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ON MINIMAL IDEALS IN SEMIGROUPS WITH RESPECT TO THEIR SUBSETS, I

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Summary. In the paper, the following concept are defined:
(i) a minimal left (right, two-sided) ideal with respect to a subset B of a semigroup S,
(ii) a kernel with respect to a subset B of a semigroup S,
and their basic properties are investigated.

Keywords: minimal left (right, two-sided) ideal with respect to a subset B of a semigroup S, kernel with respect to a subset B of a semigroup S, partial group

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In many papers concerning the algebraic theory of semigroups, properties of the following types of ideals in semigroups are investigated:

1) the minimal left (right, two-sided) ideals (see for example [3], [5], [6], [7], [8], [9], [11]);

2) the 0-minimal left (right, both-sided) ideals (see for example [4]);

3) the minimal quasi-ideals (see for example [12]);

4) the simple left (right, two-sided) ideals (see for example [8], [10]).

In this paper, the following concepts are defined:

a) a minimal left (right, two-sided) ideal with respect to a subset B of a semigroup S;

b) a kernel with respect to a subset B of a semigroup S.

An example of two semigroups, each satisfying exactly one of the following two properties, is given:

a) S_1 does not contain any minimal left (right, two-sided) ideal (it does not have a kernel), and it contains infinitely many mutually different subsets such that with respect to each of them S_1 contains minimal left (right, two-sided) ideals and the kernel.

b) S_2 contains at least one minimal left (right, two-sided) ideal, hence it contains the kernel, nonetheless it does not contain any simple left (right, two-sided) ideal and contains infinitely many mutually different subsets such that with respect to each of them S_2 has minimal, left (right, two-sided) ideals (none of them is a minimal left (right, two-sided) ideal of S) and with respect to each of them it also has the kernel.

Let S be a semigroup and let $\emptyset \neq B \subseteq S$. In this paper, basic properties of a minimal left (right, two-sided) ideal with respect to the set B of the semigroup S (under certain conditions on a subset B of a semigroup S) are investigated. The main result of this paper is Theorem 3, which is a generalization of Corollary 9 (see [3]).

After the basic assertions on minimal left (right, two-sided) ideals with respect to a set B of a semigroup S, some well known corollaries will be given, e.g. on minimal left, on 0-minimal right (if the semigroup S is a semigroup with the zero 0) and on simple left (if the semigroup S contains the kernel) ideals of a semigroup S.

Throughout the paper, the following notation will be used:

$X \subset Y$ will mean that X is a proper subset of the set Y (to distinguish it from $X \subseteq Y$ which means either $X \subset Y$ or $X = Y$).

Let S be a semigroup and let $\emptyset \neq A \subseteq S$. $L(A)$ $(R(A),\ J(A))$ is the left (right, two-sided) ideal generated by A. If $a \in S$ and $A = \{a\}$, then instead of $L(\{a\})$ we will write $L(a)$.

$\mathscr{L}$ $(\mathscr{R},\ \mathscr{J})$ is the Green $\mathscr{L}$-equivalence ($\mathscr{R}$-equivalence, $\mathscr{J}$-equivalence) on S (see [1]).

$S/\mathscr{L}$ $(S/\mathscr{J},\ S/\mathscr{R})$ is the set of all $\mathscr{L}$-classes ($\mathscr{J}$-classes, $\mathscr{R}$-classes) which belong to the equivalence $\mathscr{L}$ $(\mathscr{J}, \mathscr{R})$ on S.

L_a $(J_a,\ R_a)$ is the element of $S/\mathscr{L}$ $(S/\mathscr{J},\ S/\mathscr{R})$ containing the element $a \in S$.

$\leqslant$ is a partial ordering on $S/\mathscr{L}$ $(S/\mathscr{J},\ S/\mathscr{R})$ (see [1]). We will write $R_a < R_b$ provided $R_a \leqslant R_b$ and $R_a \neq R_b$.

$NL(A)$ $(N(A),\ NR(A))$ will denote the set of all elements $x \in S$ such that for each $a \in A$: $L_a \nleq L_x$ $(J_a \nleq J_x,\ R_a \nleq R_x)$ (see [13]).

L_B (R_B) will denote the set $\cup\{L_b \mid b \in B\}$ $(\cup\{R_b \mid b \in B\})$.

