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ONE CONFIGURATIONAL CHARACTERIZATION OF OSTROM NETS

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Summary. By the quadrileteral condition in a given net there is meant the following implication: If A_1, A_2, A_3, A_4 are arbitrary poits, no three of them lie on the same line, with coll $(A_i A_j)$ (collinearity) for any five from six couples $\{i,j\}$ then there follows the collinearity coll $(A_k A_l)$ for the remaining couple $\{k, l\}$.

In the article there is proven the every net satisfying the preceding configuration condition is necessarity the Ostrom net (i.e., the net over a field). Conversely, every Ostrom net satisfies the above configuration condition.

Keywords: Net, Ostrom net, quadrilateral closure condition.

AMS classification: 51A20.

In the present article we will prove that a net of an arbitrary degree is an Ostrom net if and only if it satisfies the quadrilateral condition. Various characterizations of Ostrom nets by several closure conditions together have been discovered recently by many authors (cf. V. D. Belousov [1abc][1]), V. Havel [2a] and J. Kadleček [3], H. Thiele [4]). Besides, V. D. Belousov proved in [1a] that every net of finite degree satisfying the quadrilateral condition is an Ostrom net over a Galois field. Let us mention that the problem of characterization of general Ostrom nets (over skew-fields) by the quadrilateral condition was suggested to me by Professor V. Havel who recommended to use direct methods instead of the special methods of V. D. Belousov (from [1e]).

Part I is devoted to fundamental geometric notions concerning nets and algebraic descriptions of nets whereas in Part II we first deduce all requested algebraic properties of nets satisfying the quadrilateral condition and then verify the validity of the quadrilateral condition in a given Ostrom net.

PART I

1. By an incidence structure we understand a couple $(\mathscr{P}, \mathscr{L})$ of non-void sets $\mathscr{P}, \mathscr{L}$ such that

$$l \in \mathscr{L} \Rightarrow l \subseteq \mathscr{P}\,, \quad \# l \geqq 2$$

[1]) The article [1c] was written together with G. B. Beljavskaja.

and

$$A, B \in a\,,\ \ b \in \mathscr{L} \Rightarrow (A = B \vee a = b)\,.$$

$\mathscr{P}$ is the set of points, $\mathscr{L}$ the set of lines of the incidence structure.

Under a net $\mathscr{N}$ we shall mean an incidence structure $(\mathscr{P}, \mathscr{L})$ such that at least three lines pass through each of its points, and for every $(A, b) \in \mathscr{P} \times \mathscr{L}$ there is just one line a through A for which either $a = b$ or $a \cap b = \emptyset$. The relation of being parallel (on $\mathscr{L}$) is then defined as follows: $a \parallel b$ for $a, b \in \mathscr{L}$ if and only if either $a = b$ or $a \cap b = \emptyset$. The equivalence classes can be termed pencils (of parallel lines) of $\mathscr{N}$. If a, b are non-parallel lines then there exists their unique common point (denoted by $a \sqcap b$). If A, B are points on the same line then we shall denote this line by $A \sqcup B$. If all points of the set $\mathscr{P}' \subseteq \mathscr{P}$ lie on the same line then they are said to be collinear. We shall designate this property by $\operatorname{coll}(\mathscr{P}')$. If all lines of the set $\mathscr{L}' \subseteq \mathscr{L}$ go through the same common point then they will be said to be concurrent and we shall write $\operatorname{conc}(\mathscr{L}')$. If necessary we view the set of all pencils as an indexed collection $(\mathscr{L}_\iota)\ \iota \in \mathscr{I}$ so that the net can be then designated by $\mathscr{N} = (\mathscr{P}, \mathscr{L}, \mathscr{I})$. If $A \in \mathscr{P}$ and $b \in \mathscr{L}_\iota$ for some $\iota \in \mathscr{I}$ then $[A, b]$ or only $[A, \iota]$ will designate the line a such that $A \in a \in \mathscr{L}\iota$. The number $\# l$ is independent of $l \in \mathscr{L}$, equal to $\# \mathscr{L}\iota$ for every $\iota \in \mathscr{I}$, and we will call it the order of $\mathscr{N}$. The number $\# \mathscr{I}$ is called the degree of $\mathscr{N}$.

2. In the sequel we make use of one general coordinatizing concept due to V. Havel (see [2b]):

Let $\mathscr{N} = (\mathscr{P}, \mathscr{L}, \mathscr{I})$ be a net. Let $\omega_{\mathscr{N}} \subseteq \underset{\iota \in \mathscr{I}}{\times} \mathscr{L}_\iota$ (cartesian product of an indexed collection $(\mathscr{L}_\iota)$, $\iota \in \mathscr{I}$ be defined by $(l_\iota)_{\iota \in \mathscr{I}} \in \omega_{\mathscr{N}} :\Leftrightarrow \operatorname{conc}((l_\iota)_{\iota \in \mathscr{I}})$. This relation $\omega_{\mathscr{N}}$ satisfies the following condition: If $\alpha, \beta \in \mathscr{I}$ with $\alpha \neq \beta$ and $a \in \mathscr{L}_\alpha$, $b \in \mathscr{L}_\beta$ then there is a unique $(c_\iota)_{\iota \in \mathscr{I}} \in \omega_{\mathscr{N}}$ such that $c_\alpha = a$, $c_\beta = b$. This leads to the following definition: Let $(S_\iota)_{\iota \in \mathscr{I}}$, $\# \mathscr{I} \geqq 3$ be an indexed collection of at least two-point sets. We say that the relation $\sigma \subseteq \underset{\iota \in \mathscr{I}}{\times} S_\iota$ is admissible if it satisfies the condition of unique solvability: If $\alpha, \beta \in \mathscr{I}$ are distinct indexes and $a \in S_\alpha$, $b \in S_\beta$ then there is a unique collection $(c_\iota)_{\iota \in \mathscr{I}}$ such that $c_\alpha = a$, $c_\beta = b$.

We see that for every net $\mathscr{N} = (\mathscr{P}, \mathscr{L}, \mathscr{I})$ the relation $\omega_{\mathscr{N}}$ is admissible in $\underset{\iota \in \mathscr{I}}{\times} \mathscr{L}_L$. Conversely, every admissible relation $\sigma \subseteq \underset{\iota \in \mathscr{I}}{\times} S_\iota$, where $\# \mathscr{I} \geqq 3$, $\# S_\iota \geqq 2$ (for all $\iota \in \mathscr{I}$), uniquely determines a net $\mathscr{N}_\sigma = (\mathscr{P}, \mathscr{L}, \mathscr{I})$ such that $\omega_{\mathscr{N}} = \sigma$. In fact, put $\mathscr{P} = \sigma$ and for all $\alpha \in \mathscr{I}$ and all $s \in S_\alpha$ construct successively $l_s = \{(s_\iota)_{\iota \in \mathscr{I}} \in \sigma \mid s_\alpha = s\}$, $\mathscr{L}_\alpha = \{l_s \mid s \in S_\alpha\}$, $\mathscr{L} = \bigcup_{\alpha \in \mathscr{I}} \mathscr{L}_\alpha$.

An admissible relation $\sigma \subseteq \underset{\iota \in \mathscr{I}}{\times} S_\iota$ can be understood as a set of 3-basic quasigroup operations $\cdot_{\alpha\beta\gamma} : S_\alpha \times S_\beta \to S_\gamma$ with mutually distinct indexes α, β, γ from $\mathscr{I}$. Here for all $(x, y) \in S_\alpha \times S_\beta$ there is a unique collection $(z_\iota)_{\iota\in\mathscr{I}} \in \sigma$ such that $z_\alpha = x$ $z_\beta = y$, $x \cdot_{\alpha\beta\gamma} y = z_\gamma$.

This general concept yields some known algebraic descriptions of a given net, namely (i) the coordinatizing admissible algebra of Havel (cf. [2b]), (ii) the ternary partial groupoid of Thiele (cf. [5]) and (iii) the orthogonal system of quasigroups ("OCK") of Belousov (cf. [1a]).

