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NONABSOLUTELY CONVERGENT SERIES

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Summary. Assume that for any t from an interval $[a, b]$ a real number $u(t)$ is given. Summarizing all these numbers $u(t)$ is no problem in case of an absolutely convergent series $\sum_{t\in[a,b]} u(t)$. The paper gives a rule how to summarize a series of this type which is not absolutely convergent, using a theory of generalized Perron (or Kurzweil) integral.

Keywords. Nonabsolutely convergent series, generalized Perron integral.

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Notation. $\mathbb{N}$ is the set of all integers, $\mathbb{R}$ is the set of all real numbers. $[a,b]$, $[a, b)$, $(c, d]$ etc. will be bounded intervals in $\mathbb{R}$. If a point $t \in \mathbb{R}$ and a set $T \in \mathbb{R}$ are given, then $\operatorname{dist}(t; T) = \inf\{|t - s|; s \in T\}$. If $x \in \mathbb{R}^n$ is an n-dimensional vector, then $(x)_j$ denotes the j-th component of the vector x.

We will make use of the notion of generalized Perron integral, which was defined in [K] in this way:

A finite sequence $A = \{\alpha_0, \tau_1, \alpha_1, \ldots, \alpha_{k-1}, \tau_k, \alpha_k\}$ is a *partition* of the interval $[a, b]$ if

(1) $\qquad a = \alpha_0 < \alpha_1 < \ldots < \alpha_{k-1} < \alpha_k = b$ and

(2) $\qquad \alpha_{i-1} \leqq \tau_i \leqq \alpha_i\,, \quad i = 1, 2, \ldots, k\,.$

An arbitrary positive function δ: $[a, b] \to (0, \infty)$ is called a *gauge* on $[a, b]$. Given a gauge δ on $[a, b]$, a partition A of the interval $[a, b]$ is called δ-fine if

(3) $$[\alpha_{i-1}, \alpha_i] \subset [\tau_i - \delta(\tau_i), \tau_i + \delta(\tau_i)]\,, \quad i = 1, 2, \ldots, k\,.$$

The set of all δ-fine partitions of $[a, b]$ will be denoted by $\mathscr{A}(\delta; a, b)$ or briefly $\mathscr{A}(\delta)$.

It is known that for any gauge δ on $[a, b]$ the set $\mathscr{A}(\delta)$ is nonempty (see [K], Lemma 1,1,1).

Assume that a function U: $[a, b] \times [a, b] \to \mathbb{R}$ and a partition $A = \{\alpha_0, \tau_1, \alpha_1, \ldots, \alpha_{k-1}, \tau_k, \alpha_k\}$ are given. The finite sum

(4) $$S(U, A) = \sum_{i=1}^{k} [U(\tau_i, \alpha_i) - U(\tau_i, \alpha_{i-1})]$$

is called the ***integral sum*** corresponding to the function U and the partition A.

A function $U: [a, b] \times [a, b] \to \mathbb{R}$ is called *integrable* over $[a, b]$ if there exists $\gamma \in \mathbb{R}$ such that for every $\varepsilon > 0$ there exists a gauge $\delta: [a, b] \to (0, \infty)$ such that for every $A \in \mathscr{A}(\delta)$ the inequality

$$|S(U, A) - \gamma| < \varepsilon$$

holds. The number $\gamma \in \mathbb{R}$ is called the *generalized Perron integral* of U over the interval $[a, b]$ and will be denoted by

$$\gamma = \int_a^b DU(\tau, t) .$$

In [K] a definition of an integral using the concept of major and minor functions is given, and it is proved that such a definition is equivalent to the definition given above.

The definition using major and minor functions may be formulated in the following way:

A function $U: [a, b] \times [a, b] \to \mathbb{R}$ is integrable over $[a, b]$ if there exists $\gamma \in \mathbb{R}$ such that for every $\varepsilon > 0$ there exists a gauge δ on $[a, b]$ and functions $M, m: [a, b] \to \mathbb{R}$ such that

(5) $$(t - \tau)(M(t) - M(\tau)) \geqq (t - \tau)(U(\tau, t) - U(\tau, \tau)) \geqq$$
$$\geqq (t - \tau)(m(t) - m(\tau)) \quad \text{whenever} \quad t, \tau \in [a, b] \quad \text{and}$$
$$|t - \tau| \leqq \delta(\tau) \quad \text{and}$$

(6) $$\gamma - \varepsilon < m(b) - m(a) \leqq M(b) - M(a) \leqq \gamma + \varepsilon .$$

Then $\gamma = \int_a^b DU(\tau, t)$.

Let a function $u: [a, b] \to \mathbb{R}$ be given. The symbol $\sum_{t \in [a,b]} u(t)$ can be met usually in the following situation: there is an at most countable set of indices $D \subset [a, b]$ such that $u(t) = 0$ for any $t \in [a, b] \setminus D$; this set D will be ordered into a sequence in an arbitrary way, say $D = \{t_1, t_2, \ldots\}$. If the series $\sum_{k=1}^{\infty} u(t_k)$ is absolutely convergent, i.e. the series $\sum_{k=1}^{\infty} |u(t_k)|$ is convergent, we have $\sum_{t \in [a,b]} u(t) = \sum_{k=1}^{\infty} u(t_k)$.

However, if the series is not absolutely convergent, then in order to obtain a reasonable theory we have to give a rule how to order the index set D. In fact, this is the aim of the present paper.

In the following we will deal only with real-valued functions u; if u is an $\mathbb{R}^n$-valued function with $n > 1$, then the sum $\sum_{t \in [a,b]} u(t)$ can be defined componentwise:

$$\Big(\sum_{t \in [a,b]} u(t)\Big)_j = \sum_{t \in [a,b]} (u(t))_j , \quad j = 1, 2, \ldots, n .$$

Definition 1. *Assume that a gauge $\delta: [a, b] \to (0, \infty)$ is given. By $I(\delta; a, b)$ or briefly $I(\delta)$ we denote the set of all finite nonempty sets $B \subset [a, b]$ such that the following holds:*

(7) *If $t, t' \in B$, $t < t'$ are neighbouring points, i.e. $(t, t') \cap B = \emptyset$, then $t' - t < \delta(t) + \delta(t')$. Denote $\bar{t} = \min B$, $\bar{\bar{t}} = \max B$; then $\bar{t} - a < \delta(\bar{t})$, $b - \bar{\bar{t}} < \delta(\bar{\bar{t}})$.*

Lemma 1. (i) *For every gauge δ on $[a, b]$ the set $I(\delta)$ is nonempty.* (ii) *If a gauge $\delta: [a, b] \to (0, \infty)$ is given and $a < c < b$, then for any two sets $B_1 \in I(\delta; a, c)$ and $B_2 \in I(\delta; c, b)$ the set $B_1 \cup B_2$ belongs to $I(\delta; a, b)$.*

Proof. (i) For every $t \in (a, b]$ such that $t < a + \delta(a)$ the set $\{a\}$ obviously belongs to $I(\delta; a, t)$. Denote

(8) $$c = \sup \{t \in (a, b],\ I(\delta; a, t) \neq \emptyset\} .$$

We have just shown that $c > a$. There is $t_0 \in (a, b]$ such that $I(\delta; a, t_0) \neq \emptyset$ and $c - \delta(c) < t_0$. If $B \in I(\delta; a, t_0)$ then $B \cup \{c\} \in I(\delta; a, c)$ because denoting $\bar{t} = \max B$ we have the estimate $c - \bar{t} = (c - t_0) + (t_0 - \bar{t}) < \delta(c) + \delta(\bar{t})$.

Let us assume that $c < b$; then for every $c' \in (c, b]$ such that $c' < c + \delta(c)$ we have $B \cup \{c\} \in I(\delta; a, c')$ and consequently the set $I(\delta; a, c')$ is nonempty, but this is impossible because of (8). It means that $c = b$ and $I(\delta; a, b) \neq \emptyset$.

(ii) Denote $t_1 = \max B_1$ and $t_2 = \min B_2$, then $c - t_1 < \delta(t_1)$ and $t_2 - c < \delta(t_2)$ by (7). Then $t_2 - t_1 < \delta(t_1) + \delta(t_2)$ and consequently the assumption (7) holds for $B_1 \cup B_2$ on the interval $[a, b]$.

