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A CHARACTERIZATION OF THE SET OF ALL SHORTEST PATHS IN A CONNECTED GRAPH

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Summary. Let G be a (finite undirected) connected graph (with no loop or multiple edge). The set $\mathscr{S}$ of all shortest paths in G is defined as the set of all paths ξ in G with the property that if ζ is an arbitrary path in G joining the same pair of vertices as ξ, then the lenght of ξ does not exceed the length of ζ. While the definition of $\mathscr{S}$ is based on determining the length of a path, Theorem 1 gives—metaphorically speaking—an "almost non-metric" characterization of $\mathscr{S}$: a characterization in which the length of a path greater than one is not considered. Two other theorems are derived from Theorem 1. One of them (Theorem 3) gives a characterization of geodetic graphs.

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Let G be a (finite undirected) graph (with no loop or multiple edge). We denote by V and E its vertex set and its edge set, respectively. Let G be connected. The letters u, v, w, x, y and z (and the same letters with indices) will be reserved for denoting elements of V. Let $\mathscr{X}$ denote the set of all sequences

$$(0) \qquad u_0, \ldots, u_k$$

where $k \geqslant 0$. Further, instead of (0) we write $u_0 \ldots u_k$. If $\alpha = v_0 \ldots v_m$ and $\beta = w_0 \ldots w_n$ $(m, n \geqslant 0)$, then we write

$$\alpha\beta = v_0 \ldots v_m w_0 \ldots w_n.$$

Let $*$ denote the empty sequence in the sense that $\alpha * = \alpha = * \alpha$ for every $\alpha \in \mathscr{X} \cup \{*\}$. The small letters of Greek alphabet (possibly with indices) will be reserved for denoting elements of $\mathscr{X} \cup \{*\}$.

A sequence $u_0 \dots u_k (k \geqslant 0)$ is called a path in G if $u_0, \dots, u_k$ are mutually distinct and $\{u_j, u_{j+1}\} \in E$ for each j, $0 \leqslant j < k$. Let $\mathscr{P}$ denote the set of all paths in G.

If $\alpha = v_0 \dots v_m (m \geqslant 0)$ is a path in G, then we put $\bar{\alpha} = v_m \dots v_0$, $A\alpha = v_0$, $B\alpha = v_m$ and $\|\alpha\| = m$ (the number $\|\alpha\|$ is called the length of α). If $\mathscr{R} \subseteq \mathscr{P}$, then we denote by $\mathscr{R}_{(u,v)}$ the set of all $\beta \in \mathscr{R}$ with the property that $A\beta = u$ and $B\beta = v$, for every u and v. Since G is connected, $\mathscr{P}_{(x,y)} \neq \emptyset$ for every x and y.

A sequence ξ is called a shortest path in G if $\xi \in \mathscr{P}$ and $\|\xi\| \leqslant \|\zeta\|$ for each $\zeta \in \mathscr{P}_{(A\xi,B\xi)}$. (Note that the notion of a shortest path is closely connected with the notion of the interval function of a graph in the sense of [3]).

Let $\mathscr{S}$ denote the set of all shortest paths in G. Consider arbitrary u and v. Clearly, $\|\varphi\| = \|\psi\|$ for every $\varphi, \psi \in S_{(u,v)}$. We put $d(u,v) = \|\xi\|$ for any $\xi \in \mathscr{S}_{(u,v)}$. (The function d is called the distance function of G. Note that a characterization of the distance function of a connected graph was given in [2]).

The definition of the set $\mathscr{S}$ of all shortest paths in G has been based on determining the length of a path. The following theorem, which is the main result of the present paper, gives—metaphorically speaking—an "almost non-metric" characterization of $\mathscr{S}$; namely a characterization of $\mathscr{S}$ in which $\|\xi\|$ is not considered for any path ξ with the property that $\|\xi\| > 1$.

A graph is called nontrivial if it has at least two vertices. In Theorem 1 (and other theorems of the present paper) all the conventions stated above will be used.