Ad (i): Let $\mathscr{N} = (\mathscr{P}, \mathscr{L}, \mathscr{I})$ be a given net. We call every quadruple $(O, \mathrm{I}, \mathrm{II}, \mathrm{III})$ formed by a point O and by mutually distinct indexes $\mathrm{I}, \mathrm{II}, \mathrm{III} \in \mathscr{I}$ a frame. Choose a frame $(O, \mathrm{I}, \mathrm{II}, \mathrm{III})$ and denote by S the point set $[O, \mathrm{I}]$. Further, choose bijections $\varkappa_{\mathrm{I}}, \varkappa_{\mathrm{II}}, \varkappa_{\mathrm{III}}$ such that (Fig. 1)

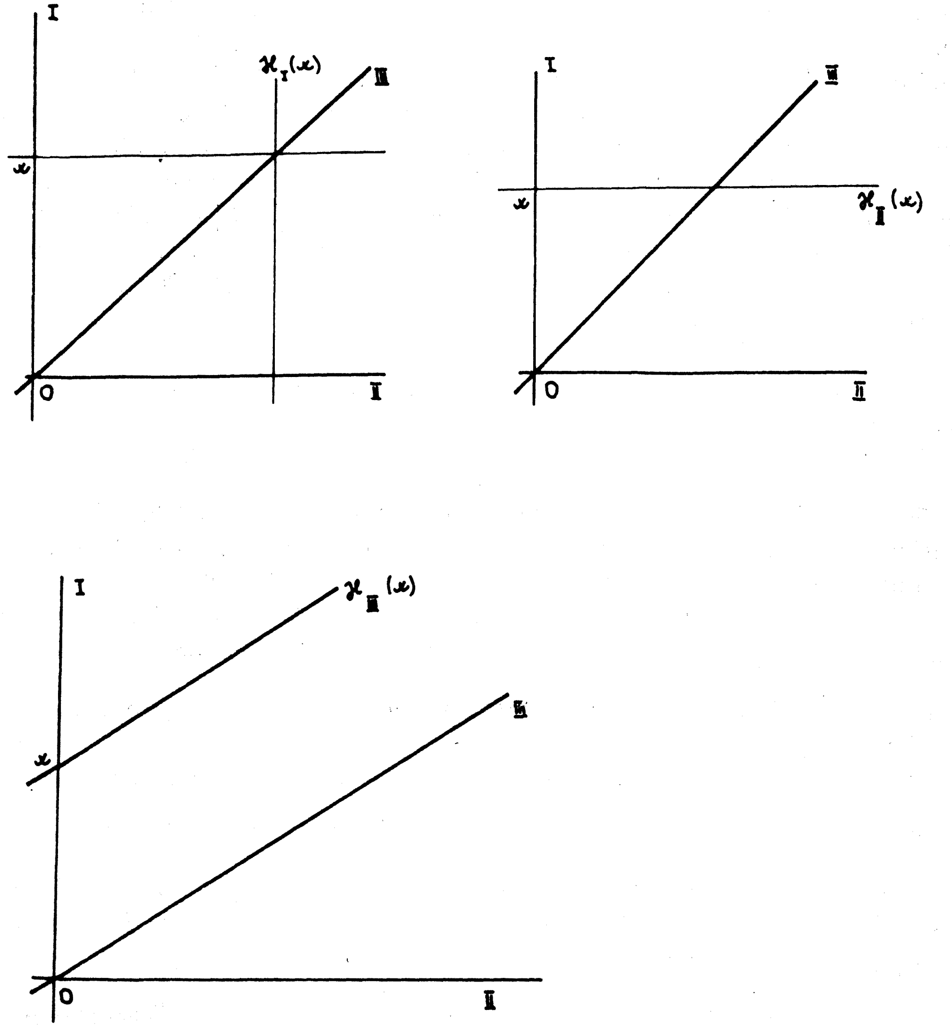

Fig. 1

$\varkappa_{\mathrm{I}}$: $S \to \mathscr{L}_{\mathrm{I}}$, $x \mapsto [([x, \mathrm{II}] \sqcap [O, \mathrm{III}]), \mathrm{I}]$,
$\varkappa_{\mathrm{II}}$: $S \to \mathscr{L}_{\mathrm{II}}$ $x \mapsto [x, \mathrm{II}]$ and
$\varkappa_{\mathrm{III}}$: $S \to \mathscr{L}_{\mathrm{III}}$, $x \mapsto [x, \mathrm{III}]$.

We change the operation $\cdot_{\mathrm{I\,II\,III}}$: $\mathscr{L}_{\mathrm{I}} \times \mathscr{L}_{\mathrm{III}} \to \mathscr{L}_{\mathrm{II}}$ isotopically onto the operation $+ : S \times S \to S$ such that (Fig. 2) $(x, z) \mapsto \varkappa_{\mathrm{II}}^{-1}(\varkappa_{\mathrm{I}}(x) \cdot_{\mathrm{I\,II\,III}} \varkappa_{\mathrm{III}}(z))$; $(S, +)$ is

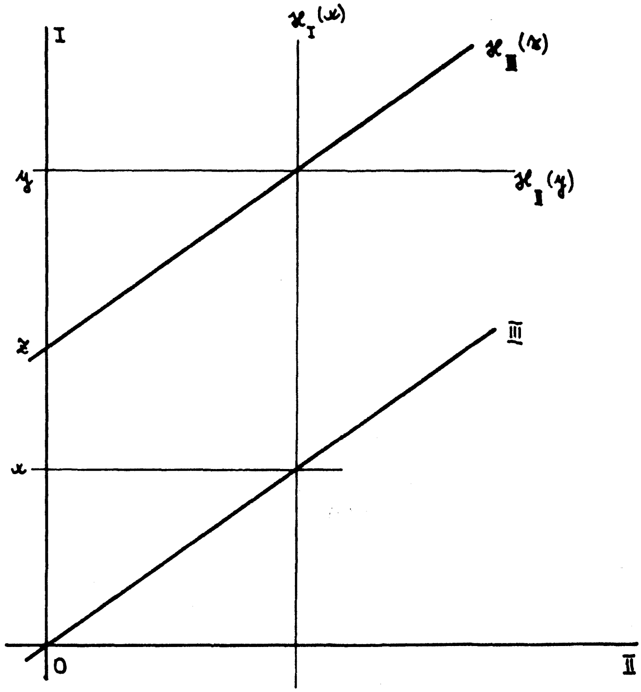

Fig. 2

a loop with neutral element 0. Further, let us distinguish 3-basic operations $\cdot_{\mathrm{I}\xi\mathrm{II}}$: $\mathscr{L}_{\mathrm{I}} \times \mathscr{L}_{\xi} \to \mathscr{L}_{\mathrm{II}}$ for all $\xi \in \mathscr{I} \setminus \{\mathrm{I}, \mathrm{II}, \mathrm{III}\}$ and change them isotopically into the operations $\square_{\xi} : S \times S \to S$ $(x, z) \mapsto \varkappa_{\mathrm{II}}^{-1}(\varkappa_{\mathrm{I}}(x) \cdot_{\mathrm{I}\xi\mathrm{II}} \varkappa_{\xi}(z))$ where $\varkappa_{\xi}$: $S \to \mathscr{L}_{\xi}$, $x \mapsto$ $\mapsto [x, \xi]$ (Fig. 3). With help of permutations φ_{ξ}: $S \to S$, $x \mapsto \varkappa_{\mathrm{II}}^{-1}(\varkappa_{\mathrm{I}}(x) \sqcap \varkappa_{\xi}(O))$ (Fig. 4) we obtain $x \square_{\xi} z = \varphi_{\xi}(x) +_{\xi} z$ where $+_{\xi}$: $S \times S \to S$ is an induced operation on S such that $(S, +_{\xi})$ is a loop with neutral element 0. Altogether, we get the coordinatizing admissible algebra $(S, O, (\varphi_{\xi})\ \xi \in \mathscr{I} \setminus \{\mathrm{I}, \mathrm{II}, \mathrm{III}\},\ +,\ (+_{\xi})\ \xi \in \mathscr{I} \setminus$ $\setminus \{\mathrm{I}, \mathrm{II}, \mathrm{III}\})$ of Havel. It has one 0-ary operation and a collection of unary operations.