$\overline{A}$ is the set $S \setminus A$.

We will use the following assertion: Let S be a semigroup and let $\emptyset \neq A \subseteq S$. Then (see [13]):

If $NL(A) \neq \emptyset$, $(N(A) \neq \emptyset,\ NR(A) \neq \emptyset)$, then $NL(A)$ $(N(A),\ NR(A))$ is a left (two-sided, right) ideal in S.

In what follows the definitions of new concepts will be mostly omitted and the theorems about them will be given only for left ideals of S. Theorems on left ideals of S will be referred to (without further notice) in case analogous theorems (concepts) concerning right (two-sided) ideals of S should be used.

Definition 1. Let S be a semigroup and let $\emptyset \neq B \subseteq S$. A left ideal L of a semigroup S will be called a *minimal left ideal* with respect to a subset B of a semigroup S (or in S), if $L \cap B \neq \emptyset$ and there is no left ideal L' in S such that $L' \subset L$ and $L' \cap B \neq \emptyset$.

Remark 1. a) If we put $B = S$ ($B = S \setminus \{0\}$) in Definition 1, then we have for each $\emptyset \neq L \subseteq S$:

L is a minimal left (0-minimal left) ideal with respect to a subset B of the semigroup S (of the semigroup S with 0) if and only if L is a minimal left ideal of the semigroup S (of the semigroup with 0).

b) Let S be a semigroup with the kernel K and let $K \neq S$. A left ideal L of the semigroup S is called a simple left ideal of the semigroup S, if $K \subset L$ and there is no left ideal L' in S such that $K \subset L' \subset L$ (see [10]). Put $B = S \setminus K$. In the paper it is shown how to get theorems on minimal left ideals with respect to the subset B of the semigroup S using theorems on simple left ideals of the semigroup with the kernel K.

Example 1. Let S_1 be the set of all real numbers $x \in \mathbb{R}$ such that $0 < x < 1$. A binary operation on S_1 will be defined in the following way: $xy = \min\{x, y\}$ for each two elements $x, y \in S_1$. Then S_1 is a semigroup.

Let $S_2 = \{a, b, c\}$ and let a binary operation on S_2 be defined in the following way:

	a	b	c
a	a	b	c
b	a	b	c
c	a	b	c

Then S_2 is a semigroup. Let $S_3 = S_1 \times S_2$ be the direct product of semigroups S_1, S_2. For each $\alpha \in (0, 1)$ put $M^\alpha = \{y \mid y \in \mathbb{R}$ and $\alpha \leqslant y < 1\}$ and $B^\alpha = M^\alpha \times S_2$. Then for each $\alpha \in (0, 1)$ the set $\{L(\alpha, u) \mid u \in S_2\}$ is the set of all minimal left ideals with respect to the set B^α of the semigroup S_3. It is easy to prove that the semigroup S_3 contains no minimal two-sided ideal. In this example instead of the set S_1 take a set S_{10} of all real numbers $x \in \mathbb{R}$ such that $0 \leqslant x < 1$. Define the binary operation on S_{10} analogously as on S_1. Then S_{10} is a semigroup. Let $S_{30} = S_{10} \times S_2$ be the direct product of semigroups S_{10}, S_2. Then we can easily prove that the semigroup S_{30} has the following properties:

a) S_{30} contains at least one minimal left and one minimal right ideal and hence S_{30} has the kernel,

b) S_{30} does not contain any simple left (two-sided) ideal,

c) S_{30} contains infinitely many mutually different subsets $(B^\alpha, \alpha \in (0,1))$ such that with respect to each of them S_{30} has minimal left ideals (none of them is a minimal ideal of S).

For each $\beta \in (0,1)$ put $N^\beta = \{y \mid y \in \mathbb{R} \text{ and } 0 < \beta < y < 1\}$ and $B^\beta = N^\beta \times S_2$. Then for each $\beta \in (0,1)$ the set of all minimal left (right, two-sided) ideals with respect to the set B^β of the semigroup S_3 is empty.

Remark 2. By means of an example it can be shown that there exists a semigroup having a kernel and containing no minimal left (right), simple left ideal, while containing infinitely many mutually different subsets such that with respect to each of them it has both a minimal left ideal and the kernel.