Theorem 1. *Let G be a nontrivial connected graph, and let $\mathscr{R} \subseteq \mathscr{P}$. Then $\mathscr{R} = \mathscr{S}$ if and only if $\mathscr{R}$ fulfils the following Axioms I–VIII (for arbitrary u, v, w, x, y, α, β, γ and δ):*

I If $\{u,v\} \in E$, then $uv \in \mathscr{R}$.

II If $\alpha \in \mathscr{R}$, then $\bar{\alpha} \in \mathscr{R}$.

III If $u\alpha v \in \mathscr{R}$, then $u\alpha \in \mathscr{R}$.

IV If $\alpha u\beta v\gamma, u\delta v \in \mathscr{R}$, then $\alpha u\delta v\gamma \in \mathscr{R}$.

V If $u \neq v$, then there exists φ such that $u\varphi v \in \mathscr{R}$.

VI If $uv\alpha w \in \mathscr{R}$, then $uw \notin \mathscr{R}$.

VII If $uv\alpha x$, $u\beta yx$, $vu\beta y \in \mathscr{R}$, then $v\alpha xy \in \mathscr{R}$.

VIII If $xy, uv\alpha x \in \mathscr{R}$, $u\varphi yx \notin \mathscr{R}$ for all φ and $uv\psi y \notin \mathscr{R}$ for all ψ, then $v\alpha xy \in \mathscr{R}$.

Proof. It is routine to prove that if $\mathscr{R} = \mathscr{S}$, then $\mathscr{R}$ fulfils Axioms I–VIII.

Conversely, let $\mathscr{R}$ fulfil Axioms I–VIII. Consider an arbitrary non-negative integer m which does not exceed the diameter of G. We will prove the following two statements:

(1_m) $\quad \mathscr{S}_{(w,z)} \subseteq \mathscr{R}_{(w,z)}$ for every pair of w and z such that $d(w,z) \leqslant m$

and

(2_m) $\quad \mathscr{R}_{(w,z)} \subseteq \mathscr{S}_{(w,z)}$ for every pair of w and z such that $d(w,z) \leqslant m$.

We proceed by induction on m.

The case when $m = 0$ follows from Axioms I and III (or from Axioms V and III). The case when $m = 1$ follows from Axioms I and VI.

Let now $m \geqslant 2$. The proof will be divided into two parts. In part A, combining (1_{m-1}) and (2_{m-1}) we will prove that (1_m) holds. In part B, combining (1_m) and (2_{m-1}) we will prove that (2_m) holds.

A. Consider arbitrary u and v such that $d(u,v) = m$. Obviously, $\mathscr{S}_{(u,v)} \neq \emptyset$. Consider an arbitrary $\xi \in \mathscr{S}_{(u,v)}$. We want to prove that $\xi \in \mathscr{R}$.

As follows from Axiom V, there exists $\zeta \in \mathscr{R}_{(u,v)}$. We distinguish the following cases and subcases.

A.1. Let ξ and ζ have a common vertex z different from u and v. Then

(3) there exist $\alpha_1, \alpha_2, \beta_1, \beta_2$ such that $\xi = u\alpha_1 z\alpha_2 v$ and $\zeta = u\beta_1 z\beta_2 v$.

As follows from (1_{m-1}), $u\alpha_1 z, z\alpha_2 v \in \mathscr{R}$. According to Axiom IV, $u\alpha_1 z\beta_2 v \in \mathscr{R}$. Similarly, we see that $\xi = u\alpha_1 z\alpha_2 v \in \mathscr{R}$.

A.2. Let ξ and ζ have no common vertex different from u and v. Put $n = ||\zeta||$. Obviously, $n \geqslant m = ||\zeta||$. There exist mutually distinct $u_1, \ldots, u_m, v_1, \ldots, v_n$ such that

(4) $$\xi = u_1 \ldots u_m v_1 \quad \text{and} \quad \zeta = u_1 v_n \ldots v_1.$$

Clearly, $u_1 = u$ and $v_1 = v$.