Ad (ii): We start again from a frame $(O, \mathrm{I}, \mathrm{II}, \mathrm{III})$ of a given net $\mathscr{N} = (\mathscr{P}, \mathscr{L}, \mathscr{I})$ and in addition choose another point $E \in [O, \mathrm{III}]$. Let $S, \varphi_{\iota}, +_{\iota}$ $(\iota \in \mathscr{I} \setminus (\mathrm{I}, \mathrm{II}, \mathrm{III}\})$ have the same meaning as above. Further, put φ_{II}: $S \to \{0\}$ and φ_{III}: $S \to S$, $x \mapsto x$ (i.e. $\varphi_{\mathrm{III}} = \mathrm{id}_S$). Now we can define a partial ternary operation T: $\mathscr{I} \setminus \{\mathrm{I}\} \times S \times$ $\times S \to S$; $(\iota, x, z) \mapsto \varphi_{\iota}(x) +_{\iota} z$, obtaining the algebraic structure (S, T) of Thiele.

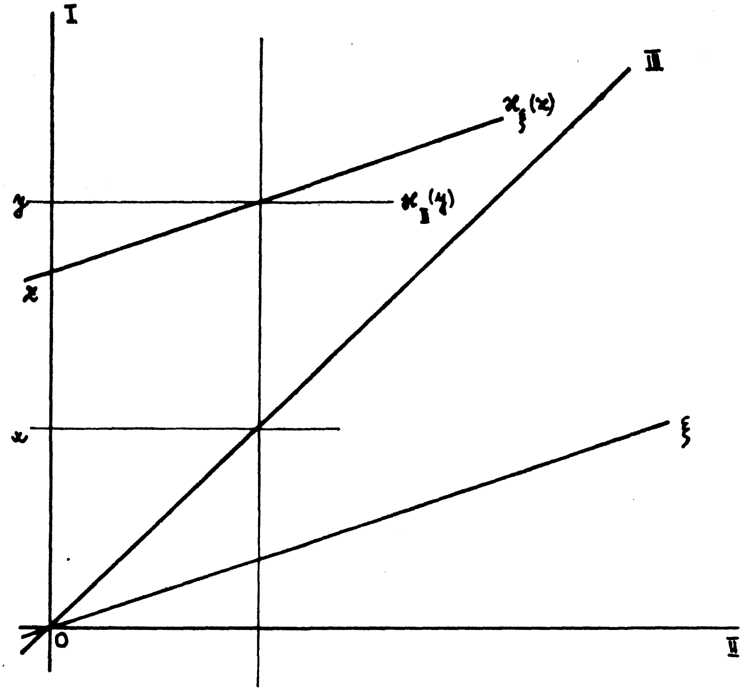
I
III
$\varkappa_\xi(z)$
y
$\varkappa_{II}(y)$
z
x
ξ
O
II

Fig. 3

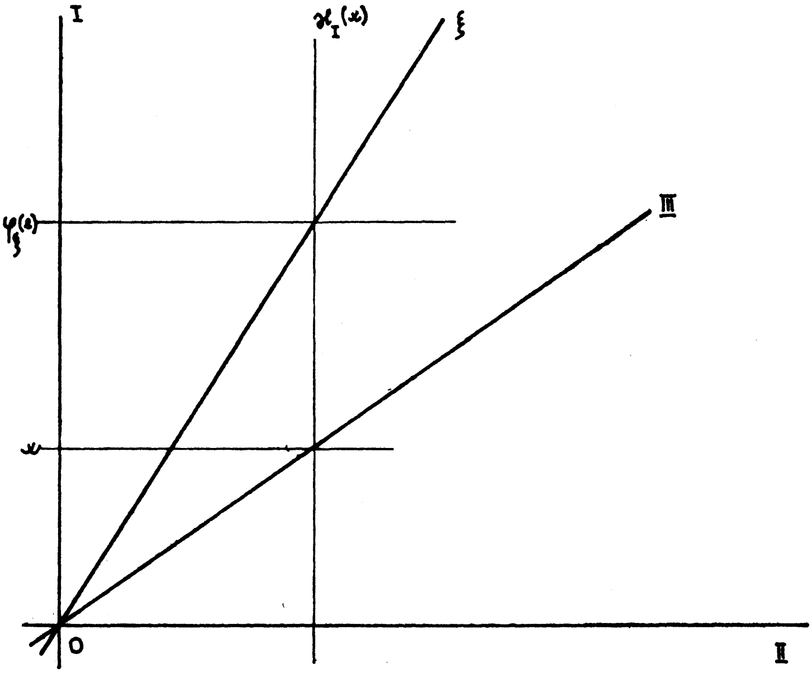
I
$\varkappa_I(x)$
ξ
$\varphi_\xi(z)$
III
x
O
II

Fig. 4

We see that Thiele obtain a ternary structure which is directly subordinated to the Hall ternary ring of an affine plane whereas Havel prefers the decomposition into nets of degree $(\mathscr{P}, \mathscr{L}_{\mathrm{I}} \cup \mathscr{L}_{\mathrm{II}} \cup \mathscr{L}_{\iota}, (\mathscr{L}_{\mathrm{I}}, \mathscr{L}_{\mathrm{II}}, \mathscr{L}_{\iota}), \iota \in \mathscr{I} \setminus \{\{\mathrm{I}, \mathrm{II}\}\})$.

Ad (iii): Let $\mathscr{N} = (\mathscr{P}, \mathscr{L}, \mathscr{I})$ be a given net and I, II distinct indexes. Take an arbitrary set S such that $\# S = \# l \, (l \in \mathscr{L})$, and bijections $\lambda_\iota: S \to \mathscr{L}_\iota$, $\iota \in \mathscr{I}$. Further, change every 3-basic operation $\cdot_{\mathrm{I}\,\mathrm{II}\,\xi}$, $\xi \in \mathscr{I} \setminus \{\mathrm{I}, \mathrm{II}\}$ isotopically into $\odot_{\mathrm{I}\,\mathrm{II}\,\xi}: S \times \times S \to S$, $(x, y) \mapsto \lambda_{\mathrm{I}}^{-1}(x) \odot_{\mathrm{I}\,\mathrm{II}\,\xi} \lambda_{\mathrm{II}}^{-1}(y)$. In this way we get the desired quasigroup system $((S, \odot_{\mathrm{I}\,\mathrm{II}\,\xi}), \xi \in \mathscr{I} \setminus \{\mathrm{I}, \mathrm{II}\})$. It results that $(S, \odot_{\mathrm{I}\,\mathrm{II}\,\alpha})$, $(S, \odot_{\mathrm{I}\,\mathrm{II}\,\beta})$ are orthogonal quasigroups whenever α, β are distinct indexes from $\mathscr{I} \setminus \{\mathrm{I}, \mathrm{II}\}$. Note that Belousov complements the above quasigroup operations by "projections" (or "selectors") E and F such that

$$E: S \times S \to S, \quad (x, y) \mapsto y; \quad F: S \times S \to S, \quad (x, y) \mapsto x.$$

3. Let $\mathscr{N} = (\mathscr{P}, \mathscr{L}, \mathscr{I})$ be a given net. Let us an ordered quadruple (A_1, A_2, A_3, A_4) of points a quadrilateral if (i) coll A_1A_2, col A_2, A_3, coll A_3A_4, coll A_4A_1[1]) and no three of the points A_1, A_2, A_3, A_4 are collinear. We will use the notation $A_1A_2A_3A_4$ instead of (A_1, A_2, A_3, A_4) for $i \neq j$; $A_i \sqcup A_j$, if it exists, will be called an edge, the edges $A_1 \sqcup A_2$, $A_2 \sqcup A_3$, $A_3 \sqcup A_4$, $A_4 \sqcup A_1$ are sides and $A_1 \sqcup A_3$, $A_2 \sqcup A_4$, if they exist, the first and second diagonal. A quadrilateral with the first diagonal or with both diagonals will be called admissible or full, respectively. If for an index quintuple $(\iota_1, \iota_2, \iota_3, \iota_4, \iota_5)$ there exists at least one admissible quadrilateral $X_1X_2X_3X_4$ such that $X_1 \sqcup X_2 \in \mathscr{L}_{\iota_1}$, $X_1 \sqcup X_3 \in \mathscr{L}_{\iota_2}$, $X_1 \sqcup X_4 \in \mathscr{L}_{\iota_3}$, $X_2 \sqcup X_3 \in \mathscr{L}_{\iota_4}$ and $X_3 \sqcup X_4 \in \mathscr{L}_{\iota_5}$ then $X_1X_2X_3X_4$ is said to be well situated.