Theorem 1. *Let S be a semigroup and let $\emptyset \neq B \subseteq S$. Then for each $\emptyset \neq L \subseteq S$ the following holds:*

(a) *L is a minimal left ideal with respect to the subset B of the semigroup S if and only if there exists an element $b \in B$ such that $L = L(b)$ and L_b is a minimal element of $\overline{NL(B)}/\mathscr{L}$.*

(b) *For each $b \in B$: $L(b)$ is a minimal left ideal with respect to the subset B of the semigroup S if and only if $L(b) \cap \overline{NL(B)} = L_b$.*

Proof. (a) I. Suppose that L is a minimal left ideal with respect to the subset B of the semigroup S. Let $b \in L \cap B$. Then $L(b) \subseteq L$ and $L(b) \cap B \neq \emptyset$. It follows from the assumption that $L = L(b)$. Let $a \in \overline{NL(B)}$ and let $L_a \leqslant L_b$. Then there exists an element $c \in B$ such that $L_c \leqslant L_a$. This implies that $L(c) = L(b)$, hence $L_a = L_b$. Therefore, L_b is a minimal element of $\overline{NL(B)}/\mathscr{L}$.

II. Let $b \in B$, $L = L(b)$ and let L_b be a minimal element of $\overline{NL(B)}/\mathscr{L}$. Let L' be a left ideal of the semigroup S such that $L' \subset L$ and $L' \cap B \neq \emptyset$. Let $c \in L' \cap B$. Hence $L(c) \subseteq L'(c) \subset L(b)$. Therefore $L_c < L_b$ and $L_b, L_c \in \overline{NL(B)}/\mathscr{L}$. This is a contradiction with the fact that L_b is a minimal element of $\overline{NL(B)}/\mathscr{L}$. Therefore $L(b)$ is a minimal ideal with respect to the subset B of the semigroup S.

(b) Let $b \in B$.

I. Suppose that $L(b)$ is a minimal left ideal with respect to the subset B of the semigroup S. Using (a) we get that $L_b \subseteq L(b) \cap \overline{NL(B)}$. Suppose that there is an element $d \in L(b) \cap \overline{NL(B)}$ such that $d \notin L_b$. Hence $L_d \subseteq \overline{NL(B)}$ and $L_d < L_b$. This is a contradiction with the fact that L_b is a minimal element of $\overline{NL(B)}/\mathscr{L}$. Therefore $L(b) \cap \overline{NL(B)} \subseteq L_b$.

II. Suppose that $L(b) \cap \overline{NL(B)} = L_b$. Further suppose that there exists a left ideal L of the semigroup S such that $L \subset L(b)$ and $L \cap B \neq \emptyset$. Then $L \cap L_b \neq \emptyset$. Hence

$L(b) \subset L$, which contradicts $L \subset L(b)$. Therefore $L(b)$ is a minimal left ideal with respect to the subset B of the semigroup S. □

Corollary 1. *Let S be a semigroup. Then for each $\emptyset \neq L \subseteq S$ the following holds:*

(a) *L is a minimal left ideal in S if and only if there exists an element $b \in S$ such that $L = L(b)$ and L_b is a minimal element in $S/\mathscr{L}$.*

(b) *For each $b \in S$: $L(b)$ is a minimal left ideal in S if and only if $L(b) = L_b$.*

Proof. Put $B = S$. Then $\overline{NL(B)} = S$. Using Theorem 1 we get Corollary 1. □

Corollary 2. *Let S be a semigroup S with zero 0. Put $B = S \setminus \{0\}$. Then for each $\emptyset \neq L \subseteq S$ the following holds:*

(a) *L is a 0-minimal left ideal of the semigroup S if and only if there exists an element $b \in B$ such that $L = L(b)$ and L_b is a minimal element in $B/\mathscr{L}$.*

(b) *For each $b \in B$, $L(b)$ is a 0-minimal left ideal of the semigroup S if and only if $L(b) = \{0\} \cup L_b$.*

Proof. From the assumption we have that $B = S \setminus \{0\}$. Then $\overline{NL(B)} = S \setminus \{0\}$. Using Theorem 1 we get Corollary 2. □

Corollary 3. *Let S be a semigroup with the kernel K and let S be not simple. Put $B = S \setminus K$. Then for each $L \subseteq S$ the following holds:*

L is a simple left ideal in S if and only if there exists an element $b \in B$ such that $L = K \cup L(b)$ and $L(b)$ is a minimal left ideal with respect to the subset B of the semigroup S.

