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INTEGRATING FACTOR

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Summary. The problem of integrating factor for ordinary differential equations is investigated. Conditions are given which guarantee that each solution of

$$\partial_1 F(x,y) + y'\partial_2 F(x,y) = 0$$

is also a solution of

$$M(x,y) + y'N(x,y) = 0$$

where $\partial_1 F = \mu M$ and $\partial_2 F = \mu N$.

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Introduction. Let G be an open set in the Euclidean plane and let f be a function differentiable on a one-dimensional interval I such that the graph of f is a part of G. Let F be a function differentiable on G; let $\partial_1 F$ and $\partial_2 F$ be the first order partial derivatives of F. It follows from well-known elementary theorems that

$$\partial_1 F\bigl(x, f(x)\bigr) + f'(x)\partial_2 F\bigl(x, f(x)\bigr) = 0$$

for each $x \in I$ if and only if the function $F\bigl(x, f(x)\bigr)$ $(x \in I)$ is constant. Thus, the differential equation

$$\partial_1 F(x,y) + y'\partial_2 F(x,y) = 0 \tag{1}$$

is, in this sense, equivalent to the "non-differential" equation

$$F(x,y) = c. \tag{2}$$

Now suppose that M, N and μ are functions on G such that

$$\partial_1 F = \mu M, \quad \partial_2 F = \mu N. \tag{3}$$

It is obvious that each solution of

$$M(x,y) + y'N(x,y) = 0 \tag{4}$$

is also a solution of (1). The present note investigates the question whether each solution of (1) is also a solution of (4). The function μ is usually called an *integrating factor* (i.f.) for the equation (4).

Authors of elementary textbooks dealing with differential equations caution us that, in general, it is not easy to find an i.f. for a given equation. We could easily correct this statement writing "nontrivial i.f." instead of "i.f.", but this would not remove the main difficulty. If we wish to go from (2) to (4), we must cancel μ. We might, of course, suppose that $\mu(z) \neq 0$ for each $z \in G$, but this would be a loss of generality. A "reasonable" i.f. still may have "many" zeros. If, e.g., G is the whole plane and f a function differentiable on $(-\infty, \infty)$, if, further

$$\mu(x,y) = y - f(x) \tag{5}$$

and if (3) holds, then, clearly, f is a solution of (1), but it is not obvious whether f is also a solution of (4). It follows, however, from the theorems proved in this note that this is the case, i.e. that we may cancel even integrating factors like (5). (The equation $y + 2xy' = 0$ and the function $F(x,y) = xy^2$ may serve as an illustration.) We suppose only that the functions M and N are continuous and that the set of zeros of μ has no interior point. (It is easy to see that this condition is fulfilled by each "practical" nontrivial i.f.) Under this condition, equations (4) and (1) are equivalent, i.e. we can solve (4) by means of (2).

Conventions. The word function means a mapping to the set $\mathbb{R} = (-\infty, \infty)$. Let M and N be functions on a set $G \subset \mathbb{R} \times \mathbb{R}$. By a solution of the equation (4) we mean a function f differentiable on a nondegenerate interval $I \subset \mathbb{R}$ such that $\langle x, f(x)\rangle \in G$ and $M(x, f(x)) + f'(x)N(x, f(x)) = 0$ for each $x \in I$. (If x belongs to the boundary of I, then, as usual, $f'(x)$ denotes the corresponding unilateral derivative.) The meaning of a statement like "f is a solution of (4) on $(0, 1]$" is obvious. When we say that an equation

$$M_1(x,y) + y'N_1(x,y) = 0 \tag{6}$$

is equivalent to (4), we mean that M_1 and N_1 are defined on G and that the system of all solutions of (6) is the same as the system of all solutions of (4).

Lemma. *Let a, $b \in \mathbb{R}$ and α, β, $\lambda \in (0,\infty)$. Let F be a function continuous on the set*

$$S = [a-\alpha, a+\alpha] \times [b-\beta, b+\beta]$$

and let $|F(z)| \leqslant \lambda$ for each $z \in S$. Let $0 < \sigma < \min(\alpha, \beta/\lambda)$; define $J = [a-\sigma, a+\sigma]$. Then there are $\delta \in (0,\sigma)$, $\varepsilon \in (0,\beta)$ with the following property: If $|a_1 - a| < \delta$, $|b_1 - b| < \varepsilon$, then there is a function f on J such that $f(a_1) = b_1$ and that

$$|f(x) - b| < \beta, \quad f'(x) = F\big(x, f(x)\big)$$

for each $x \in J$.