Definition 2. *Assume that a function $u: [a, b] \to \mathbb{R}$ is given. We say that the series $\sum_{t \in [a,b]} u(t)$ is convergent and that its sum is equal to $u \in \mathbb{R}$, if for every $\varepsilon > 0$ there is a gauge δ on $[a, b]$ such that for every finite set of indices $\{t_1, t_2, \ldots, t_m\}$ belonging to $I(\delta)$ the inequality*

(10) $$\left|\sum_{n=1}^{m} u(t_n) - u\right| < \varepsilon$$

holds. The series $\sum_{t \in [a,b)} u(t)$ is defined as the series $\sum_{t \in [a,b]} u(t)$ with $u(b) = 0$, similarly $\sum_{t \in (a,b]} u(t)$, $\sum_{t \in (a,b)} u(t)$.

Remark. For a given series $\sum_{t \in [a,b]} u(t)$ and for any set $B = \{t_1, t_2, \ldots, t_m\} \subset [a, b]$ let us denote

$$s(B) = \sum_{n=1}^{m} u(t_n) .$$

Then (10) can be written in the form

(10)′ $$|s(B) - u| < \varepsilon$$

for every $B \in I(\delta)$.

Lemma 2. *Let a finite set $B_0 \subset [a, b]$ and a gauge δ on $[a, b]$ be given. Assume that*

(11) $$\delta(\tau) \leqq \operatorname{dist}(\tau; B_0 \setminus \{\tau\}) \quad \textit{for every} \quad \tau \in [a, b] .$$

Then every set $B \in I(\delta)$ includes B_0.

Proof. The condition (11) can be written also in the form

$|\tau - \sigma| \geqq \delta(\tau)$ holds for any $\sigma \in B_0$ and $\tau \in [a, b]$ such that $\tau \neq \sigma$.

Assume that there are $B \in I(\delta)$ and $\sigma \in B_0$ such that $\sigma \notin B$. Let us find neighbouring points $t', t'' \in B$ such that $t' < \sigma < t''$. Then

$$\delta(t'') + \delta(t') > t'' - t' = (t'' - \sigma) + (\sigma - t') \geqq \delta(t'') + \delta(t'),$$

which is a contradiction.

Proposition 1. *Let real functions u, v: $[a, b] \to \mathbb{R}$ be given. Assume that there are points $s_1, s_2, \ldots, s_k \in [a, b]$ such that*

(12) $$u(t) = v(t) \quad \textit{for every} \quad t \in [a, b] \setminus \{s_1, s_2, \ldots, s_k\} .$$

If at least one of the series $\sum_{t \in [a,b]} u(t)$, $\sum_{t \in [a,b]} v(t)$ is convergent, then the other is also convergent and the equality

$$\sum_{t \in [a,b]} u(t) - \sum_{j=1}^{k} u(s_j) = \sum_{t \in [a,b]} v(t) - \sum_{j=1}^{k} v(s_j)$$

holds.

Proof. Assume for instance that the series $\sum_{t \in [a,b]} u(t) = u$ is convergent. Then for every $\varepsilon > 0$ there is a gauge δ such that (10)′ holds for every $B \in I(\delta)$. Let us define

$$\delta'(\tau) = \min \{\delta(\tau), \operatorname{dist}(\tau; C \setminus \{\tau\})\} \quad \text{where} \quad C = \{s_1, s_2, \ldots, s_k\} .$$

Lemma 2 implies that an arbitrary set $B = \{t_1, t_2, \ldots, t_m\} \in I(\delta')$ includes all the points $s_1, s_2, \ldots, s_k$.

From (12) it follows that $u(t_n) = v(t_n)$ for every $t_n \in B$ which does not belong to C. We have an estimate

$$|\sum_{n=1}^{m} v(t_n) - [\sum_{j=1}^{k} v(s_j) - \sum_{j=1}^{k} u(s_j) + u]| \leqq$$

$$\leqq |\sum_{n=1}^{m} v(t_n) - \sum_{j=1}^{k} u(s_j) + \sum_{n=1}^{m} u(t_n)]| + |\sum_{n=1}^{m} u(t_n) - u| =$$

$$= |\sum_{\substack{n=1 \\ t_n \notin C}}^{m} v(t_n) - \sum_{\substack{n=1 \\ t_n \notin C}}^{m} u(t_n)| + |\sum_{n=1}^{m} u(t_n) - u| = |\sum_{n=1}^{m} u(t_n) - u| < \varepsilon .$$

Since the set $B \in I(\delta')$ was arbitrary, we get the equality

$$\sum_{t\in[a,b]} v(t) = \sum_{j=1}^{k} v(s_j) - \sum_{j=1}^{k} u(s_j) + u .$$

The proof of the other implication is analogous.

Corollary. *Let a function $u: [a, b] \to R$ be given. Then*

$$\sum_{t\in(a,b]} u(t) = \sum_{t\in[a,b)} u(t) + u(b) = u(a) + \sum_{t\in[a,b]} u(t)$$

provided at least one of the three series is convergent.

Proof. By Definition 2 the series $\sum_{t\in[a,b)} u(t)$ is identical with a series $\sum_{t\in[a,b]} v(t)$ where $v(t) = u(t)$ for $t \in [a, b)$, $v(b) = 0$, and the series $\sum_{t\in(a,b]} u(t)$ is defined as a series $\sum_{t\in[a,b]} w(t)$ where $w(t) = u(t)$ for $t \in (a, b]$, $w(a) = 0$.

Proposition 1 implies that

$$\sum_{t\in[a,b]} u(t) - u(a) - u(b) = \sum_{t\in[a,b]} v(t) - v(a) - v(b) =$$

$$= \sum_{t\in[a,b]} w(t) - w(a) - w(b) , \quad \text{i.e.}$$

$$\sum_{t\in[a,b]} u(t) - u(a) - u(b) = \sum_{t\in[a,b)} u(t) - u(a) = \sum_{t\in(a,b]} u(t) - u(b)$$

provided at least one of the series $\sum_{t\in[a,b]} u(t)$, $\sum_{t\in[a,b]} v(t)$, $\sum_{t\in[a,b]} w(t)$ is convergent.

Proposition 2. *The series $\sum_{t\in[a,b]} u(t)$ is convergent if and only if for every $\varepsilon > 0$ there is a gauge $\delta: [a, b] \to (0, \infty)$ such that for every two sets $B_1, B_2 \in I(\delta)$ the inequality*

$$(13) \qquad |s(B_1) - s(B_2)| < \varepsilon$$

holds.

Proof. 1. If the series $\sum_{t\in[a,b]} u(t)$ is convergent and has the sum u, then for every $\varepsilon > 0$ there is a gauge δ such that for every $B \in I(\delta)$ the inequality $|s(B) - u| < \varepsilon/2$ holds. Then

$$|s(B_1) - s(B_2)| \leqq |s(B_1) - u| + |s(B_2) - u| < \varepsilon$$

for every $B_1, B_2 \in I(\delta)$.

2. Assume that for every $n \in N$ there is a gauge δ_n on $[a, b]$ such that the inequality

$$(14) \qquad |s(B_1) - s(B_2)| < \frac{1}{n}$$

holds for every $B_1, B_2 \in I(\delta_n)$. We may assume that

$$\delta_1(\tau) \geqq \delta_2(\tau) \geqq \delta_3(\tau) \geqq \ldots, \quad \tau \in [a, b].$$

For every $n \in N$ let us choose a set $B_n \in I(\delta_n)$; then also $B_n \in I(\delta_k)$ for every $k \leqq n$.