Proof. I. Let L be a simple left ideal in S. Let $b \in L \setminus K$. Then $K \cup L(b) \subseteq L$ and $K \cup L(b)$ is a left ideal containing the kernel K. Then the assumption implies that $L = K \cup L(b)$. Suppose that L_b is not a minimal element of $B/\mathscr{L}$. There exists an element $c \in B$ such that $L_c < L_b$. Then $L(b) \setminus L_b \neq \emptyset$ and $\big(L(b) \setminus L_b\big) \cap B \neq \emptyset$. Then $L_1 = K \cup \big(L(b) \setminus L_b\big)$ is a left ideal of the semigroup S and $K \subset L_1 \subset L$. This is a contradiction with the fact that L is a simple left ideal of S. It follows that L_b is a minimal element in $B/\mathscr{L}$. Using Theorem 1 we get that $L = K \cup L(b)$ and $L(b)$ is a minimal ideal with respect to the subset B of the semigroup S.

II. Let $L = K \cup L(b)$ and let $L(b)$ be a minimal left ideal with respect to the subset B of the semigroup S. Suppose that there exists a left ideal L' in S such that $K \subset L' \subseteq L$. Let $d \in L' \cap L_b$. Then $L_b = L_d \subseteq L(d) \subseteq L'$. We get $L \subseteq L'$. Hence $L' = L$. Therefore L is a simple left ideal of S. □

Definition 2. We will say that a semigroup S satisfies the condition m_{LB} (m_B) if $\emptyset \neq B \subseteq S$ and the set of all minimal left (two-sided) ideals with respect to the subset B in S is nonempty.

Let S be a semigroup and let $\emptyset \neq B \subseteq S$. A minimal left ideal L with respect to the subset B of the semigroup S will be called a left mB-ideal of the semigroup S if L has the following property: for each left ideal L' of S the following holds: If $L' \subset L$ and $c \in S$ then $L'c \cap \overline{NL(B)} = \emptyset$.

Lemma 1. *Let a semigroup S satisfy the condition m_{LB}. Let either $NL(B) = \emptyset$, or let $NL(B)$ be a two-sided ideal of S. Then its every minimal left ideal with respect to the subset B of the semigroup S is a left mB-ideal of the semigroup S.*

The proof is clear.

Let S be a semigroup without zero (with zero 0). Put $B = S$ ($B = S \setminus \{0\}$). Let S satisfy the condition m_{LB}.Then each minimal (0-minimal) left ideal with respect to the set $B = S$ ($B = S \setminus \{0\}$) of the semigroup is a left mB-ideal of S.

It can be shown by means of an example that there is a semigroup S and its nonempty subset $B \subsetneq S$ with the following properties:

a) $\overline{NL(B)} \neq S$ and $NL(B)$ is not a two-sided ideal of S,

b) S satisfies the condition m_{LB},

c) S contains a minimal left ideal with respect to the subset B of S that is its left mB-ideal and contains a minimal left ideal with respect to the subset B of S that is left mB-ideal of S.

E x a m p l e 2. Let $S = \{0, \alpha, \beta, u, v, e\}$. Define on S a binary operation as follows:

	α	β	u	v	e
α	α	0	0	v	e
β	0	β	u	0	0
u	u	0	0	β	u
v	0	v	e	0	0
e	e	0	0	v	e

Then S is a semigroup. Put $B = \{\alpha, \beta\}$. Then $\overline{NL(B)} \neq \emptyset$, $\overline{NL(B)}$ is not a two-sided ideal of S. S satisfies the condition m_{LB} and constains a minimal left ideal with respect to the subset of S that is not its left mB-ideal of S and contains a minimal left ideal that is its left mB-ideal of S.