The quadrilateral closure condition Q in a net $\mathscr{N} = (\mathscr{P}, \mathscr{L}, \mathscr{I})$. For every index quintuple $(\iota_1, \iota_2, \iota_3, \iota_4, \iota_5)$ admitting at least one well situated quadrilateral there exists a unique index ι_6 such that every well situated quadrilateral is full and its second diagonal belongs to $\mathscr{L}_{\iota_6}$. A net satisfying the condition Q will be called a Q-net. Now we choose again a frame $(O, \mathrm{I}, \mathrm{II}, \mathrm{III})$ put $S := [O, \mathrm{II}]$ and define the coordinate map $S \times S \to \mathscr{P}$, $(a, b) \mapsto \varkappa_{\mathrm{I}}(a) \sqcap \varkappa_{\mathrm{II}}(b)$ (Fig. 5) where $\varkappa_{\mathrm{I}}$ and $\varkappa_{\mathrm{II}}$ are bijections from § 2. The maps φ_ξ, $\xi \in \mathscr{I} \setminus \{\mathrm{I}, \mathrm{II}, \mathrm{III}\}$ also defined in § 2, can be complemented by maps $\varphi_{\mathrm{III}} = \mathrm{id}_S$, and $\varphi_{\mathrm{II}}: S \to \{0\}$ and we have then the set $\Phi := \{\varphi_\iota \mid \iota \in \mathscr{I} \setminus \{\mathrm{I}\}\}$. It will be useful for us to abbreviate $\mathscr{I}^\wedge = \mathscr{I} \setminus \{\mathrm{I}, \mathrm{II}\}$. For every $\iota \in \mathscr{I}^\wedge$ define a binary operation $+_\iota: S \times S \to S$ by $\varphi_\iota(a) +_\iota b = \varkappa_{\mathrm{II}}^{-1}(\varkappa_{\mathrm{I}}(a) \sqcap \varkappa_\iota(b))$, (where $\varkappa_\iota$ has been defined in § 2). Every groupoid $(S, +_\iota)$, $\iota \in \mathscr{I}^\wedge$ is a loop with neutral element 0 and all lines of $\mathscr{N}$ are described as follows: $[(0, a), \iota] = \{(x, y) \mid y = \varphi_\iota(x) +_\iota a\}$ for all $a \in S$ and $\iota \in \mathscr{I}^\wedge$, $[(0, a), \mathrm{II}] = \{(x, y) \mid y = a\}$ for all $a \in S$ and $[(a, 0), \mathrm{I}] = \{(x, y) \mid x = a\}$ for all $a \in S$.

4. Let $\mathscr{N} = (\mathscr{P}, \mathscr{L}, \mathscr{I})$ and $\mathscr{N}' = (\mathscr{P}', \mathscr{L}', \mathscr{I}')$ be nets. A map $\pi: \mathscr{P} \to \mathscr{P}'$ is called an isomorphism of $\mathscr{N}$ onto $\mathscr{N}'$ if π is a bijection with coll $AB \Leftrightarrow$ coll $\pi(A)\,\pi(B)$ for all $A, B \in \mathscr{P}$ and $\{\pi(X) \mid X \in l\} \in \mathscr{L}'$ for all $l \in \mathscr{L}$. An isomorphism of $\mathscr{N}$ onto $\mathscr{N}$

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[1]) coll AB is an abbreviation for coll (A, B).

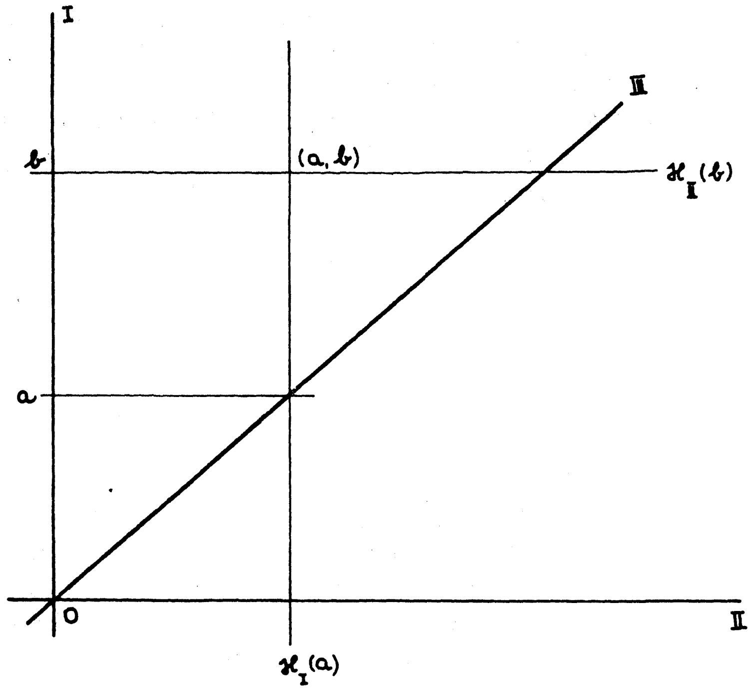

Fig. 5

is called an automorphism of $\mathcal{N}$. An automorphism of $\mathcal{N}$ is called an ι-translation if every point X together with its image lies on some line from $\mathcal{L}_\iota$ and if either all points are fixed or no point is fixed under the given automorphism. An Ostrom net over a skew-field $\mathbb{F}$ (cf. [4], [5]) is defined using a non-trivial left vector space $\mathcal{V}$ over $\mathbb{F}$ Points are ordered couples $(\mathbf{v}_1, \mathbf{v}_2) \in \mathcal{V} \times \mathcal{V}$ and lines are of the form $l_c = \{(\mathbf{c}, \mathbf{y}) \mid \mathbf{y} \in \mathcal{V}\}$ for all $\mathbf{c} \in \mathcal{V}$, $l_{\alpha,\mathrm{v}} = \{(\mathbf{x}, \alpha\mathbf{x} + \mathbf{v}) \mid \mathbf{x} \in \mathcal{V}\}$ for all $(\alpha, \mathbf{v}) \in \mathbb{F} \times \mathcal{V}$. ($\alpha$ is the slope and $\mathbf{v}$ the intercept of a "non-vertical" line $l_{\alpha,\mathrm{v}}$ whereas "∞" can be viewed as the common slope of all "vertical" lines l_c, thus we can take slopes as indexes of line pencils of the Ostrom net).

PART II

5. In what follows let $\mathcal{N} = (\mathcal{P}, \mathcal{L}, \mathcal{I})$ be a Q-net with a selected frame $(O, \mathrm{I}, \mathrm{II}, \mathrm{III})$ and with the coordinatizing admissible algebra $(S, O, (\varphi_\iota)\ \iota \in \mathcal{I}^\wedge, (+_\iota)\ \iota \in \mathcal{I}^\wedge)$ from § 3. First, some preliminary remarks: The operation $+_{\mathrm{III}}$ will be briefly denoted by $+$. The sixth (concluding) join line in condition Q will be drawn in figures by a dashed line. For every $a \in S \setminus \{0\}$ the quadrilateral $(0, 0), (a, 0), (a, a), (0, a)$ is well situated with respect to the index quintuple $(\mathrm{II}, \mathrm{III}, \mathrm{I}, \mathrm{I}, \mathrm{II})$ so that by condition Q

the concluding line $(0, a) \sqcup (a, 0)$ must exist and belongs to some pencil $\mathscr{L}_{\text{IV}}$ where the index IV depends only on I, II, III (and not on the element a). If $ABCD$ is an admissible quadrilateral with $A \sqcup B$, $C \sqcup D \in \mathscr{L}_\alpha$, $A \sqcup D$, $B \sqcup C \in \mathscr{L}_\beta$ and $A \sqcup C \in \mathscr{L}_\gamma$ for some indexes α, β, γ then by condition Q used for the index quintuple $(\alpha, \gamma, \beta, \beta, \alpha)$ the concluding line $B \sqcup D$ exists and belongs to some pencil $\mathscr{L}_\delta$ where δ depends only on α, β, γ. In the sequel we will not mention explicitly the corresponding index quintuples when applying condition Q as they will be clear from the context.