For a given $\eta > 0$ let us find $n_0 \in N$ such that $1/n_0 \leqq \eta$. For every $m, n \in N$ such that $m > n \geqq n_0$ we have an estimate

$$(15) \qquad |s(B_n) - s(B_m)| < \frac{1}{n} \leqq \eta.$$

This means that $\{s(B_n)\}_{n=1}^{\infty}$ is a Cauchy sequence in R, which has a limit $u \in R$. Passing to the limit with $m \to \infty$ in (15) we get

$$|s(B_n) - u| \leqq \frac{1}{n}.$$

Let $\varepsilon > 0$ be given. Let us find $n' \in N$ such that $1/n' \leqq \varepsilon/2$; then for every $B \in I(\delta_{n'})$ we have the inequality

$$|s(B) - u| \leqq |s(B) - s(B_{n'})| + |s(B_{n'}) - u| < \frac{2}{n'} \leqq \varepsilon.$$

Consequently $\sum_{t \in [a,b]} u(t) = u$.

Lemma 3. *Assume that a convergent series* $\sum_{t \in [a,b]} u(t)$ *is given; for* $\varepsilon > 0$ *let a gauge* δ *on* $[a, b]$ *be given such that the inequality* (13) *holds for every* $B_1, B_2 \in I(\delta; a, b)$. *Then*

$$|s(C_1) - s(C_2)| < \varepsilon \quad \textit{for every interval} \quad [c, d] \subset [a, b] \quad \textit{and every} \quad C_1, C_2 \in I(\delta; c, d).$$

Proof. Assume that $C_1 = \{s_1, s_2, \ldots, s_k\}$, $C_2 = \{t_1, t_2, \ldots, t_m\}$. Let us choose sets $B = \{\tau_1, \ldots, \tau_p\} \in I(\delta; a, c)$ and $D = \{\sigma_1, \ldots, \sigma_q\} \in I(\delta; d, b)$ (if $a = c$ or $d = b$ then $B = \emptyset$ or $D = \emptyset$, respectively). According to Lemma 1 (ii) the sets $B \cup C_1 \cup D$ and $B \cup C_2 \cup D$ belong to $I(\delta; a, b)$. By (13) we get the inequality

$$|s(C_1) - s(C_2)| = |\sum_{i=1}^{k} u(s_i) - \sum_{i=1}^{m} u(t_i)| =$$
$$= |[\sum_{i=1}^{k} u(s_i) + \sum_{i=1}^{p} u(\tau_i) + \sum_{i=1}^{q} u(\sigma_i)] -$$
$$- [\sum_{i=1}^{m} u(t_i) + \sum_{i=1}^{p} u(\tau_i) + \sum_{i=1}^{q} u(\sigma_i)]| =$$
$$= |s(B \cup C_1 \cup D) - s(B \cup C_2 \cup D)| < \varepsilon.$$

Proposition 3. (i) *If the series* $\sum_{t\in[a,b]} u(t)$ *is convergent, then* $\sum_{t\in[c,d]} u(t)$ *is convergent for every interval* $[c, d] \subset [a, b]$.

(ii) *For* $\varepsilon > 0$ *let a gauge* δ *be given such that* $\left|s(B) - \sum_{t\in[a,b]} u(t)\right| < \varepsilon$ *holds for every* $B \in I(\delta; a, b)$. *Then* $\left|s(C) - \sum_{t\in[c,d]} u(t)\right| \leqq \varepsilon$ *holds for every* $C \in I(\delta; c, d)$, *where* $[c, d] \subset [a, b]$.

Proof. This is a consequence of Proposition 2 and Lemma 3.

Theorem 1. *Assume that a convergent series* $\sum_{t\in[a,b]} u(t)$ *is given. Let us define*

$$f(a) = u(a)\,, \quad f(\tau) = \sum_{t\in[a,\tau]} u(t) \quad \text{for} \quad \tau \in (a, b]\,. \tag{16}$$

Then the function f is regulated (i.e. has one-sided limits) and

$$\lim_{s\to\tau-} f(s) = f(\tau) - u(\tau)\,, \quad \tau \in (a, b]\,, \tag{17}$$
$$\lim_{s\to\tau+} f(s) = f(\tau)\,, \quad \tau \in [a, b)\,.$$

Proof. Let $\varepsilon > 0$ be given. Let us find a gauge δ on $[a, b]$ such that $\left|s(B) - \sum_{t\in[a,b]} u(t)\right| < \varepsilon$ holds for every $B \in I(\delta; a, b)$.

a) Assume that $\tau \in (a, b]$. Let $s \in [a, \tau)$ be such that $\tau - \delta(\tau) < s$. Take any set $B \in I(\delta; a, s)$ such that $s \in B$. Since $\{\tau\} \in I(\delta; s, \tau)$, by Lemma 1 the set $B \cup \{\tau\}$ belongs to $I(\delta; a, \tau)$. According to Proposition 3 (ii) the following estimate holds:

$$\begin{aligned} |f(\tau) - u(\tau) - f(s)| &\leqq |f(\tau) - [u(\tau) + s(B)]| + |f(s) - s(B)| = \\ &= |f(\tau) - s(B \cup \{\tau\})| + |f(s) - s(B)| \leqq 2\varepsilon\,. \end{aligned} \tag{18}$$

b) Assume that $a \leqq \tau < b$, let $C \in I(\delta; a, \tau)$ be such a set that $\tau \in C$ (if $\tau = a$ then $C = \{\tau\}$). For every $s \in (\tau, b]$ such that $s < \tau + \delta(\tau)$ the set $\{\tau\}$ belongs to $I(\delta; \tau, s)$ and consequently $C \in I(\delta; a, s)$. Then

$$|f(s) - f(\tau)| \leqq |f(s) - s(C)| + |f(\tau) - s(C)| \leqq 2\varepsilon\,. \tag{19}$$

The relations (18), (19) imply (17).

Corollary 1. *If the series* $\sum_{t\in[a,b]} u(t)$ *is convergent, then the set* $\{t \in [a, b];\ u(t) \neq 0\}$ *is at most countable.*

Proof. Since the function f defined by (16) is regulated, it can be discontinuous only in an at most countable set; according to (17)

$$f(\tau-) \neq f(\tau) \quad \text{if and only if} \quad u(\tau) \neq 0\,.$$

Corollary 2. *If the series* $\sum_{t\in[a,b]} u(t)$ *is convergent then*

$$\lim_{s\to\tau} u(s) = 0 \quad \textit{for every} \quad \tau \in [a, b]\,.$$

Proof. Let $\tau \in (a, b]$ and $\varepsilon > 0$ be given. There is $\lambda > 0$ such that the following holds: If $\tau - \lambda < s < \tau$, then $|f(\tau-) - f(s)| \leqq \varepsilon$. Then also $|f(\tau-) - f(s-)| \leqq \varepsilon$ for every $s \in (\tau - \lambda, \tau)$. Hence

$$|u(s)| = |f(s) - f(s-)| \leqq |f(s) - f(\tau-)| + |f(\tau-) - f(s-)| \leqq 2\varepsilon\,,$$

if $s \in (\tau - \lambda, \tau)$. This means that $\lim_{s\to\tau-} u(s) = 0$. Similarly $\lim_{s\to\tau+} u(s) = 0$ for every $\tau \in [a, b)$.

Corollary 3. *Assume that the series* $\sum_{t\in[a,b]} u(t)$ *is convergent. Let us define*

$$g(a) = 0\,, \quad g(\tau) = \sum_{t\in[a,\tau]} u(t) \quad \textit{for} \quad \tau \in (a, b]\,. \tag{20}$$

Then the function g is regulated and

$$\lim_{s\to\tau-} g(s) = g(\tau)\,, \quad \tau \in (a, b]\,, \tag{21}$$

$$\lim_{s\to\tau+} g(s) = g(\tau) + u(\tau)\,, \quad \tau \in [a, b)\,.$$

Proof. By Proposition 1 we have $g(\tau) = f(\tau) - u(\tau)$ for every $\tau \in [a, b]$. If $\tau \in (a, b]$ then

$$\lim_{s\to\tau-} g(s) = \lim_{s\to\tau-} f(s) - \lim_{s\to\tau-} u(s) = f(\tau-) = f(\tau) - u(\tau) = g(\tau)\,;$$

if $\tau \in [a, b)$ then

$$\lim_{s\to\tau+} g(s) = \lim_{s\to\tau+} f(s) + \lim_{s\to\tau+} u(s) = f(\tau) = g(\tau) + u(\tau)\,.$$