Lemma 2. *Let a semigroup S satisfy the condition m_{LB}. Let L be a left mB-ideal of the semigroup S. Then for each $c \in \overline{NL(B)}$ the following holds: If $Lc \cap \overline{NL(B)} \neq \emptyset$, then Lc is a minimal left ideal with respect to the subset B of the semigroup S.*

Proof. Let $c \in \overline{NL(B)}$, and let $Lc \cap B \neq \emptyset$. Suppose there exists a left ideal L^* of S such that $L^* \subset Lc$ and $L^* \cap B \neq \emptyset$. By L_1 we will denote the set of all elements $a \in L$ such that $ac \in L^*$. Then by the assumption we get that $L_1 \neq \emptyset$ and $L_1 \cap \overline{NL(B)} \neq \emptyset$. If $s \in S$ and $a \in L_1$, then $(sa)c = s(ac) \in sL^* \subseteq L^*$. Hence L_1 is a left ideal of S. Due to the assumption we have $L_1 = L$. Hence $Lc = L_1c \subseteq L^*$. This is a contradiction with $L^* \subset Lc$. Therefore Lc is a minimal left ideal with respect to the subset B of the semigroup S. □

Corollary 4. (See [3].) *Let L be a minimal left ideal of a semigroup S and let $c \in S$. Then Lc be a minimal left ideal of the semigroup S.*

Proof. Put $B = S$. Then L is a left mB-ideal of S. By Lemmas 1 and 2 we get Corollary 4. □

Corollary 5. (See [4].) *Let S be a semigroup with zero 0. Let L be a 0-minimal left ideal of S, and let $c \in S$. Then either $Lc = \{0\}$ or Lc is a 0-minimal left ideal of S.*

Proof. Put $B = S \setminus \{0\}$. Then $\overline{NL(B)} = S \setminus \{0\}$ and $NL(B) = \{0\}$. Due to Lemmas 1 and 2 we get Corollary 5. □

Corollary 6. (See [10].) *Let S be a semigroup with the kernel K and let L be a simple left ideal of S. Let $c \in S$. Then the set $K \cup Lc$ is either a simple left ideal of S or $K = K \cup Lc$.*

Proof. Put $B = S \backslash K$. Then using Corollary 3 and Lemma 2 we get Corollary 6. □

Let a semigroup S satisfy the condition m_{LB}. By ${}_*B$ we will denote the set of all elements of B such that for each minimal left ideal with respect to the subset B there exists exactly one element $b \in {}_*B$ such that $L = L(b)$ (see Theorem 1) and $L(b)$ is a minimal left ideal with respect to the subset B of S for each $b \in {}_*B$. The set ${}_*B$ will be called the left lower basic (minimal) set of the subset B of the semigroup S. Clearly ${}_*B$ is such a minimal subset of the set B that the sets of all minimal ideals with respect to B and of those with respect to ${}_*B$ coincide.

Definition 3. We will say that a semigroup S satisfies the condition m^*_{LB} if S satisfies the condition m_{LB} and the left lower basic set ${}_*B$ of the set B has the following properties:

i) If $b \in {}_*B$, $c \in S$ and $L(b)c \cap \overline{NL(B)} = \emptyset$, then there exists an element $d \in {}_*B$ such that $L(b)c \subseteq L(d)$.

ii) $\overline{NL(B)} = \overline{N(B)}$.

R e m a r k 4. It is easy to prove that the following assertion holds:

(a) Let a semigroup S contain at least one minimal left ideal. Put $B = S$. Then the semigroup S satisfies the condition m^*_{LB}.

(b) Let a semigroup S with 0 contain at least one 0-minimal left ideal. Put $B = S \setminus \{0\}$. Then the semigroup S satisfies the condition m^*_{LB}.

Lemma 3. *Let a semigroup S satisfy the condition m^*_{LB}. Then the set union of all minimal left ideals with respect to the subset B of S is a two-sided ideal of S.*

P r o o f. Put $M = \cup\{L(b) \mid b \in {}_*B\}$. Let $a \in M$ and $c \in S$. There exists an element $d \in {}_*B$ such that $a \in L(d)$. Then either α) $L(d)c \cap \overline{NL(B)} = \emptyset$, or β) $L(d)c \cap \overline{NL(B)} \neq \emptyset$. First suppose that α) holds. Then by the assumption, there exists an element $d' \in {}_*B$ such that $L(d)c \subseteq L(d')$. It follows that $ac \in M$. In the case β), due to Lemma 2 we get that there exists $h \in {}_*B$ such that $L(b)c = L(h)$. It follows that M is a right ideal of S. Clearly M is a left ideal of S. Hence M is a two-sided ideal of S. □

Definition 4. We will say that a semigroup S satisfies the condition m^{**}_{LB} if S satisfies the condition m^*_{LB} and for each $b, c \in {}_*B$ there exists an element $d \in \overline{NL(B)}$ such that $L(b)d = L(c)$.