6. Let $a \in S \setminus \{0\}$. We investigate an admissible quadrilateral $(0, 0)\,(a, 0)\,(a, a)\,(0, a)$ and apply condition Q (Fig. 6). The concluding line $(0, a) \sqcup (a, 0) = k$ mus

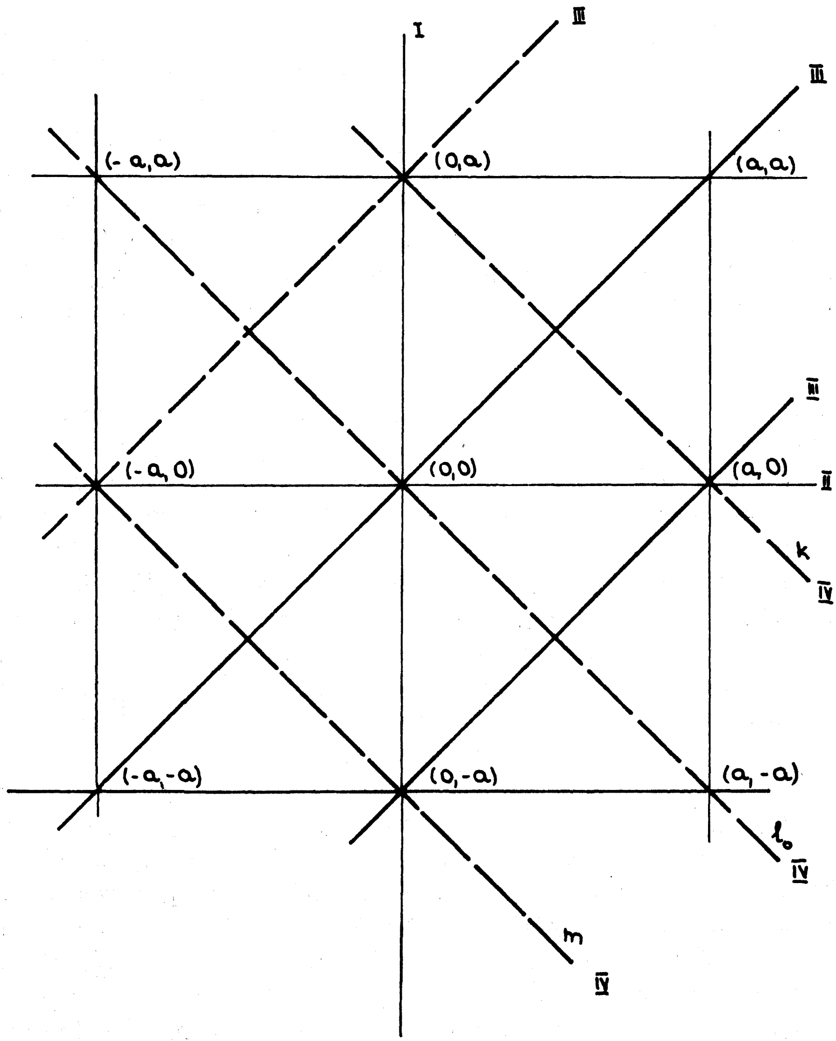

Fig. 6

belong to some $\mathscr{L}_{IV}$. Put $[(a, 0), III] \sqcap [(0, 0), I] = (0, b)$. As $\varphi_{III} = id_S$ we obtain $\varphi_{III}(a) + b = a + b = 0$. We can write $b = a^-$. If $(S, +)$ is an additive loop with neutral element 0 then we write $b = a^-$ and $a = -b$ if $a + b = 0$. Thus we have an admissible quadrilateral $(0, a^-)(a, a^-)(a, 0)(0, 0)$ and by applying condition Q we obtain the concluding line $(0, 0) \sqcup (a, a^-) = l_0$ belonging to $\mathscr{L}_{IV}$. Further application of condition Q to admissible quadrilaterals $(a^-, a^-)(0, a^-)(0, 0)(a^-, 0)$ and $(a, a)(a, a^-)(a^-, a^-)(a^-, a)$ gives the concluding lines $(0, a^-) \sqcup (a^-, 0)$ and $(a^-, a) \sqcup (a, a^-) \in \mathscr{L}_{IV}$, respectively. The latter concluding line is of course l_0 again. It remains to apply condition Q to the admissible quadrilateral $(0, 0)(0, a) . (a^-, a)(a^-, 0)$. Then the concluding line $(0, a) \sqcup (a^-, 0)$ belongs to $\mathscr{L}_{III}$ and we have the whole configuration in Fig. 6.

The line $(0, a) \sqcup (a^-, 0)$ has the equation $y = x + a$. As $(a^-, 0)$ is a point of this line, we have $(a^-) + a = 0$. Thus $a^- = -a$ and we obtain the additively inverse element of a.

The equations of the lines $[(0, a), IV]$, $[(0, 0), IV]$, $[(0, -a), IV]$ are $y = \varphi_{IV}(x) +_{IV} a$, $y = \varphi_{IV}(x)$ and $y = \varphi_{IV}(x) +_{IV} (-a)$, respectively. As the point $(a, 0)$ lies on the first line, the point $(a, -a)$ on the second and the point $(-a, a)$ on the second, we have $0 = \varphi_{IV}(a) +_{IV} a$, $-a = \varphi_{IV}(a)$, $a = \varphi_{IV}(-a)$ so that $a = \varphi_{IV}(\varphi_{IV}(a))$. From $(-a, 0) \in m$ it follows that $0 = \varphi_{IV}(-a) +_{IV} (-a)$. Thus $0 = a +_{IV} (-a)$ and $-a$ is the additively inverse element of the element a with respect to $+_{IV}$.

7. Let $a, b \in S \setminus \{0\}$; $a \neq b$. Using the already known properties of $\mathscr{N}$ we get the configuration in Fig. 7. The quadrilateral $(0, 0)(a + b, 0)(a + b, a + b) . (0, a + b)$ is admissible and condition Q yields the concluding line $(0, a + b) \sqcup (a + b, 0) = l_1 \in \mathscr{L}_{IV}$. Successively, we investigate admissible quadrilaterals $(0, b)(a, b)(a, a + b)(0, a + b)$, $(a, a)(b, a)(b, b)(a, b)$, $(a, b)(a, 0)(a + b, 0) . (a + b, b)$ and $(0, a + b)(0, a)(b, a)(b, a + b)$. The concluding lines by condition Q are $l_1 \in \mathscr{L}_{IV}$, $l_1 \in \mathscr{L}_{IV}$, $(a, 0) \sqcup (a + b, b) \in \mathscr{L}_{III}$ and $(0, a) \sqcup (b, a + b) \in \mathscr{L}_{III}$, respectively. The equations of l_1 and of the two last lines are $y = \varphi_{IV}(x) +_{IV} (a + b)$, $y = x + (-a)$ and $y = x + a$ respectively. From $(b, a + b) \in [(0, a), III]$ it follows (by substituting $x = b$, $y = a + b$ into the equation $y = x + a$) that $a + b = b + a$. Thus the operation $+$ is commutative.