Theorem 2. *Assume that a function* $u\colon [a, b] \to \mathbb{R}$ *is given. Let us define a function* $U\colon [a, b] \times [a, b] \to \mathbb{R}$ *by*

$$U(\tau, t) = u(t) \quad \textit{for} \quad \tau < t\,,$$
$$U(\tau, t) = 0 \quad \textit{for} \quad \tau = t\,,$$
$$U(\tau, t) = -u(\tau) \quad \textit{for} \quad \tau > t\,.$$

Then the series $\sum_{t\in[a,b]} u(t)$ *is convergent if and only if* $U(\tau, t)$ *is integrable over* $[a, b]$. *We have the equality*

$$\int_a^b \mathrm{D}U(\tau, t) = \sum_{t\in(a,b]} u(t)\,.$$

Proof. (i) Assume that the function U is integrable and denote

$$\gamma = \int_a^b DU(\tau, t) .$$

For a given $\varepsilon > 0$ there is a gauge δ on $[a, b]$ such that

$$|S(U, A) - \gamma| < \varepsilon$$

holds for every $A \in \mathscr{A}(\delta; a, b)$. Let us define

$$\delta'(\tau) = \min \{\delta(\tau), b - \tau, \tau - a\} \quad \text{for} \quad \tau \in (a, b),$$

$$\delta'(\tau) = \min \{\delta(\tau), b - a\} \quad \text{for} \quad \tau = a, b .$$

Let an arbitrary finite set $B = \{t_1, t_2, \dots, t_m\} \in I(\delta')$ be given. By Lemma 2 the set B contains the points a, b. Assume that

$$a = t_1 < t_2 < \dots < t_m = b .$$

For any $i = 1, 2, \dots, m - 1$ we have by (7)

$$i_{i+1} - t_i < \delta'(t_i) + \delta'(t_{i+1}), \quad \text{i.e.} \quad t_{i+1} - \delta'(t_{i+1}) < t_i + \delta'(t_i) .$$

Hence the open interval $(t_i, t_{i+1}) \cap (t_{i+1} - \delta'(t_{i+1}), t_i + \delta'(t_i))$ is nonempty. Corollary 1 of Theorem 1 implies that there is $\alpha_i \in (t_i, t_{i+1}) \cap (t_{i+1} - \delta'(t_{i+1}), t_i + \delta'(t_i))$ such that $u(\alpha_i) = 0$. Denote $\alpha_0 = a$, $\alpha_m = b$.

The set $A = \{\alpha_0, t_1, \alpha_1, \dots, t_m, \alpha_m\}$ obviously belongs to $\mathscr{A}(\delta'; a, b)$. Consequently

$$|\sum_{n=1}^{m} u(t_n) - [u(a) + \gamma]| = |\sum_{n=2}^{m} u(t_n) - \gamma| =$$

$$= |[\sum_{n=2}^{m} u(t_n) + \sum_{n=1}^{m-1} u(\alpha_n)] - \gamma| =$$

$$= |[\sum_{\alpha_{n-1} < t_n} u(t_n) + \sum_{t_n < \alpha_n} u(\alpha_n)] - \gamma| = |S(U, A) - \gamma| < \varepsilon .$$

According to Definition 2 the series $\sum_{t \in [a,b]} u(t)$ is convergent and $\sum_{t \in [a,b]} u(t) = u(a) + \gamma$. Hence $\gamma = \sum_{t \in (a,b]} u(t)$.

(ii) Assume that the series $\sum_{t \in [a,b]} u(t) = u$ is convergent. For every gauge δ and $t \in (a, b]$ let us denote by $I_t(\delta)$ the set of all $B \in I(\delta; a, t)$ such that $t \in B$. For $t = a$ the set $I_t(\delta)$ will consist of a single element $\{a\}$.

Let $\varepsilon > 0$ be given. There is a gauge δ on $[a, b]$ such that

(22) $$|s(B) - u| < \varepsilon \quad \text{holds for any} \quad B \in I(\delta; a, b) .$$

Let us define $m(t) = \inf_{B \in t_t(\delta)} s(B)$, $M(t) = \sup_{B \in I_t(\delta)} s(B)$, $t \in [a, b]$. Let us notice that $m(a) = u(a)$, $M(a) = u(a)$. From (22) it follows that $u - \varepsilon < s(B) < u + \varepsilon$ for every $B \in I_b(\delta) \subset I(\delta; a, b)$, and consequently

$$u - \varepsilon \leqq m(b) \leqq M(b) \leqq u + \varepsilon\,,$$

(23) $$u - u(a) - \varepsilon \leqq m(b) - m(a) \leqq M(b) - M(a) \leqq u - u(a) + \varepsilon\,.$$

Assume that $a \leqq \tau < t \leqq b$ and $t < \tau + \delta(\tau)$. For arbitrary $\lambda > 0$ there are $B_1, B_2 \in I_\tau(\delta)$ such that

$$s(B_1) < m(\tau) + \lambda\,, \quad s(B_2) > M(\tau) - \lambda\,.$$

Since $\{\tau, t\} \in I(\delta; \tau, t)$, by Lemma 1 (ii) thes sets $B_1 \cup \{\tau, t\} = B_1 \cup \{t\}$ and $B_2 \cup \cup \{\tau, t\} = B_2 \cup \{t\}$ belong to $I(\delta; a, t)$; these sets also belong to $I_t(\delta)$ because they contain t. Hence

$$m(t) \leqq s(B_1 \cup \{t\}) = s(B_1) + u(t) < m(\tau) + \lambda + u(t)\,,$$
$$M(t) \geqq s(B_2 \cup \{t\}) = s(B_2) + u(t) > M(\tau) - \lambda + u(t)\,.$$

Since the number $\lambda > 0$ was arbitrary, we get inequalities

(24) $$m(t) - m(\tau) \leqq u(t) = U(\tau, t) - U(\tau, \tau) \leqq M(t) - M(\tau)\,.$$

Similarly, if $a \leqq t < \tau \leqq b$ where $\tau - \delta(\tau) < t$, then for an arbitrary $\eta > 0$ we can find $C_1, C_2 \in I_t(\delta)$ such that

$$s(C_1) < m(t) + \eta\,, \quad s(C_2) > M(t) - \eta\,.$$

Since $\{\tau\} \in I(\delta; t, \tau)$, the sets $C_1 \cup \{\tau\}$, $C_2 \cup \{\tau\}$ belong to $I_\tau(\delta)$ and consequently

$$m(\tau) \leqq s(C_1 \cup \{\tau\}) = s(C_1) + u(\tau) < m(t) + \eta + u(\tau)\,,$$
$$M(\tau) \geqq s(C_2 \cup \{\tau\}) = s(C_2) + u(\tau) > M(t) - \eta + u(\tau)\,.$$

We get the inequality

(25) $$m(\tau) - m(t) \leqq u(\tau) = U(\tau, \tau) - U(\tau, t) \leqq M(\tau) - M(t)\,.$$

According to the definition of integral using major and minor functions (see (5), (6)) it follows from (23), (24), (25) that the function U is integrable over $[a, b]$ and

$$\int_a^b \mathrm{D}U(\tau, t) = u - u(a) = \sum_{t \in (a,b]} u(t)\,.$$

Theorem 3. *Assume that real functions $u, v\colon [a, b] \to \mathbb{R}$ are given. Let us define a function $V\colon [a, b] \times [a, b] \to \mathbb{R}$ by*

(26) $$V(\tau, t) = u(t) + v(\tau) \quad \text{for} \quad \tau < t\,,$$
$$V(\tau, t) = 0 \quad \text{for} \quad \tau = t\,,$$
$$V(\tau, t) = -u(\tau) - v(t) \quad \text{for} \quad \tau > t\,.$$

Then the series $\sum_{t \in [a,b]} (u(t) + v(t))$ is convergent if and only if the function V is integrable over $[a, b]$. We have the equality

$$\int_a^b \mathrm{D}V(\tau, t) = v(a) + \sum_{t \in (a,b)} (u(t) + v(t)) + u(b)\,.$$