E x a m p l e 3. Let a semigroup S contain at least one minimal left ideal. Put $B = S$. Then $\overline{NL(B)} = S$. Let ${}_*B$ be the left lower basic set of the subset of the set B ($\subseteq S$). Then it is easy to prove that the semigroup S satisfies the condition m^{**}_{LB}.

Theorem 2. *Let a semigroup S satisfy the condition m^{**}_{LB}. Then:*

(a) *For each two-sided ideal M of the semigroup S the following holds: If $M \cap {}_*B \neq \emptyset$, then $L({}_*B) \subseteq M$.*

(b) *The set $L({}_*B) = \cup\{L(b) \mid b \in {}_*B\}$ is a minimal two-sided ideal with respect to the subset ${}_*B$ of the semigroup S.*

P r o o f. (a) Let $b \in M \cap {}_*B$. Suppose that $c \in {}_*B$ and $c \notin M$. By the assumption there exists an element $d \in \overline{NL(B)}$ such that $L(b)d = L(c)$. This is a contradiction with $L(b) \subseteq M$ and $c \notin M$. Hence $L({}_*B) \subseteq M$.

(b) By the assumption and Lemma 3, $L({}_*B)$ is a two-sided ideal of the semigroup S. Suppose that there exists a two-sided ideal M' of the semigroup S such that $M' \subset L({}_*B)$ and $M' \cap {}_*B \neq \emptyset$. Using (a) we get $L({}_*B) \subseteq M'$. This contradicts the assumption. □

Corollary 7. *Let a semigroup S contain at least one minimal left ideal. Then the set union of all minimal left ideals of the semigroup S is its minimal two-sided ideal (for the kernel of the semigroup S see e.g. [3], [9]).*

R e m a r k 5. Let S be a semigroup in Example 2 and $B = \{\alpha, \beta\}$. Then

a) S satisfies the condition m^*_{LB} and does not satisfy the condition m^{**}_{LB}.

b) The set union of all minimal ideals with respect to the set B of a semigroup S is not a minimal two-sided ideal of S and $L_B \neq R_B$.

Definition 5. Let S be a semigroup and let $\emptyset \neq B \subseteq S$. Denote by K_B the intersection of all two-sided ideals N of the semigroup S such that $N \cap B \neq \emptyset$. If $K_B \neq \emptyset$ then the two-sided ideal K_B of S will be called the kernel with respect to the subset B of the semigroup S.

Clearly the following holds: If $B = S$ and $K_B \neq \emptyset$, then K_B is the kernel of the semigroup S.

Corollary 8. *Let a semigroup S satisfy the condition m^{**}_{LB}. Then $L({}_*B)$ is the kernel with respect to the subset ${}_*B$ of the semigroup S.*

We get Corollary 8 using Theorem 2.

E x a m p l e 4. Let S_1, S_2, S_3, S_{10}, S_{30} be semigroups from Example 1. Let for each $\alpha \in (0, 1)$, M^α and B^α be the sets from Example 1. It is easy to show that each semigroup S_3 (S_{30}) satisfies the condition m^{**}_{LB} for each $\alpha \in (0, 1)$ ($\alpha \in \langle 0, 1)$). The semigroup S_3 (S_{30}) has the kernel with respect to its every subset B^α, $\alpha \in (0, 1)$ ($\alpha \in \langle 0, 1)$), contains the kernel and does not contain any simple left (right, two-sided) ideal.

Definition 6. Let S be a semigroup and let $\emptyset \neq B \subseteq S$. We will say that the semigroup S satisfies the condition mu^{**}_{LB} (mu^{**}_{RB}) if it satisfies the condition m^{**}_{LB} (m^{**}_{RB}) and for each a, $b \in {}_*B$ (a, $b \in B_*$) we have $L_a b = L_b$ ($bR_a = R_b$).