Recall that we want to prove that $\xi \in \mathscr{R}$. Suppose to the contrary that $\xi \notin \mathscr{R}$.

Put $\xi_1 = \xi$, $\zeta_1 = \zeta$,

$$\xi_i = v_{n-i+2} \ldots v_n u_1 \ldots u_{m-i+2} \text{ and } \zeta_i = v_{n-i+2} \ldots v_1 u_m \ldots u_{m-i+2}$$

for each $i \in \{2, \ldots, m+1\}$. Clearly,

(5) $$\zeta_{m+1} = v_{n-m+1} \ldots v_1 u_m \ldots u_1.$$

If $\zeta_{m+1} \in \mathscr{R}$, then Axioms II and III imply that $\xi = u_1 \ldots u_m v_1 \in \mathscr{R}$, which is a contradiction. Hence $\zeta_{m+1} \notin \mathscr{R}$.

Since $\xi_1 \notin \mathscr{R}$ and $\zeta_1 \in \mathscr{R}$, there exists $j \in \{1, \ldots, m\}$ such that (a) $\xi_j \notin \mathscr{R}$, $\zeta_j \in \mathscr{R}$ and (b) either $\xi_{j+1} \in \mathscr{R}$ or $\zeta_{j+1} \notin \mathscr{R}$. There exist mutually distinct $x_1, \ldots, x_m$, $y_1, \ldots, y_n$ such that

(6) $$\xi_j = x_1 \ldots x_m y_1 \quad \text{and} \quad \zeta_j = x_1 y_n \ldots y_1.$$

Clearly, $\{x_1, \ldots, x_m, y_1, \ldots, y_n\} = \{u_1, \ldots, u_m, v_1, \ldots, v_n\}$. It is obvious that $d(x_1, y_1) \leqslant m$.

Let first $d(x_1, y_1) < m$. Since $\zeta_j \in \mathscr{R}$, it follows from (2_{m-1}) that $\zeta_j \in S$. Hence $m > d(x_1, y_1) = ||\zeta_j|| = n \geqslant m$, which is a contradiction.

Let now $d(x_1, y_1) = m$. Then $\xi_j \in \mathscr{S}$. As follows from (1_{m-1}), $x_1 \ldots x_m \in \mathscr{R}$. Since $\xi_j \notin \mathscr{R}$, Axiom IV implies that $x_1 \varphi x_m y_1 \notin \mathscr{R}$ for all φ.

A.2.1. Suppose there exists ψ such that $x_1 y_n \psi x_m \in \mathscr{R}$. Since $\xi_j \in \mathscr{S}$, we have $d(x_1, x_m) = m - 1$. According to (2_{m-1}), $x_1 y_n \psi x_m \in \mathscr{S}$. Thus $y_n \psi x_m \in \mathscr{S}$ and $||y_n \psi x_m|| = m - 2 = d(y_n, x_m)$. This means that $d(y_n, y_1) \leqslant m - 1$. Since $y_n \ldots y_1 \in \mathscr{R}$, it follows from (2_{m-1}) that $y_n \ldots y_1 \in \mathscr{S}$. If $d(y_n, y_1) \leqslant m - 2$, then $n \leqslant m - 1$, which is a contradiction.

Assume that $d(y_n, y_1) = m - 1$. Since $y_n \psi x_m \in \mathscr{S}$ and $||y_n \psi x_m|| = m - 2$, we have $y_n \psi x_m y_1 \in \mathscr{S}$. Since $d(y_n, y_1) = m - 1$, it follows from (1_{m-1}) that $y_n \psi x_m y_1 \in \mathscr{R}$. Since $x_1 y_n \ldots y_1 \in \mathscr{R}$, Axiom IV implies that $x_1 y_n \psi x_m y_1 \in \mathscr{R}$. Since $x_1 \ldots x_m \in \mathscr{R}$, Axiom IV implies that $\xi_j = x_1 \ldots x_m y_1 \in \mathscr{R}$, which is a contradiction.