Proof. Let us define

$$R(\tau, t) = u(t) + v(t) \quad \text{for} \quad \tau < t\,,$$
$$R(\tau, t) = 0 \quad \text{for} \quad \tau = t\,,$$
$$R(\tau, t) = -u(\tau) - v(\tau) \quad \text{for} \quad \tau > t\,.$$

By Theorem 2 the series $\sum_{t\in[a,b]} (u(t) + v(t))$ is convergent if and only if R is integrable over $[a, b]$, and

(27) $$\int_a^b \mathrm{D}R(\tau, t) = \sum_{t\in(a,b]} (u(t) + v(t))$$

holds. Using the definition of the generalized Perron integral, it can be easily proved that the function $V(\tau, t) - R(\tau, t) = v(\tau) - v(t)$ is integrable over $[a, b]$, and

(28) $$\int_a^b \mathrm{D}[V(\tau, t) - R(\tau, t)] = v(a) - v(b)\,.$$

Then the function V is integrable if and only if R is integrable. From (27), (28) we obtain

$$\int_a^b \mathrm{D}V(\tau, t) = \int_a^b \mathrm{D}R(\tau, t) + \int_a^b \mathrm{D}[V(\tau, t) - R(\tau, t)] =$$
$$= \{\sum_{t\in(a,b)} (u(t) + v(t)) + (u(b) + v(b))\} + (v(a) - v(b)) =$$
$$= v(a) + \sum_{t\in(a,b)} (u(t) + v(t)) + u(b)\,.$$

Corollary 4. *The series* $\sum_{t\in[a,b]} u(t)$ *is convergent if and only if the function* $U'\colon [a, b] \times [a, b] \to \mathbb{R}$ *defined by*

$$U'(\tau, t) = u(\tau) \quad \textit{for} \quad \tau < t\,,$$
$$U'(\tau, t) = 0 \quad \textit{for} \quad \tau = t\,,$$
$$U'(\tau, t) = -u(t) \quad \textit{for} \quad \tau > t$$

is integrable over $[a, b]$; *the equality*

$$\int_a^b \mathrm{D}U'(\tau, t) = \sum_{t\in[a,b)} u(t)$$

is satisfied.

Theorem 4. *Assume that functions* $u, v\colon [a, b] \to \mathbb{R}$ *are given. Let us define a function* $W\colon [a, b] \times [a, b] \to \mathbb{R}$ *by*

$$W(\tau, t) = v(\tau) \quad \textit{for} \quad \tau < t\,,$$
$$W(\tau, t) = 0 \quad \textit{for} \quad \tau = t\,,$$
$$W(\tau, t) = -u(\tau) \quad \textit{for} \quad \tau > t\,.$$

If the function W is integrable over $[a, b]$, then the series $\sum_{t\in[a,b]} (u(t) + v(t))$ *is convergent, and the equality*

$$\int_a^b \mathrm{D}W(\tau, t) = v(a) + \sum_{t\in(a,b)} (u(t) + v(t)) + u(b)$$

holds.

Proof. Denote $\int_a^b \mathrm{D}W(\tau, t) = \gamma$. Since the values $u(a)$, $v(b)$ have no influence on the values of $W(\tau, t)$, we can assume that

$$u(a) = v(b) = 0\,. \tag{29}$$

For a given $\varepsilon > 0$ there is a gauge δ such that $|S(W, A) - \gamma| < \varepsilon$ holds for every $A \in \mathscr{A}(\delta; a, b)$. Let us define

$$\delta'(\tau) = \min\{\delta(\tau), b - \tau, \tau - a\} \quad \text{for} \quad \tau \in (a, b)\,,$$
$$\delta'(\tau) = \min\{\delta(\tau), b - a\} \quad \text{fot} \quad \tau = a, b\,.$$

Let an arbitrary set $\{t_1, t_2, \ldots, t_m\} \in I(\delta'; a, b)$ be given. Lemma 2 implies that this set includes the points a, b. We can assume that

$$a = t_1 < t_2 < \ldots < t_m = b\,.$$

Define $\alpha_0 = a$, $\alpha_m = b$; for every $i = 2, 3, \ldots, m - 1$ it follows from (7) that there exists a point $\alpha_i \in (t_i, t_{i+1}) \cap (t_{i+1} - \delta(t_{i+1}), t_i + \delta(t_i))$ similarly as in the proof of Theorem 2. Then $A = \{\alpha_0, t_1, \alpha_1, \ldots, \alpha_{m-1}, t_m, \alpha_m\} \in \mathscr{A}(\delta; a, b)$. Let us note that $\alpha_0 = t_1 < \alpha_1$; $\alpha_{m-1} < t_m = \alpha_m$; $\alpha_{i-1} < t_i < \alpha_i$ for $i = 2, \ldots, m - 1$. We have the estimate

$$\varepsilon > |S(W, A) - \gamma| = |[W(t_1, \alpha_1) - W(t_1, t_1) +$$
$$+ \sum_{i=2}^{m-1} (W(t_i, \alpha_i) - W(t_i, \alpha_{i-1})) + W(t_m, t_m) - W(t_m, \alpha_{m-1})] - \gamma| =$$
$$= |[v(t_1) + \sum_{i=2}^{m-1} (v(t_i) + u(t_i)) + u(t_m)] - \gamma| =$$
$$= |\sum_{i=1}^{m} (u(t_i) + v(t_i)) - \gamma|\,.$$

Consequently,

$$\gamma = \sum_{t\in[a,b]} (u(t) + v(t)) = (u(a) + v(a)) + \sum_{t\in(a,b)} (u(t) + v(t)) +$$
$$+ (u(b) + v(b)) = v(a) + \sum_{t\in(a,b)} (u(t) + v(t)) + u(b)$$

(we take (29) into consideration).

If we use the known properties of the integrals of functions U or U' as defined in Theorem 2 or Corollary 4, we can obtain several properties of the series $\sum_{t\in[a,b]} u(t)$:

Proposition 4. *Let $\alpha \in \mathbb{R}$ be given. If the series $\sum_{t\in[a,b]} u(t)$ is convergent then the series $\sum_{t\in[a,b]} (\alpha\, u(t))$ is convergent and*

$$\sum_{t\in[a,b]} (\alpha\, u(t)) = \alpha \sum_{t\in[a,b]} u(t)\,.$$

(See [S], Th. 1.5.)

Proposition 5. *If the series $\sum_{t\in[a,b]} u(t)$, $\sum_{t\in[a,b]} v(t)$ are convergent, then*

$$\sum_{t\in[a,b]} (u(t) + v(t)) = \sum_{t\in[a,b]} u(t) + \sum_{t\in[a,b]} v(t)\,.$$

(See [S], Th. 1.6.)

Proposition 6. *If $c \in (a, b)$ and the series $\sum_{t\in[a,c]} u(t)$ and $\sum_{t\in[c,b]} u(t)$ are convergent then*

$$\sum_{t\in[a,b]} u(t) = \sum_{t\in[a,c]} u(t) + \sum_{t\in(c,b]} u(t)\,.$$

(See [S], Th. 1.10.)

Proposition 7. *Assume that for every $c \in (a, b)$ the series $\sum_{t\in[a,c]} u(t)$ is convergent and that there exists a finite limit $\lim_{c\to b-} \sum_{t\in[a,c]} u(t) = \alpha$. Then the series $\sum_{t\in[a,b]} u(t)$ is convergent and $\alpha = \sum_{t\in[c,b]} u(t)$.*

(See [S], Th. 1.13.)

Proposition 8. *Assume that for every $c \in (a, b)$ the series $\sum_{t\in[c,b]} u(t)$ is convergent and that there exists a finite limit $\lim_{c\to a+} \sum_{t\in[a,b]} u(t) = \beta$. Then the series $\sum_{t\in[a,b]} u(t)$ is convergent and $\beta = \sum_{t\in(a,b]} u(t)$.*

(See [S], Remark 1.14.)