Since $(a, b), (b, a) \in l_1$ we obtain analogously $(I_{a,b})$ $b = \varphi_{IV}(a) +_{IV} (a + b)$. Now take $(I_{\varphi_{IV}(a),b})$ and use $\varphi_{IV}^2 = id_S$ so that $(II_{a,b})$ $b = a +_{IV} (\varphi_{IV}(a) + b)$. We will investigate the equation $\varphi_{IV}(a) + (a +_{IV} b) = c$. Add (by $+_{IV}$) the element a from the left. One obtains $a +_{IV} (\varphi_{IV}(a) + (a +_{IV} b)) = a +_{IV} c$, where the left hand side is equal to $a +_{IV} b$ by $(II_{a,a+_{IV}b})$. As $(S, +_{IV})$ is a loop, $a +_{IV} b = a +_{IV} c$ implies $b = c$. Thus $b = \varphi_{IV}(a) + (a +_{IV} b)$. From $(a + b, b) \in [(0, a), III]$ it follows that $b = (a + b) + (-a)$. By $-a = \varphi_{IV}(a)$ and by commutativity of $+$ we get $b = \varphi_{IV}(a) + (a + b)$. From $b = \varphi_{IV}(a) + (a +_{IV} b) = \varphi_{IV}(a) + (a + b)$ we deduce $a +_{IV} b = a + b$. Hence $+ = +_{IV}$.

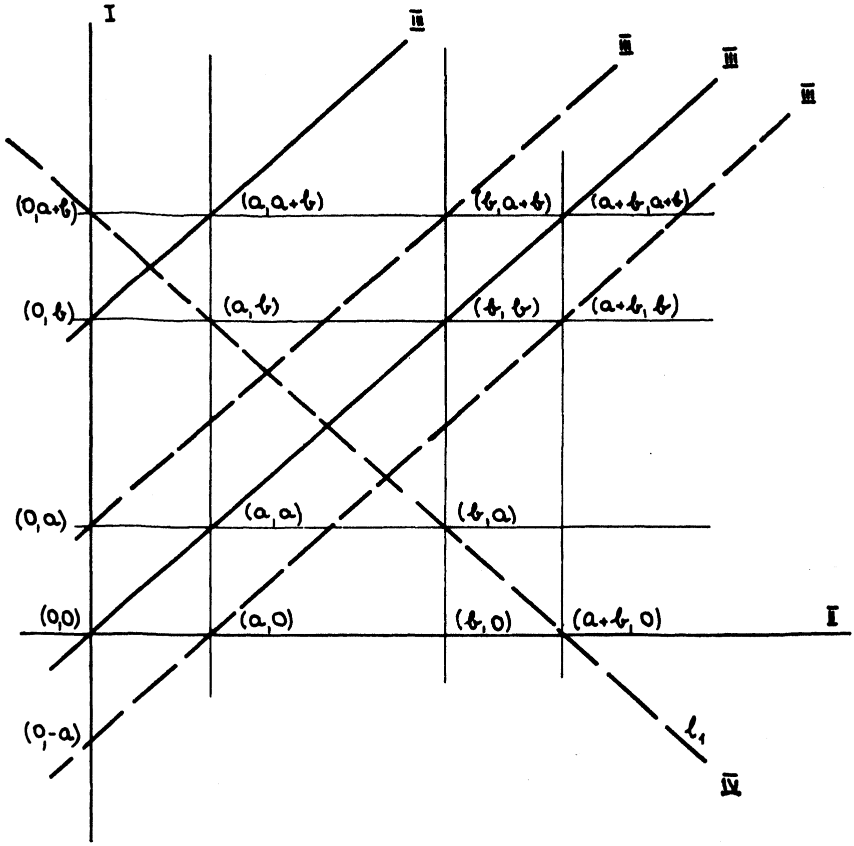

Fig. 7

Now take $(\mathrm{I}_{a+b,\varphi_{\mathrm{IV}}(a)})$, i.e. $\varphi_{\mathrm{IV}}(a) = \varphi_{\mathrm{IV}}(a + b) + (\varphi_{\mathrm{IV}}(a) + (a + b))$. The second term on the right hand side is equal to b. By adding $\varphi_{\mathrm{IV}}(b)$ we obtain $\varphi_{\mathrm{IV}}(b) + \varphi_{\mathrm{IV}}(a) = \varphi_{\mathrm{IV}}(b) + (\varphi_{\mathrm{IV}}(a + b) + b)$. Here the right hand side is equal to $\varphi_{\mathrm{IV}}(a + b)$ so that by commutativity of $+$ we obtain $\varphi_{\mathrm{IV}}(a) + \varphi_{\mathrm{IV}}(b) = \varphi_{\mathrm{IV}}(a + b)$.

8. Let $a, b, c \in S$. Set $A = (\varphi_{\mathrm{IV}}(b), a)$, $B = (\varphi_{\mathrm{IV}}((a + b) + c), a)$, $C = ((\varphi_{\mathrm{IV}}(a + b) + c), \varphi_{\mathrm{IV}}(c))$, $D = (\varphi_{\mathrm{IV}}(b), \varphi_{\mathrm{IV}}(c))$. As $A \sqcup B$, $C \sqcup D$ exist and belong to $\mathscr{L}_{\mathrm{II}}$, $A \sqcup D$, $B \sqcup C$ exist and belong to $\mathscr{L}_{\mathrm{I}}$ and $A \sqcup C$ exists and belongs to $\mathscr{L}_{\mathrm{III}}$, $ABCD$ is an admissible quadrilateral and condition Q applied to $ABCD$ gives the concluding line $B \sqcup D = \{(x, \varphi_{\mathrm{IV}}(x) + v) \mid x \in S\} \in \mathscr{L}_{\mathrm{IV}}$ with a convenient $v \in S$. Since D lies on this line we have $\varphi_{\mathrm{IV}}(c) = \varphi_{\mathrm{IV}}(\varphi_{\mathrm{IV}}(b)) + v$. Using $\varphi_{\mathrm{IV}}^2 = \mathrm{id}_S$ and adding $\varphi_{\mathrm{IV}}(b)$ we get $\varphi_{\mathrm{IV}}(b) + \varphi_{\mathrm{IV}}(c) = \varphi_{\mathrm{IV}}(b) + (b + v)$. Using the additivity of φ_{IV} and $(\mathrm{I}_{b,v})$ we obtain $\varphi_{\mathrm{IV}}(b + c) = v$ so that $B \sqcup D = \{(x, \varphi_{\mathrm{IV}}(x) + \varphi_{\mathrm{IV}}(b + c)) \mid x \in S\}$. As the point B lies on this line we can write $a = \varphi_{\mathrm{IV}}(\varphi_{\mathrm{IV}}((a + b) + c)) + \varphi_{\mathrm{IV}}(b + c)$. By $\varphi_{\mathrm{IV}}^2 = \mathrm{id}_S$ we have $a = ((a + b) +$

$+ c) + \varphi_{\mathrm{IV}}(b + c)$. Adding $b + c$ and using commutativity of $+$ we obtain $a + (b + c) = (b + c) + (\varphi_{\mathrm{IV}}(b + c) + ((a + b) + c))$. Finally by $(\mathrm{II}_{b+c,(a+b)+c})$ we conclude $a + (b + c) = (a + b) + c$, the associativity for $+$. So $(S, +)$ is a commutative group.