A.2.2. Suppose $x_1 y_n \psi x_m \notin \mathscr{R}$ for all ψ. Since $x_1 \varphi x_m y_1 \notin \mathscr{R}$ for all φ and $x_1 y_n \ldots y_1 \in \mathscr{R}$, it follows from Axiom VIII that $\zeta_{j+1} = y_n \ldots y_1 x_m \in \mathscr{R}$. The fact that $\zeta_{j+1} \in \mathscr{R}$ implies that $\xi_{j+1} = y_n x_1 \ldots x_m \in \mathscr{R}$. Since $x_1 y_n \ldots y_1$, $y_n \ldots y_1 x_m \in \mathscr{R}$, it follows from Axiom VII that $\xi_j = x_1 \ldots x_m y_1 \in \mathscr{R}$, which is a contradiction.

Thus $\xi \in \mathscr{R}$ and (1_m) holds.

B. Consider arbitrary u and v such that $d(u, v) = m$. According to Axiom V, $\mathscr{R}_{(u,v)} \neq \emptyset$. Consider an arbitrary $\zeta \in \mathscr{R}_{(u,v)}$. We want to prove that $\zeta \in \mathscr{S}$. Clearly, there exists $\xi \in \mathscr{S}_{(u,v)}$. We distinguish the following cases and subcases.

B.1. Let ξ and ζ have a common vertex z different from u and v. Then (3) holds. As follows from (2_{m-1}), $u\beta_1 z, z\beta_2 v \in \mathscr{S}$. We can see that $\zeta = u\beta_1 z\beta_2 v \in \mathscr{S}$.

B.2. Let ξ and ζ have no common vertex different from u and v. Put $n = ||\zeta||$. Obviously, $n \geqslant m$. There exist mutually distinct $u_1, \ldots, u_m, v_1, \ldots, v_n$ such that (4) holds. We wish to prove that $n = m$, and therefore, $\zeta \in \mathscr{S}$. Suppose to the contrary that $n > m$.

Define $\xi_1, \zeta_1, \ldots, \xi_{m+1}, \zeta_{m+1}$ in the same way as in A.2. Note that for ζ_{m+1}, (5) holds. Clearly, $v_1 \ldots v_n u_1 \in \mathscr{R}$. If $\zeta_{m+1} \in \mathscr{R}$, then Axiom IV implies that

$$v_{n-m+1} \ldots v_2 v_1 v_2 \ldots v_n u_1 \in \mathscr{R},$$

which contradicts the fact that $\mathscr{R} \subseteq \mathscr{P}$. Hence $\zeta_{m+1} \notin \mathscr{R}$.

Since $\xi_1 \in \mathscr{S}$ and $\zeta_1 \in \mathscr{R}$, there exists $j \in \{1, \ldots, m\}$ such that (a) $\xi_j \in \mathscr{S}, \zeta_j \in \mathscr{R}$ and (b) either $\xi_{j+1} \notin \mathscr{S}$ or $\zeta_{j+1} \notin \mathscr{R}$. There exist mutually distinct $x_1, \ldots, x_m$, $y_1, \ldots, y_n$ such that (6) holds. According to (1_m), $x_1, \ldots x_m y_1 \in \mathscr{R}$.

B.2.1. Suppose $d(y_n, x_m) \leqslant m - 1$. Then $d(y_n, y_1) \leqslant m$. If $d(y_n, y_1) \leqslant m - 1$, then (2_{m-1}) implies that $y_n \ldots y_1 \in \mathscr{S}$, and therefore $n \leqslant m$, which is a contradiction. Thus we have $d(y_n, y_1) = m$. Since $d(y_n, x_m) \leqslant m - 1$, we see that $d(y_n, x_m) = m - 1$ and there exists φ such that $y_n \varphi x_m y_1 \in \mathscr{S}$.