Proposition 9. *Assume that φ: $[a, b] \to [c, d]$ is a continuous strictly monotone function such that $\varphi(a) = c$, $\varphi(b) = d$, or $\varphi(a) = d$, $\varphi(b) = c$. If one of the series $\sum_{t\in[c,d]} u(t)$, $\sum_{t\in[a,b]} u(\varphi(t))$ is convergent, then also the other is convergent and*

$$\sum_{t\in[c,d]} u(t) = \sum_{t\in[a,b]} u(\varphi(t))\,.$$

(See [S], Th. 1.24.)

Theorem 5. *Assume that a convergent series $\sum_{t\in[a,b]} u(t) = u$ is given. Then there is a sequence $\{t_n\}_{n=1}^{\infty}$ of pairwise different points from $[a, b]$, such that*

$$\sum_{t\in[a,b]} u(t) = \sum_{n=1}^{\infty} u(t_n)$$

and $\{t \in [a, b];\ u(t) \neq 0\} \subset \{t_1, t_2, t_3, \ldots\}$.

Proof. Let us denote $M = \{t \in [a, b];\ u(t) \neq 0\}$. Since the set M is at most countable, there is a sequence $\{\sigma_n\}_{n=1}^{\infty} \subset [a, b]$ such that $M \subset \{\sigma_1, \sigma_2, \sigma_3, \ldots\}$. Let us denote $C_k = \{\sigma_1, \sigma_2, \ldots, \sigma_k\}$ for every $k \in \mathbb{N}$. For any $k = 1, 2, 3, \ldots$ there is a gauge δ_k on $[a, b]$ such that

$$(30) \qquad |s(B) - u| < \frac{1}{k} \quad \text{holds for any finite set} \quad B \in I(\delta_k)\,.$$

Let us choose a set $B_1 \in I(\delta_1)$. There is an integer p_1 such that $B_1 \cap M \subset C_{p_1}$. Let us define

$$\Delta_2(\tau) = \min\{\delta_2(\tau), \operatorname{dist}(\tau; B_1 \cup C_{p_1} \setminus \{\tau\})\} \quad \text{for any} \quad \tau \in [a, b]\,.$$

Let us choose a set $B_2 \in I(\Delta_2)$; then $B_2 \subset B_1 \cup C_{p_1}$ holds according to Lemma 2.

Further, if the set B_k has been defined for an integer k, we can find an integer p_k such that $B_k \cap M \subset C_{p_k}$, and we will denote

$$\Delta_{k+1}(\tau) = \min\{\delta_{k+1}(\tau), \Delta_k(\tau), \operatorname{dist}(\tau; B_k \cup C_{p_k} \setminus \{\tau\})\}$$
$$\text{for any} \quad \tau \in [a, b]\,.$$

Then let us choose a set $B_{k+1} \in I(\Delta_{k+1})$.

In this way we can obtain a sequence $\{p_k\}$ of integers, a sequence $\{\Delta_k\}$ of gauges and a sequence of finite sets $B_1 \subset B_2 \subset \ldots \subset B_k \subset B_{k+1} \subset \ldots \subset [a, b]$ such that $B_k \in I(\Delta_k)$ and

$$(31) \qquad B_k \cap M \cap C_{p_k} \subset B_{k+1}$$

hold for any integer k.

Let us denote the elements of B_1 by $t_1 < t_2 < \ldots < t_{m_1}$. If $t_1, t_2, \ldots, t_{m_k}$ have been defined for an integer k, let us denote the elements of $B_{k+1} \setminus B_k$ by $t_{m_k+1} < < t_{m_k+2} < \ldots < t_{m_{k+1}}$. We obtain a sequence of pairwise different points $\{t_n\}_{n=1}^{\infty}$ such that $B_k = \{t_1, t_2, \ldots, t_{m_k}\}$. (31) implies that

$$\{t_1, t_2, t_3, \ldots\} = \bigcup_{k=1}^{\infty} B_k \subset \bigcup_{k=1}^{\infty} C_{p_k} = M\,.$$

Let us prove that $\sum_{n=1}^{\infty} u(t_n) = u$. For a given $\varepsilon > 0$ let us find an integer k_0 such that $1/k_0 \leqq \varepsilon$. If an arbitrary integer $N \geqq m_{k_0}$ is given, we will find such $k \geqq k_0$ that $m_k < N \leqq m_{k+1}$. In case that $N = m_{k+1}$, the set $\{t_1, t_2, \ldots, t_N\}$ coincides with B_{k+1} which belongs to $I(\Delta_{k+1})$; hence

$$\left|\sum_{n=1}^{\infty} u(t_n) - u\right| = |s(B_{k+1}) - u| < \frac{1}{k+1} < \frac{1}{k_0} \leqq \varepsilon\,.$$

Now assume that $N < m_{k+1}$. Let t_r be the neighbour of t_N inside $B_{k+1} \cap [t_N, b]$, i.e a point from B_{k+1} satisfying $(t_N, t_r) \cap B_{k+1} = \emptyset$. Then $t_r - t_N < \Delta_{k+1}(t_r) + \Delta_{k+1}(t_N)$ according to Definition 1. There is $c \in (t_N, t_r)$ such that $t_r - \Delta_{k+1}(t_r) < c < t_N + \Delta_{k+1}(t_N)$.

It is quite evident that $\{t_1, t_2, \ldots, t_N\} \cap [a, c] \in I(\Delta_{k+1}; a, c)$, while $\{t_1, t_2, \ldots, t_N\} \cap [c, b] = \{t_1, t_2, \ldots, t_{m_k}\} \cap [c, b] \in I(\Delta_k; c, b)$. According to Lemma 1 (ii) we can conclude that $\{t_1, t_2, \ldots, t_N\} \in I(\Delta_k; a, b)$; consequently

$$\left|\sum_{n=1}^{N} u(t_n) - u\right| < \frac{1}{k} \leqq \frac{1}{k_0} \leqq \varepsilon$$

holds by (30).

Proposition 10. *Assume that a convergent series of real numbers* $\sum_{n=1}^{\infty} \alpha_n$ *is given. If* $\{t_n\}_{n=1}^{\infty} \subset [a, b]$ *is any increasing sequence and we define*

$$u(t) = \alpha_n \quad \text{for} \quad t = t_n\,,$$
$$u(t) = 0 \quad \text{for} \quad t \in [a, b] \setminus \{t_1, t_2, \ldots\}\,,$$

then the series $\sum_{t \in [a,b]} u(t)$ *is convergent and* $\sum_{t \in [a,b]} u(t) = \sum_{n=1}^{\infty} \alpha_n$.

Proof. Denote $\sum_{n=1}^{\infty} \alpha_n = \alpha$. Since the sequence $\{t_n\}$ is increasing in the compact interval $[a, b]$, it has a limit $c \in (a, b]$. For any $\varepsilon > 0$ there is an integer N such that

$$\left|\sum_{n=1}^{m} \alpha_n - \alpha\right| < \varepsilon \quad \text{holds for any} \quad m \geqq N\,. \tag{32}$$

Let us define

$$\begin{aligned}
\delta(\tau) &= t_1 - \tau \quad \text{for} \quad \tau \in [a, t_1)\,;\\
\delta(t_1) &= t_2 - t_1\,;\\
\delta(\tau) &= \min\{\tau - t_n, t_{n+1} - \tau\} \quad \text{for} \quad \tau \in (t_n, t_{n+1})\,, \quad n \in \mathbb{N}\,;\\
\delta(t_n) &= \min\{t_{n+1} - t_n, t_n - t_{n-1}\} \quad \text{for} \quad n \geqq 2\,;\\
\delta(c) &= c - t_N\,;\\
\delta(\tau) &= \tau - c \quad \text{for} \quad \tau \in (c, b]\,.
\end{aligned}$$

Let an arbitrary set $B \in I(\delta; a, b)$ be given. Since $\delta(\tau) \leqq |\tau - c|$ holds for any $\tau \in [a, b] \setminus \{c\}$ and $\delta(\tau) \leqq |\tau - t_N|$ holds for any $\tau \in [a, b] \setminus \{t_N\}$, the points t_N and c belong to B.