9. Now we shall modify the preceding investigations by replacing the frame $(O, \mathrm{I}, \mathrm{II}, \mathrm{III})$ by a new frame $(O, \mathrm{I}, \mathrm{II}, \alpha)$, $\alpha \neq \mathrm{III}$. The second coordinates rest without change whereas the first coordinates are now determined with help of the line $[0, \alpha]$ instead of $[0, \mathrm{III}]$ (cf. Fig. 8). Instead of the operation $+ = +_{\mathrm{III}}$ we

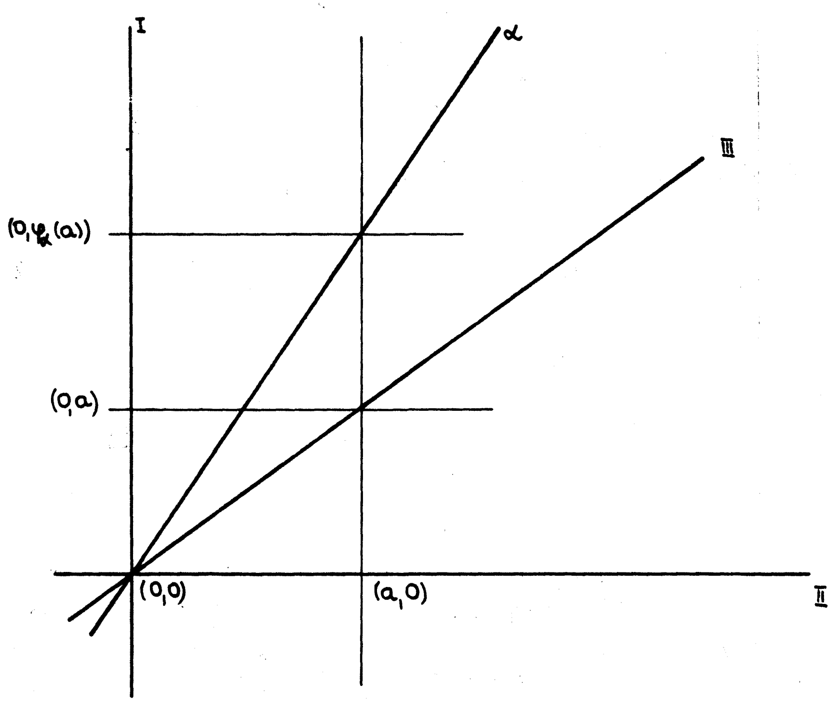

Fig. 8

obtain the operation $+_\alpha$ which is again commutative and associative. Hence $(S, +_\alpha)$ is a commutative group for all $\alpha \in \mathscr{I}^\wedge$.

10. Now we choose arbitrary elements $a, b, c \in S$ with $c \neq a$ and arbitrary indexes $\alpha, \beta \in \mathscr{I}^\wedge$. Put $[(a, b), \mathrm{II}] = l_0$, $[(a, b), \mathrm{I}] = l_1$, $[(c, b), \mathrm{I}] = l_2$, $[(a, b), \alpha] = \{(x, \varphi_\alpha(x) +_\alpha b -_\alpha \varphi_\alpha(a)) \mid x \in S\} = l_3$, $[(c, b), \beta] = \{(x, \varphi_\beta(x) +_\beta b -_\beta \varphi_\beta(c)) \mid x \in S\} = l_4$, $l_3 \sqcap l_2 = (c, \varphi_\alpha(c) +_\alpha b -_\alpha \varphi_\alpha(a)) = A$, $l_4 \sqcap l_1 = (a, \varphi_\beta(a) +_\beta b -_\beta \varphi_\beta(c)) = B$. The quadrilateral $(a, b)\, B (c, b)\, A$ is admissible and condition Q applied to it gives the concluding line $A \sqcup B \in \mathscr{L}_\gamma$ for some index $\gamma \neq$ I. We shall distinguish two cases: either $\gamma =$ II or $\gamma \in \mathscr{I}^\wedge$.

The case $\gamma =$ II: As $A \sqcup B = \{(x, y) \mid y = \text{const.}\}$, we have $\varphi_\alpha(c) +_\alpha b -_\alpha \varphi_\alpha(a) = \varphi_\beta(a) +_\beta b -_\beta \varphi_\beta(c)$. For $b := 0$ we obtain $\varphi_\alpha(c) -_\alpha \varphi_\alpha(a) = \varphi_\beta(a) -_\beta \varphi_\beta(c) := d$. If a, c run over S then also d runs over S and using the commutativity of $+_\alpha$ and $+_\beta$ we get $d +_\alpha b = d +_\beta b$ (identically for $d, b \in S$).

The case $\gamma \in \mathscr{I}^\wedge$: As $A \sqcup B = \{(x, \varphi_\gamma(x) +_\gamma (\varphi_\beta(a) +_\beta b -_\beta \varphi_\beta(c)) -_\gamma \varphi_\gamma(a)) \mid x \in S\}$ and A is one point of this line, it follows that

(i) $\varphi_\alpha(c) +_\alpha b -_\alpha \varphi_\alpha(a) = \varphi_\gamma(c) +_\gamma (\varphi_\beta(a) +_\beta b -_\beta \varphi_\beta(c)) -_\gamma \varphi_\gamma(a)$. As a consequence we obtain the following assertion.

$$\forall \alpha, \beta \in \mathscr{I}^\wedge \quad \exists \gamma \in \mathscr{I}^\wedge \quad \forall a, b, c \in S:$$

(ii) $(\varphi_\alpha(c) +_\alpha b -_\alpha \varphi_\alpha(a)) +_\gamma \varphi_\gamma(a) = \varphi_\gamma(c) +_\gamma (\varphi_\beta(a) +_\beta b -_\beta \varphi_\beta(c))$.
For $c := 0$ and $b := \varphi_\alpha(a)$ we get

(iii) $\forall \alpha, \beta \in \mathscr{I}^\wedge$ $\exists \gamma \in \mathscr{I}^\wedge$ $\forall a \in S$: $\varphi_\alpha(a) +_\beta \varphi_\beta(a) = \varphi_\gamma(a)$. By setting $b := \varphi_\beta(c)$ we deduce from (ii)

$$(\varphi_\alpha(c) +_\alpha \varphi_\beta(c) -_\alpha \varphi_\alpha(a)) +_\gamma \varphi_\gamma(a) = \varphi_\gamma(c) +_\gamma (\varphi_\beta(a) +_\beta \varphi_\beta(c) -_\beta \varphi_\beta(c)).$$

Consequently $(\varphi_\gamma(c) -_\alpha \varphi_\alpha(a)) +_\gamma \varphi_\gamma(a) = \varphi_\gamma(c) +_\gamma \varphi_\beta(a)$ (we have used the equation (iii) with changed indexes α, β). As a and c are arbitrary, we can suppose that $\varphi_\gamma(c) = \varphi_\alpha(a)$ which yields.

(iv) $\forall \alpha, \beta \in \mathscr{I}^\wedge$ $\exists \gamma \in \mathscr{I}^\wedge$ $\forall a \in S$: $\varphi_\alpha(a) +_\gamma \varphi_\beta(a) = \varphi_\gamma(a)$.
Substituting $b := \varphi_\beta(c)$ in (i) and arranging the left hand side by means of (iii) (with changed α, β) we get

$$(\varphi_\gamma(c) -_\alpha \varphi_\alpha(a)) +_\gamma \varphi_\gamma(a) = \varphi_\gamma(c) +_\gamma \varphi_\beta(a).$$

From this identity we successively obtain by (iv) and by the properties of $+_\gamma$:

$$(\varphi_\gamma(c) -_\alpha \varphi_\alpha(a)) +_\gamma \varphi_\alpha(a) +_\gamma \varphi_\beta(a) = \varphi_\gamma(c) +_\gamma \varphi_\beta(a),$$
$$(\varphi_\gamma(c) -_\alpha \varphi_\alpha(a)) +_\gamma \varphi_\alpha(a) = \varphi_\gamma(c); \quad \varphi_\gamma(c) -_\alpha \varphi_\alpha(a) = \varphi_\gamma(c) -_\gamma \varphi_\alpha(a).$$

Putting $\varphi_\gamma(c) := u$, $\varphi_\alpha(a) := v$ we get $u -_\alpha v = u -_\gamma v$. Analogously we obtain $u -_\beta v = u -_\gamma v$ so that $u -_\alpha v = u -_\beta v$ (identically for $u, v \in S$). For $u := 0$ we have $-_\alpha v = -_\beta v$ so that $x +_\alpha y = x +_\beta y$ (it suffices to set $u := x$, $v := -_\alpha y = -_\beta y$). Thus also here the operations α, β are the same.