According to (1_m), $y_n\varphi x_m y_1 \in \mathscr{R}$. Since $x_1 y_n \dots y_1 \in \mathscr{R}$, Axiom IV implies that $x_1 y_n \varphi x_m y_1 \in \mathscr{R}$. This means that $x_1 y_n \varphi x_m \in \mathscr{R}$. Since $\xi_j \in \mathscr{S}$, we have $d(x_1, x_m) = m - 1$. As follows from (2_{m-1}), $x_1 y_n \varphi x_m \in \mathscr{S}$. We get $y_n \varphi x_m \in \mathscr{S}$ and therefore, $d(y_n, x_m) = \|y_n \varphi x_m\| = m - 2$, which is a contradiction.

B.2.2. Suppose $d(y_n, x_m) \geqslant m$. Then $d(y_n, x_m) = m$ and $y_n x_1 \dots x_m \in \mathscr{S}$. By virtue of (1_m), $y_n x_1 \dots x_m \in \mathscr{R}$. Since $x_1 \dots x_m y_1$, $x_1 y_n \dots y_1 \in \mathscr{R}$, it follows from Axiom VII that $y_n \dots y_1 x_m \in \mathscr{R}$. Clearly, $\xi_{j+1} = y_n x_1 \dots x_m$ and $\zeta_{j+1} = y_n \dots y_1 x_m$. We have $\xi_{j+1} \in \mathscr{S}$ and $\zeta_{j+1} \in \mathscr{R}$, which is a contradiction.

Thus $\zeta \in \mathscr{S}$ and (2_m) holds. The proof of the theorem is complete. □

If a nontrivial connected graph G is bipartite, then a simpler "almost non-metric" characterization of $\mathscr{S}$ can be given.

Theorem 2. *Let G be a nontrivial connected bipartite graph, and let $\mathscr{R} \subseteq \mathscr{P}$. Then $\mathscr{R} = \mathscr{S}$ if and only if $\mathscr{R}$ fulfils Axioms I–IV and the following Axiom IX (for arbitrary u, v, w, α, β, γ and δ):*

IX If $vw \in \mathscr{R}$ and $v \neq u \neq w$, then there exists φ such that either $u\varphi vw \in \mathscr{R}$ or $u\varphi wv \in \mathscr{R}$.

Proof. Let $\mathscr{R} = \mathscr{S}$. Theorem 1 implies that $\mathscr{R}$ fulfils Axioms I–IV. It is routine to show that $\mathscr{R}$ fulfils Axiom IX.

Conversely, let $\mathscr{R}$ fulfil Axioms I–IV and IX. In the sections of the proof designated as (v)–(viii) we will show that $\mathscr{R}$ fulfils Axioms V–VIII, respectively.

(v) Consider arbitrary u and v such that $u \neq v$. We want to prove that $\mathscr{R}_{(u,v)} \neq \emptyset$. If $d(u, v) = 1$, then the result follows from Axiom I. Let $d(u, v) \geqslant 2$. There exists w such that $vw \in \mathscr{R}$. According to Axiom IX, there exists φ such that either $u\varphi wv \in \mathscr{R}$ or $u\varphi vw \in \mathscr{R}$. If $u\varphi wv \in \mathscr{R}$, then $\mathscr{R}_{(u,v)} \neq \emptyset$. If $u\varphi vw \in \mathscr{R}$, then the same result follows from Axiom III. Hence $\mathscr{R}$ fulfils Axiom V.

(vi) Consider arbitrary u, v, w and α such that $uv\alpha w \in \mathscr{R}$. We want to prove that $uw \notin \mathscr{R}$. On the contrary, let $uw \in \mathscr{R}$. As follows from Axiom IX, there exists φ such that either $v\varphi uw \in \mathscr{R}$ or $v\varphi wu \in \mathscr{R}$. Let first $v\varphi uw \in \mathscr{R}$. Since $uv\alpha w \in \mathscr{R}$, Axiom IV implies that $v\varphi uv\alpha w \in \mathscr{R}$, which contradicts the fact that $\mathscr{R} \subseteq \mathscr{P}$. Let now $v\varphi wu \in \mathscr{R}$. Combining Axioms II and IV, we get $uw\bar{\varphi}v\alpha w \in \mathscr{R}$, which is a contradiction, too. We get $uw \notin \mathscr{R}$. Hence $\mathscr{R}$ fulfils Axiom VI.