Let us denote $m = \max\{n \in \mathbb{N};\ t_n \in B\}$. Then $m \geqq N$. The gauge δ is defined so that

$$\delta(\tau) \leqq \operatorname{dist}(\tau; \{t_1, t_2, \ldots, t_m\} \setminus \{\tau\})$$

holds for any $\tau \in [a, t_m]$. By Lemma 2 the set B contains all points $t_1, t_2, \ldots, t_m$, consequently

$$s(B) = \sum_{n=1}^{m} u(t_n) = \sum_{n=1}^{m} \alpha_n . \quad \text{Since} \quad m \geqq N , \quad (32) \text{ yields}$$

$$|s(B) - \alpha| = \left|\sum_{n=1}^{m} \alpha_n - \alpha\right| < \varepsilon .$$

Theorem 6. *Let an absolutely convergent series $\sum_{n=1}^{\infty} \alpha_n$ of real numbers and a sequence of pairwise different points $\{s_n\}_{n=1}^{\infty} \subset [a, b)$ be given. Let us define $u(t) = \alpha_n$ if $t = s_n$, $n \in \mathbb{N}$, $u(t) = 0$ if $t \in [a, b] \setminus \{s_n\}_{n=1}^{\infty}$. Then the series $\sum_{t\in[a,b]} u(t)$ is convergent, the function W: $[a, b] \times [a, b] \to \mathbb{R}$ defined by*

$$W(\tau, t) = u(\tau) \quad \text{if} \quad \tau < t , \quad W(\tau, t) = 0 \quad \text{if} \quad \tau \geqq t$$

is integrable over $[a, b]$, and

$$\int_a^b \mathrm{D}W(\tau, t) = \sum_{t\in[a,b]} u(t) = \sum_{n=1}^{\infty} \alpha_n .$$

Proof. Denote $\alpha = \sum_{n=1}^{\infty} \alpha_n$. Let $\varepsilon > 0$ be given. There is an integer n_0 such that $\sum_{n=n_0+1}^{\infty} |\alpha_n| < \varepsilon$. Let us define

$$\text{(33)} \qquad \delta(\tau) = \min\{|\tau - s_n|;\ n = 1, 2, \ldots, n_0\} \quad \text{for} \quad \tau \in [a, b] \setminus \{s_n\}_{n=1}^{n_0} ;$$
$$\delta(\tau) = \min\{|\tau - s_n|;\ n = 1, 2, \ldots, n_0,\ n \neq k\} \quad \text{for} \quad \tau = s_k ,$$
$$k = 1, 2, \ldots, n_0 .$$

Let a partition $A \in \mathscr{A}(\delta; a, b)$ be given, $A = \{\alpha_0, \tau_1, \ldots, \tau_k, \alpha_k\}$. Lemma 2 implies that the set $\{s_1, s_2, \ldots, s_{n_0}\}$ is contained in the set $\{\tau_1, \tau_2, \ldots, \tau_k\}$. Moreover, for every s_n, $n = 1, 2, \ldots, n_0$ there is an integer i such that $s_n = \tau_i < \alpha_i$ (if $s_n = \tau_i = \alpha_i < \tau_{i+1}$ then $s_n \in (\tau_{i+1} - \delta(\tau_{i+1}), \tau_{i+1})$ which contradicts (33)). Denote $J = \{n \in \mathbb{N};\ s_n = \tau_i < \alpha_i \text{ for some } i\}$; then $J \subset \{s_1, s_2, \ldots, s_{n_0}\}$. We have the estimate

$$|S(W, A) - \alpha| = \Big|\sum_{\substack{i=1 \\ \tau_i<\alpha_i}}^{k} u(\tau_i) - \alpha\Big| = \Big|\sum_{n\in J} u(s_n) - \sum_{n=1}^{\infty} \alpha_n\Big| =$$

$$= \Big|\sum_{\substack{n=1 \\ n\notin J}}^{\infty} \alpha_n\Big| \leqq \sum_{n=n_0+1}^{\infty} |\alpha_n| < \varepsilon .$$

Consequently, the function W is integrable over $[a, b]$ and $\int_a^b \mathrm{D}W(\tau, t) = \alpha$. Theorem 4 (with $u(\tau)$ and 0 instead of $v(\tau)$ and $u(\tau)$) implies that the series $\sum_{t\in[a,b)} u(t)$ is convergent and has the sum α.

Theorem 7. *Assume that functions* $u, v\colon [a, b] \to \mathbb{R}$ *satisfy* $|u(t)| \leqq v(t)$ *for* $t \in [a, b]$. *If the series* $\sum_{t\in[a,b]} v(t)$ *is convergent, then*

(i) *the series* $\sum_{t\in[a,b]} u(t)$ *is convergent and* $|\sum_{t\in[a,b]} u(t)| \leqq \sum_{t\in[a,b]} v(t)$;

(ii) *for every sequence of pairwise different points* $\{s_n\}_{n=1}^{\infty} \subset [a, b]$ *such that* $\{t \in [a, b];\ u(t) \neq 0\} \subset \{s_1, s_2, s_3, \ldots\}$ *the equality*

$$\sum_{n=1}^{\infty} u(s_n) = \sum_{t\in[a,b]} u(t)$$

holds.

Proof. (i) Let $\varepsilon > 0$ be given. By Proposition 2 there is a gauge δ on $[a, b]$ such that

$$|\sum_{n=1}^{m} v(t_n) - \sum_{j=1}^{k} v(\tau_j)| < \varepsilon \quad \text{holds for every two sets}$$

$$\{t_1, t_2, \ldots, t_m\},\ \{\tau_1, \tau_2, \ldots, \tau_k\} \in (\delta)\,.$$

Let $B_0 = \{t_1, t_2, \ldots, t_m\} \in I(\delta)$ be fixed. Let us denote

$$\delta'(\tau) = \min\{\delta(\tau), \operatorname{dist}(\tau; B_0 \setminus \{\tau\})\} \quad \text{for any} \quad \tau \in [a, b]\,.$$

Then by Lemma 2 arbitrary sets $\{s_1, s_2, \ldots, s_k\}$, $\{\sigma_1, \sigma_2, \ldots, \sigma_l\} \in I(\delta')$ contain all points from B_0. We have an estimate

$$|\sum_{i=1}^{k} u(s_i) - \sum_{j=1}^{l} u(\sigma_j)| = |\sum_{\substack{i=1\\ s_i\notin B_0}}^{k} u(s_i) - \sum_{\substack{j=1\\ \sigma_j\notin B_0}}^{l} u(\sigma_j)| \leqq$$

$$\leqq |\sum_{\substack{i=1\\ s_i\notin B_0}}^{k} u(s_i)| + |\sum_{\substack{j=1\\ \sigma_j\notin B_0}}^{l} u(\sigma_j)| \leqq \sum_{\substack{i=1\\ s_i\notin B_0}}^{k} v(s_i) + \sum_{\substack{j=1\\ \sigma_j\notin B_0}}^{l} v(\sigma_j) =$$

$$= [\sum_{i=1}^{k} v(s_i) - \sum_{n=1}^{m} v(t_n)] + [\sum_{j=1}^{l} v(\sigma_j) - \sum_{n=1}^{m} v(t_n)] < 2\varepsilon\,.$$

According to Proposition 2 the series $\sum_{t\in[a,b]} u(t)$ is convergent. Since for every finite set $\{t_1, t_2, \ldots, t_m\} \subset [a, b]$ the inequality

$$|\sum_{n=1}^{m} u(t_n)| \leqq \sum_{n=1}^{m} v(t_n)$$

holds, we conclude that

$$|\sum_{t\in[a,b]} u(t)| \leqq \sum_{t\in[a,b]} v(t)\,.$$

(ii) By Theorem 5 there is a sequence $\{t_n\}_{n=1}^{\infty} \subset [a, b]$ of pairwise different points such that

$$\{t \in [a, b];\ v(t) \neq 0\} \subset \{t_1, t_2, t_3, \ldots\} \quad \text{and} \quad \sum_{t\in[a,b]} v(t) = \sum_{n=1}^{\infty} v(t_n)\,.$$

