) : 0 < x < 1,\ y \in R^2\}$$

where $a(x) \in L(0^+, 1^-)$, $a(x) \geqslant 0$ and

$$\sup\{|g(x, |y_1|, |y_2|)| : |y_1| + |y_2| \leqslant \rho\} \in L(0, 1) \tag{13}$$

for each $\rho \in (0, +\infty)$. In our case these conditions are not fulfilled, because if we put e.g. $a(x) \equiv 0$ and $g(x, |y_1|, |y_2|) \equiv 10(x-1)^{-2}|y_2| + 1$, then the condition (13) is not valid.

Remark 3. Some classes of special singular problems were recently studied in ([16–17]). For example, in [17] the problem

$$\begin{aligned} y_1' &= -(n-1)x^{-1}y_1 + F(y_2, x), \\ y_2' &= y_1, \end{aligned} \tag{14}$$

$$(15) \qquad \begin{aligned} y_1(0) &= 0, \\ y_2(R) &= -\alpha \leqslant 0 \end{aligned}$$

is considered where n is an integer, $n \geqslant 2$, $0 < r < R$, $F: (-\infty, 0) \times (0, \infty) \to (0, \infty)$, $F \in C^1$. The work contains e.g. the following result: Let $0 \leqslant d \leqslant l \leqslant nR^{-1}$, $0 < k < sR^{-1}$ and $0 < F(y_2, x) < (n - lx)k \exp(-lx)$ hold for some constants d, l, R, k, s if $\psi_1(x) \equiv -\alpha - s(R - x)\exp(-dx) < y_2 < \psi_2(x) \equiv -\alpha$ and $0 < x < R$. Then the problem (14), (15) has a one-parametric family of solutions $y(x) = (y_1(x), y_2(x))$, which satisfy the inequalities $\varphi_1(x) \equiv 0 < y_1(x) < \varphi_2(x) \equiv kx\exp(-lx)$, $\psi_1(x) < y_2(x) < \psi_2(x)$ on $(0, R)$.

We note that our problem (1), (2) is more general than the one given above. Theorem 1 (if we put $f_1 \equiv -(n-1)x^{-1}y_1 + F(y_2, x)$, $f_2 \equiv y_2$, $a = 0$, $b = R$, $A = 0$, $B = -\alpha$ and $\varphi_i(x)$, $\psi_i(x)$, $i = 1, 2$ as above) yields that there is at least one solution of the problem (14), (15) with the properties mentioned above. We remark that the above conclusion about the existence of a one-parametric family of such solutions is not correct in the case of $lR = 2$, $n > 2$, $F(y_2, x) \equiv \varepsilon$ =const and $0 < \varepsilon < (n-2)k\exp(-2)$. Then the general solution of the system (14) has the form $y_1(x) = Mx^{1-n} + \varepsilon n^{-1}x$, $y_2(x) = M(2-n)^{-1}x^{2-n} + \varepsilon(2n)^{-1}x^2 + N$ where M, N are arbitrary constants, but only one solution, corresponding to $M = 0$ and $N = -\varepsilon R^2(2n)^{-1} - \alpha$, satisfies the conditions (15).

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Souhrn

SINGULÁRNÍ CAUCHY-NICOLETTIHO ÚLOHA PRO SYSTÉM DVOU OBYČEJNÝCH DIFERENCIÁLNÍCH ROVNIC

JOSEF DIBLÍK

V práci je studována singulární Cauchy-Nicolettiho úloha pro systém dvou obyčejných diferenciálních rovnic. Jsou dokázány nové postačující podmínky řešitelnosti této úlohy. V důkazech je aplikována topologická metoda. V práci je také provedeno porovnání se známými výsledky.

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