(This is a special case of Corollary, p. 23, [1].)

Theorem 1. *Let G be an open set in $\mathbb{R} \times \mathbb{R}$. Let F, M, N, μ be functions on G. Let F be differentiable, let M, N be continuous and let $\partial_1 F = \mu M$, $\partial_2 F = \mu N$ on G. Let f be a solution of the equation (1) on an open interval I. Suppose that $a \in I$ and that*

$$M\big(a, f(a)\big) + f'(a) N\big(a, f(a)\big) \neq 0. \tag{7}$$

Then $\mu = 0$ on some neighborhood of the point $\langle a, f(a)\rangle$.

Proof. Set $b = f(a)$, $P = \langle a, b\rangle$. We distinguish two cases.

1) $N(P) \neq 0$. For each $z \in G$ for which $N(z) \neq 0$ set $V(z) = -M(z)/N(z)$. Further set $\Delta = V(P) - f'(a)$, $\eta = |\Delta|/4$. By (7) we have $\eta > 0$. There are α, $\beta > 0$ such that $[a-\alpha, a+\alpha] \subset I$ and that

$$\begin{aligned} &|f(x) - b - (x-a)f'(a)| \leqslant \eta|x-a| \quad (|x-a| \leqslant \alpha),\\ &|V(x,y) - V(P)| \leqslant \eta \quad (|x-a| \leqslant \alpha,\ |y-b| \leqslant \beta). \end{aligned}$$

Let $0 < \sigma < \min\big(\alpha, \beta/(|V(P)| + \eta)\big)$; set $J = [a-\sigma, a+\sigma]$. According to Lemma there are $\delta \in (0,\sigma)$, $\varepsilon \in (0,\beta)$ with the following property: If $|a_1 - a| < \delta$ and $|b_1 - b| < \varepsilon$, then there is a function g on J such that $g(a_1) = b_1$ and that

$$|g(x) - b| < \beta, \quad g'(x) = V\big(x, g(x)\big) \quad (x \in J). \tag{8}$$

We may suppose that $\varepsilon + \delta|V(P)| < \eta\sigma$. Set $U = (a-\delta, a+\delta) \times (b-\varepsilon, b+\varepsilon)$. Let $\langle a_1, b_1\rangle \in U$, let g be as above and let $x \in J$. Define

$$\begin{aligned} \xi_1 &= (a-a_1)V(P) + \int_{a_1}^{x} \big(g'(t) - V(P)\big)\,\mathrm{d}t,\\ \xi_2 &= f(x) - b - (x-a)f'(a), \end{aligned}$$

$\xi = b_1 - b + \xi_1 - \xi_2$. As $g(x) = b_1 + (x-a)V(P) + \xi_1$, we have

$$g(x) - f(x) = b_1 - b + (x-a)\big(V(P) - f'(a)\big) + \xi_1 - \xi_2 = (x-a)\Delta + \xi.$$

It is easy to see that

$$|\xi| < \varepsilon + \delta|V(P)| + 2\eta\sigma + \eta\sigma < 4\eta\sigma = \sigma|\Delta|.$$

Therefore one of the numbers $g(a+\sigma) - f(a+\sigma)$, $g(a-\sigma) - f(a-\sigma)$ is positive and the other negative. It follows that $f(x_0) = g(x_0)$ for some $x_0 \in J$. According to (8), the function $F\big(x, g(x)\big)$ $(x \in J)$ is constant; since f is a solution of (1), the function $F\big(x, f(x)\big)$ $(x \in J)$ is constant as well. Thus $F(a_1, b_1) = F\big(a_1, g(a_1)\big) = F\big(x_0, g(x_0)\big) = F\big(x_0, f(x_0)\big) = F\big(a, f(a)\big)$. We see that F is constant on U. Since $N \neq 0$ on U, we have $\mu = N^{-1}\partial_2 F = 0$ there.