Let an arbitrary sequence of pairwise different points $\{s_j\}_{j=1}^{\infty} \subset [a, b]$ be given such that

$$\{t \in [a, b];\ u(t) \neq 0\} \subset \{s_1, s_2, s_3, \ldots\}\ .$$

For a given $\varepsilon > 0$ there is such an integer N that

$$\Big|\sum_{t\in[a,b]} v(t) - \sum_{n=1}^{m} v(t_n)\Big| < \varepsilon$$

holds for any $m \geqq N$. There is such an integer K that

$$\{s_1, s_2, \ldots, s_K\} \cap \{t_n\}_{n=1}^{\infty} \subset \{t_1, t_2, \ldots, t_N\}\ .$$

Let us mention that if $t \notin \{t_n\}_{n=1}^{\infty}$ then $v(t) = 0$. For any $k \geqq K$ we have

$$[a, b] \setminus (\{s_1, s_2, \ldots, s_k\} \cap \{t_n\}_{n=1}^{\infty}) \subset [a, b] \setminus \{t_1, t_2, \ldots, t_N\}\ .$$

Then

$$\Big|\sum_{t\in[a,b]} u(t) - \sum_{j=1}^{k} u(s_j)\Big| = \Big|\sum_{t\in[a,b]\setminus\{s_j\}_1^k} u(t)\Big| \leqq \sum_{t\in[a,b]\setminus\{s_j\}_1^k} v(t) =$$

$$= \sum_{t\in[a,b]\setminus(\{s_j\}_1^k\cap\{t_n\}_1^{\infty})} v(t) \leqq \sum_{t\in[a,b]\setminus\{t_n\}_1^N} v(t) = \sum_{t\in[a,b]} v(t) - \sum_{n=1}^{N} v(t_n) < \varepsilon\ .$$

Consequently $\sum_{j=1}^{\infty} u(s_j) = \sum_{t\in[a,b]} u(t)$.

Definition 3. *Assume that for every α from some index set C a series $\sum_{t\in[a,b]} u^\alpha(t)$ is given. We say that the series $\sum_{t\in[a,b]} u^\alpha(t) = u_\alpha$, $\alpha \in C$ are equiconvergent, if for every $\varepsilon > 0$ there is a gauge δ on $[a, b]$ such that*

$$\Big|\sum_{n=1}^{m} u^\alpha(t) - u_\alpha\Big| < \varepsilon \quad \textit{for every} \quad \{t_1, t_2, \ldots, t_m\} \in I(\delta) \quad \textit{and} \quad \alpha \in C\ .$$

Theorem 8. *Let for every $\alpha \in C$ a series $\sum_{t\in[a,b]} u^\alpha(t)$ be given. Assume that there are convergent series $\sum_{t\in[a,b]} v(t) = v$, $\sum_{t\in[a,b]} w(t) = w$ such that $v(t) \leqq u^\alpha(t) \leqq w(t)$ for every $t \in [a, b]$, $\alpha \in C$. Then the series $\sum_{t\in[a,b]} u^\alpha(t)$, $\alpha \in C$ are equiconvergent and there is a sequence $\{t_n\}_{n=1}^{\infty}$ such that*

$$\{t_n\}_{n=1}^{\infty} \subset \{t \in [a, b];\ u^\alpha(t) \neq 0 \textit{ for some } \alpha \in C\}\ ;$$

$$t_n \neq t_m \quad \textit{if} \quad n \neq m\ ; \qquad \sum_{t\in[a,b]} u^\alpha(t) = \sum_{n=1}^{\infty} u^\alpha(t_n) \quad \textit{for every} \quad \alpha \in C\ .$$

Proof. Let $\varepsilon > 0$ be given. Let δ_0 be a gauge such that

$$\Big|\sum_{n=1}^{k} v(t_n) - v\Big| < \varepsilon \quad \text{and} \quad \Big|\sum_{n=1}^{k} w(t_n) - w\Big| < \varepsilon \quad \text{for all}$$

$\{t_1, t_2, \ldots, t_k\} \in I(\delta_0)$.

Let $S = \{s_1, s_2, \ldots, s_p\} \in I(\delta_0)$ be a fixed set. Let us define

$$\delta(\tau) = \min\{\delta_0(\tau),\ \mathrm{dist}(\tau; \{s_1, \ldots, s_p\} \setminus \{\tau\})\} \quad \text{for} \quad \tau \in [a, b].$$

An arbitrary set $\{t_1, t_2, \ldots, t_m\} \in I(\delta)$ includes all the points $s_1, s_2, \ldots, s_p$. Then for every $\alpha \in C$ we have estimates

$$\sum_{n=1}^{m} u^\alpha(t_n) - \sum_{k=1}^{p} u^\alpha(s_k) = \sum_{\substack{n=1\\ t_n \notin S}}^{m} u^\alpha(t_n) \leqq \sum_{\substack{n=1\\ t_n \notin S}}^{m} w(t_n) =$$

$$= \sum_{n=1}^{m} w(t_n) - \sum_{k=1}^{p} w(s_k) = \left(\sum_{n=1}^{m} w(t_n) - w\right) + \left(w - \sum_{k=1}^{p} w(s_k)\right) < 2\varepsilon.$$

Analogously $\sum_{n=1}^{m} u^\alpha(t_n) - \sum_{k=1}^{p} u^\alpha(s_k) \geqq \left(\sum_{n=1}^{m} v(t_n) - v\right) + \left(v - \sum_{k=1}^{p} v(s_k)\right) > -2\varepsilon.$

Consequently

$$(34) \qquad \left|\sum_{k=1}^{m} u^\alpha(t_n) - \sum_{n=1}^{p} u^\alpha(s_k)\right| < 2\varepsilon.$$

Proposition 2 implies that $\sum_{t\in[a,b]} u^\alpha(t)$ is a convergent series and has a sum u_α. From (34) it follows that $\left|\sum_{n=1}^{m} u^\alpha(t_n) - u_\alpha\right| \leqq 2\varepsilon$, hence the series $\sum_{t\in[a,b]} u^\alpha(t)$, $\alpha \in C$ are equi-convergent.

By Theorem 5 and Corollary 1 there is a sequence $\{t_n\}_{n=1}^{\infty}$ such that $t_n \neq t_m$ for $n \neq m$,

$$(35) \qquad \sum_{t\in[a,b]} v(t) = \sum_{n=1}^{\infty} v(t_n),$$

$$(36) \qquad \{t_n\}_{n=1}^{\infty} \subset \{t \in [a, b];\ v(t) \neq 0\},$$

and

$$(37) \qquad \{t_n\}_{n=1}^{\infty} \subset \{t \in [a, b];\ w(t) \neq 0\}.$$

Let $\alpha \in C$. Then $u^\alpha(t) = v(t) + (u^\alpha(t) - v(t))$ where $u^\alpha(t) - v(t) \geqq 0$. By Proposition 5 the series $\sum_{t\in[a,b]} (u^\alpha(t) - v(t))$ is convergent. Since $u^\alpha(t) - v(t) \geqq 0$ for $t \in [a, b]$ and $u^\alpha(t) - v(t) = 0$ for every $t \in \{t_n\}_{n=1}^{\infty}$ according to (36), (37), Theorem 7 implies that

$$\sum_{t\in[a,b]} (u^\alpha(t) - v(t)) = \sum_{n=1}^{\infty} (u^\alpha(t_n) - v(t_n)).$$

Then

$$\sum_{t\in[a,b]} u^\alpha(t) = \sum_{t\in[a,b]} v(t) + \sum_{t\in[a,b]} (u^\alpha(t) - v(t)) =$$

$$= \sum_{n=1}^{\infty} v(t_n) + \sum_{n=1}^{\infty} (u^\alpha(t_n) - v(t_n)) = \sum_{n=1}^{\infty} u^\alpha(t_n).$$

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Souhrn

NEABSOLUTNĚ KONVERGENTNÍ ŘADY

DANA FRAŇKOVÁ

Nechť pro každé t z intervalu $[a, b]$ je dáno reálné číslo $u(t)$. Není problém sečíst všechna tato čísla $u(t)$ v případě, že řada $\sum_{t \in [a,b]} u(t)$ je absolutně konvergentní. Článek podává návod, jak sečíst řadu tohoto typu, která však není absolutně konvergentní. Používá se zde teorie zobecněného Perronova (neboli Kurzweilova) integrálu.

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