Further, we denote by $D_l(B)$ ($D_r(B)$) the set of all elements $b \in B$ such that $bB = B$ ($Bb = B$).

Definition 7. A semigroup S will be called a partial group if and only if $D_r(S) \neq \emptyset$ and $D_r(S) = D_l(S)$ (see [2]).

Further, we will use the following lemma (its proof see e.g. [1], [2]).

Lemma 4. *Let S be a partial group. Then*

(a) *$D_r(S) = S$ if and only if S is a group.*

(b) *If $D_r(S) \neq S$, then $S \setminus D_r(S)$ is a two-sided ideal of S and $D_r(S)$ is a group.*

(c) *The unit of the group $D_r(S)$ is a unit of the semigroup S.*

A nonempty subset H of the semigroup S will be called a filter of the semigroup S if for each two elements a, $b \in S$ the following holds: $ab \in H$ (a, $b \in S$) if and only if $a \in H$, $b \in H$. If H is filter of the semigroup S and $S \setminus H \neq \emptyset$, then $S \setminus H$ is a two-sided ideal in S.

Lemma 5. *Let a semigroup S satisfy the conditions mu^{**}_{LB}, mu^{**}_{RB}. Let $L_{\ast B} = R_{B\ast}$ and let c, d be arbitrary elements of $L_{\ast B}$. Put $G = R(c)L(d)$ and $D = G \cap L_B$. Then*

(a) *$L_{\ast B}$ is a filter in $L(_\ast B)$,*

(b) *$D \neq \emptyset$,*

(c) *$D \subseteq R_c \cap L_d$,*

(d) *$D = D_r(G) = D_l(G)$.*

Proof. By the assumption and Theorem 2 we get that $L(_\ast B)$ is a two-sided ideal of S and $R(B_\ast) \subseteq L(_\ast B)$, $L(_\ast B) \subseteq R(B_\ast)$. Therefore $L(_\ast B) = R(B_\ast)$. By the assumption, we get that $L(_\ast B) \setminus L_{\ast B} = R(B_\ast) \setminus R_{B_\ast}$. Put $K = L(_\ast B) \setminus L_{\ast B}$. Then $K = \cup\{L_b \cup L(b) \setminus L_b \mid b \in {_\ast B}\} \setminus \cup\{L_b \mid b \in {_\ast B}\} = \cup\{L(b) \setminus L_b \mid b \in {_\ast B}\}$. Hence either (i) $L(b) \setminus L_b = \emptyset$ for all $b \in {_\ast B}$, or (ii) there exists an element $b \in B$ such that $L(b) \setminus L_b \neq \emptyset$. Suppose that (ii) holds. Then $K \neq \emptyset$ and K is a two-sided ideal of S. Let a and b be elements of $L_{\ast B}$. Then by the assumption, $L_a b = L_b \subseteq L_{\ast B}$. It follows that $L_{\ast B}$ is a filter in $L(_\ast B)$ (in the case (i) we have $L(_\ast B) = L_{\ast B}$).

b) Let c, d be elements of L_B. Then $cd \in R(c)L(d) = G$ and by (a) we get $cd \in L_{\ast B}$. Hence $D \neq \emptyset$.

c) Since $G \cap L_{\ast B} = [R(c)L(d)] \cap L_{\ast B} \subseteq [R(c) \cap L(d)] \cap L_{\ast B} = [R(c) \cap L_{\ast B}] \cap [L(d) \cap L_{\ast B}]$, the assumption and Theorem 1 yield that $D \subseteq R_c \cap L_d$.

d) Let g be an element of D. By (c) we get $g \in R_c$ and $g \in L_d$. By the assumption we get that $L_d = L_g = L_d g \subseteq L(d)g \subseteq L(d)L(d) \subseteq L(d)$. Then $L(d) = L(d)g$. Analogously $gR(c) = R(c)$. Hence $gG = gR(c)L(d) = G$ and $Gg = R(c)L(d)g = G$.