2) $N(P) = 0$. Then, by (7), $M(P) \neq 0$. For each $z \in G$ for which $M(z) \neq 0$ set $W(z) = -N(z)/M(z)$. Define $A = |f'(a)| + 1$. There are $\alpha, \beta > 0$ such that

$$\begin{aligned} &|f(x) - b| \leqslant A|x-a| \quad (|x-a| \leqslant \alpha), \\ &|W(x,y)| \leqslant (3A)^{-1} \quad (|x-a| \leqslant \alpha,\ |y-b| \leqslant \beta). \end{aligned}$$

Let $0 < \tau < \min(\beta, \alpha A)$; set $K = [b-\tau, b+\tau]$, $L = (a - \tau A^{-1}, a + \tau A^{-1})$. According to Lemma there are $\delta \in (0, \alpha)$, $\varepsilon \in (0, \tau)$ with the following property: If $|a_1 - a| < \delta$ and $|b_1 - b| < \varepsilon$, then there is a function h on K such that $h(b_1) = a_1$ and that

$$(9) \qquad |h(y) - a| < \alpha, \quad h'(y) = W\big(h(y), y\big) \quad (y \in K).$$

We may suppose that $\delta < \tau/(3A)$. Let $U = (a-\delta, a+\delta) \times (b-\varepsilon, b+\varepsilon)$. Let $\langle a_1, b_1 \rangle \in U$ and let h be as above. For $y \in K$ we have $|h(y) - a| \leqslant |h(y) - h(b_1)| + |a_1 - a| < 2\tau/(3A) + \delta < \tau/A$, thus $h(y) \in L$. For $x \in L$ we have $|f(x) - b| < \tau$. Set $S_j = \big\{\langle x, y\rangle\ ;\ x \in L, \operatorname{sgn}\big(y - f(x)\big) = j\big\}$ $(j = -1, 0, 1)$, $T = \{\langle h(y), y\rangle\ ;\ y \in K\}$. The sets S_{-1}, S_1 are open and disjoint; the set T is connected, $\langle h(b+\tau), b+\tau\rangle \in T \cap S_1$, $\langle h(b-\tau), b-\tau\rangle \in T \cap S_{-1}$, $T \subset S_{-1} \cup S_0 \cup S_1$. It follows that $T \cap S_0 \neq \emptyset$. This means that there are $x_0 \in L$, $y_0 \in K$ such that $y_0 = f(x_0)$, $x_0 = h(y_0)$. According to (9), the function $F\big(h(y), y\big)$ $(y \in K)$ is constant; the function $F\big(x, f(x)\big)$ $(x \in L)$ is constant as well. Thus $F(a_1, b_1) = F\big(h(b_1), b_1\big) = F\big(h(y_0), y_0\big) = F\big(x_0, f(x_0)\big) = F\big(a, f(a)\big)$. We see that F is constant on U. Since $M \neq 0$ on U, we have $\mu = M^{-1}\partial_1 F = 0$ there. $\square$

Theorem 2. *Let G, F, M, N, μ be as in Theorem 1. Let the set $\{z;\ \mu(z) \neq 0\}$ be dense in G. Then* (1) *is equivalent to* (4).

P r o o f. It is obvious that each solution of (4) is also a solution of (1). Now suppose that I_0 is a nondegenerate interval, $a_0 \in I_0$, f is a solution of (1) on I_0 and that $M\big(a_0, f(a_0)\big) + f'(a_0)N\big(a_0, f(a_0)\big) \neq 0$. Let I be the interior of I_0. It is easy to see that there are $a_1, a_2, \ldots \in I$ such that $a_n \to a_0$ and $f'(a_n) \to f'(a_0)$. If we choose $a = a_n$ with n sufficiently large, then (7) holds. From Theorem 1 we get a contradiction. □

R e m a r k. We may easily construct "unreasonable" integrating factors as follows: Let G and F be as in Theorem 1 and let ψ be a function differentiable on $\mathbb{R}$. It is easy to see that $\psi'\big(F(\cdot)\big)$ is an integrating factor for (1). Let e.g., $\psi(t) = t^3$ for $t > 0$, $\psi = 0$ on $(-\infty, 0]$ and let $M = 0$, $N = 1$ on $\mathbb{R} \times \mathbb{R}$. Then (4) becomes $y' = 0$ and the function $\mu(x, y) = \psi'(y)$ is an i.f. Even the well known equation $N\partial_1\mu - M\partial_2\mu = (\partial_2 M - \partial_1 N) \cdot \mu$ is fulfilled. Multiplying by μ we get $y'\psi'(y) = 0$ which is satisfied, for instance, by each nonpositive function differentiable on $\mathbb{R}$.

References

[1] *W.T. Reid*: Ordinary differential equations. Wiley, New York, 1971.

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