(vii) Consider arbitrary u, v, x, y, α and β such that $uv\alpha x$, $u\beta yx$, $vu\beta y \in \mathscr{R}$. Axiom IX implies that there exists φ such that either $v\varphi yx \in \mathscr{R}$ or $v\varphi xy \in \mathscr{R}$. Let first $v\varphi yx \in \mathscr{R}$. Axiom IV implies that $vu\beta yx \in \mathscr{R}$. Since $uv\alpha x \in \mathscr{R}$, Axiom IV implies that $vuv\alpha x \in \mathscr{R}$, which is a contradiction. Let now $v\varphi xy \in \mathscr{R}$. Since $uv\alpha x \in \mathscr{R}$, it follows from Axioms II–IV that $v\alpha xy \in \mathscr{R}$. Hence $\mathscr{R}$ fulfils Axiom VII.

(viii) Assume that there exist u, v, x, y and α such that $xy, uv\alpha x \in \mathscr{R}$, $u\varphi yx \notin \mathscr{R}$ for all φ and $uv\psi y \notin \mathscr{R}$ for all ψ. Combining Axioms II and IX, we get that there exist β and γ such that $u\beta xy$, $vu\gamma y \in \mathscr{R}$. Axiom IV implies that $vu\beta xy \in \mathscr{R}$. Since $uv\alpha x \in \mathscr{R}$, it follows from Axiom IV that $vuv\alpha xy \in \mathscr{R}$, which is a contradiction. This means that $\mathscr{R}$ fulfils Axiom VIII.

As follows from Theorem 1, $\mathscr{R} = \mathscr{S}$, which completes the proof. □

Note that a result very similar to Theorem 2 was originally proved by the present author in [4].

A graph G is called geodetic if it is connected and there exists exactly one path in $\mathscr{S}_{(u,v)}$, for each pair of vertices u and v. (Cf. [1], p. 55, for example).

We will give a characterization of geodetic graphs:

Theorem 3. *A nontrivial connected graph G is geodetic if and only if there exists $\mathscr{R} \subseteq \mathscr{P}$ such that $\mathscr{R}$ fulfils Axioms I, II, III and the following Axioms X and XI (for arbitrary u, v, x, y and α):*

X If $u \neq v$, then there exists exactly one φ such that $u\varphi v \in \mathscr{R}$.

XI If $xy, uv\alpha x \in \mathscr{R}$, $y \neq v$ and $uv\psi y \notin \mathscr{R}$ for all ψ, then $v\alpha xy \in \mathscr{R}$.

Proof. Let G be geodetic. Put $\mathscr{R} = \mathscr{S}$. Then it is easy to see that $\mathscr{R}$ fulfils Axioms I, II, III, X and XI.

Conversely, suppose there exists $\mathscr{R} \subseteq \mathscr{P}$ such that $\mathscr{R}$ fulfils Axioms I, II, III, X and XI. Axiom X implies that $\mathscr{R}$ fulfils Axioms IV, V and VI. Axiom XI implies that $\mathscr{R}$ fulfils Axiom VIII.

Suppose there exist u, v, x, y, α and β such that $uv\alpha x$, $u\beta yx$, $vu\beta y \in \mathscr{R}$. According to Axiom X, $uv\alpha x = u\beta yx$. Hence there exists γ such that $uv\gamma yx \in \mathscr{R}$. Axioms II and III imply that $v\gamma y \in \mathscr{R}$. According to Axiom X, $v\gamma y = vu\beta y$. Therefore $uvu\beta yx \in \mathscr{R}$, which is a contradiction. This means that $\mathscr{R}$ fulfils Axiom VII.

It follows from Theorem 1 that $\mathscr{R} = \mathscr{S}$. Axiom X implies that G is geodetic, which completes the proof. □

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