Let g be an element of G such that $g \notin D$. Then $g \in L(d)$ and $g \notin L_{\ast B}$. Therefore $g \in K$. By (a), $L_{\ast B}$ is a filter in $L(_\ast B)$ and $K \neq \emptyset$, hence K is a two-sided ideal in $L(_\ast B)$. It follows that $Gg \cap L_{\ast B} = \emptyset$ and $gG \cap L_{\ast B} = \emptyset$. According to (b) we get $Gg \neq G$ and $gG \neq G$. The above considerations imply that the assertion (d) of Lemma 5 holds. □

Theorem 3. *Let the assumptions of Lemma 5 hold. Then:*

(a) *G is a partial group.*

(b) *$L(d) = Se$, $R(c) = eS$ and $G = R(c) \cap L(d) = eSe$ where e is the unit of the partial group G.*

(c) *$D = R_c \cap L_d$.*

Proof. (a) Since $L(d)$ is a left ideal of the semigroup S, we get that $GG = R(c)L(d)R(c)L(d) \subseteq R(c)L(d) = G$. According to (b) and (d) of Lemma 5 we get that G is a partial group.

(b) Let e be the unit of the partial group G. Then by Lemmas 4 and 5 we have $e \in R_c$ and $e \in L_d$. It means that $R(c) = eS$ and $L(d) = Se$. Then $eSe \subseteq eL(d) \subseteq L(d)$ and $eSe \subseteq R(c)e \subseteq R(c)$. It follows that $eSe \subseteq R(c) \cap L(d)$. Let x are an arbitrary element of $R(c) \cap L(d)$. Then there exists elements $u, v \in S$ such that $x = eu = ve$. Then $x = eu = e(eu) = e(ev) = eve$, i.e. $x \in eSe$. Hence $R(c) \cap L(d) \subseteq eSe$.

Clearly $R(c)L(d) \subseteq L(d)$ and $R(c)L(d) \subseteq R(c)$. Therefore $G \subseteq R(c) \cap L(d)$. Let x be an element of $R(c) \cap L(d)$. By (b) there exists an element $u \in S$ such that $x = ue$. Then $xe = (ue)e = ue = x$. Therefore $x \in R(c)L(e) = R(c)L(d)$. Hence $G = R(c) \cap L(d) = eSe$, where e is the unit element of the partial group G.

(c) By (b), $R_c \cap L_d \subseteq R(c) \cap L(d) = G$. Since $R_c \subseteq L_{*B}$ and $L_d \subseteq L_{*B}$, we get that $R_c \cap L_d \subseteq G \cap L_{*B}$. Lemma 7 implies that $D = R_c \cap L_d$. □

Corollary 9. (See [3].) *Let $L(d)$ be a minimal left ideal and $R(c)$ a minimal right ideal of a semigroup S ($c, d \in S$). Put $B = S$, $G = R(c)L(d)$ and $D = G \cap L_{*B}$. Then:*

(a) G is a group.

(b) $R(c) = eS$, $L(d) = Se$ and $G = R(c) \cap L(d) = eSe$, where e is the unit of the group G.

(c) $G = R_c \cap L_d$.

Proof. By the assumption, the semigroup S satisfies the conditions m_{LB}, m_{RB}, where $B = S$. Let ${}_*B$ (B_*) be the left (right) lower basic set of the subset B of the semigroup S. Then it is easy to prove that the semigroup S satisfies the assumptions of Theorem 3. Because by the assumption, $L(d)$ is a minimal ideal of the semigroup S, using Theorem 1 ($B = S$) we have $L(d) = L_d$. It follows that $G = R(c)L(d) \subseteq L(d) = L_d \cap L_{*B}$. Therefore $D = G$. Using (d) of Lemma 6 we conclude that G is a group. □

Example 5. Let $S_1 = \{0, 1, 2, 4, 5, 7, 8, 10, 11\}$ be a semigroup of the semigroup $S_{12} = \{0, 1, 2, \dots, 11\}$ mod 12. S_2 is the semigroup from Example 1. Let $S_3 = S_1 \times S_2$ be the direct product of S_1, S_2. Put $B_1 = \{2\} \times S_2$ and $B_2 = \{1\} \times S_2$. Then:

a) If $B = B_1$ then the semigroup S_3 satisfies the condition m^{**}_{LB} and does not satisfy the condition mu^{**}_{LB}.

b) If $B = B_2$ then ${}_*B = \{1\} \times S_2$, $B_* = \{(1, a)\}$, $L_{*B} = R_{*B}$ and the semigroup S satisfies the condition mu^{**}_{LB}, mu^{**}_{RB}.

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