Both cases yield the result that all operations $+_\iota$, $\iota \in \mathscr{I}^\wedge$ coincide (and can be denoted by the same symbol $+$). Then the new formulation of assertion (iii) reads as follows:

(iii') $\forall \alpha, \beta \in \mathscr{I}^\wedge$ $\exists \gamma \in \mathscr{I}^\wedge$ $\forall a \in S$: $\varphi_\alpha(a) + \varphi_\beta(a) = \varphi_\gamma(a)$. [1])
So we can define an addition $+$ on Φ (the set of all φ_ι as introduced in § 3) such that

[1]) As follows from the properties of the coordinatizing algebra γ is uniquely determined by α, β.

$\varphi_\alpha + \varphi_\beta = \varphi_\gamma$ for all $\alpha, \beta \in \mathscr{I}^\wedge$ by (iii') and $\varphi_{\mathrm{II}} + \varphi_\iota := \varphi_\iota$, $\varphi_\iota + \varphi_{\mathrm{II}} = \varphi_\iota$ for all $\iota \in \mathscr{I} \setminus \{\mathrm{I}\}$ (we know that $\varphi_{\mathrm{II}}(x) := 0$ for all $x \in S$).

11. Now we shall verify that $(\Phi, +)$ is a commutative group. From the definition of φ_{II} and the operation $+$ it is clear that φ_{II} is the neutral element of the groupoid $(\Phi, +)$. The associativity or commutativity for $+$ follows at once from the definition of $+$ and from the associativity or commutativity of $+$, respectively. It remains to prove that for every $\varphi \in \Phi$ there exists a unique $\varphi' \in \Phi$ such that $\varphi + \varphi' = \varphi_{\mathrm{II}}$. If $\varphi = \varphi_{\mathrm{II}}$ then $\varphi' = \varphi_{\mathrm{II}}$, too Let $\iota \in \mathscr{I}^\wedge$ and $a \in S \setminus \{0\}$. We will consider the quadrilateral $(0, 0)\,(a, 0)\,(a, \varphi_\iota(a))\,(0, \varphi_\iota(a))$ (Fig. 9). It is admissible and we can

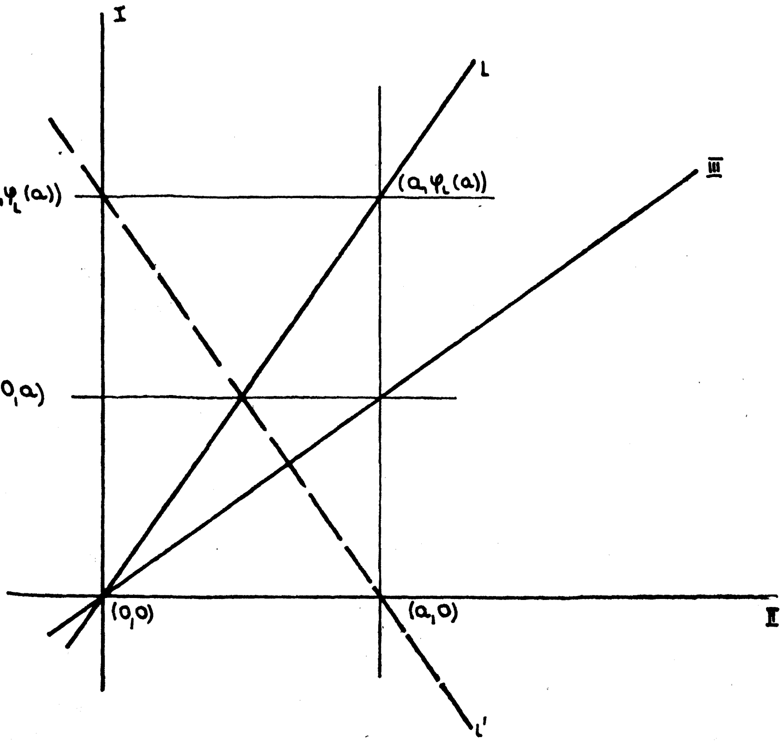

Fig. 9

apply condition Q to it obtaining the concluding line $(a, 0) \sqcup (0, \varphi_\iota(a)) \in \mathscr{L}_{\iota'}$ for some index $\iota' \in \mathscr{I}$. This line has the equation $y = \varphi_{\iota'}(x) + \varphi_\iota(a)$. As $(a, 0)$ is one point of the line we have $0 = \varphi_{\iota'}(a) + \varphi_\iota(a)$, i.e. $(\varphi_{\iota'} + \varphi_\iota)(a) = 0 = \varphi_{\mathrm{II}}(a)$. As a is arbitrary (the case $a = 0$ is trivial) and $+$ is commutative we have obtained the requested assertion.

12. Choose $\alpha, \beta \in \mathscr{I}^\wedge$ and $a \in S \setminus \{0\}$. The quadrilateral $(0, 0)\,(0, \varphi_\alpha(a))\,(a, \varphi_\alpha(a))\,\cdot$ $.\,(a, 0)$ (Fig. 10) is admissible. Using condition Q for a convenient index quintuple we get the concluding line $(a, 0) \sqcup (0, \varphi_\alpha(a))$.

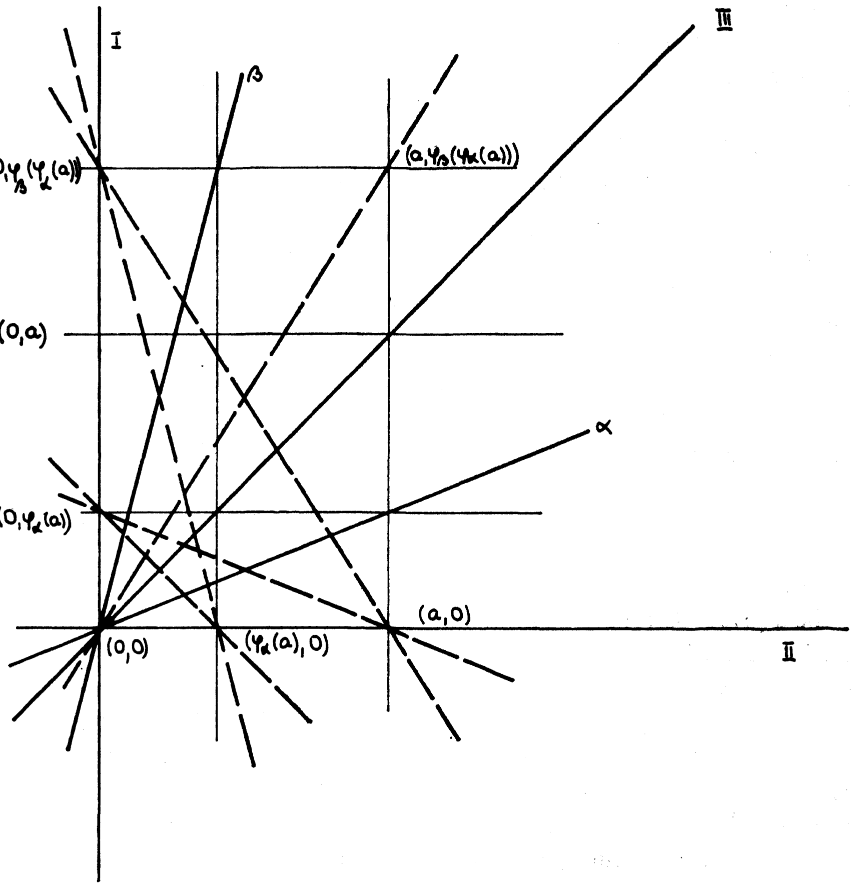

Fig. 10

Thus successively get further admissible quadrilaterals $(0, 0)\,(\varphi_\alpha(a), 0)\,.$ $.\,(\varphi_\alpha(a), \varphi_\beta(\varphi_\alpha(a)))\,(0, \varphi_\beta(\varphi_\alpha(a)))$ (condition Q yields the concluding line $(\varphi_\alpha(a), 0) \sqcup$ $\sqcup\,(0, \varphi_\beta(\varphi_\alpha(a))))$, $(0, \varphi_\alpha(a))\,(a, 0)\,(\varphi_\alpha(a), 0)\,(0, \varphi_\beta(\varphi_\alpha(a)))$ (condition Q yields the concluding line $(a, 0) \sqcup (0, \varphi_\beta(\varphi_\alpha(a))))$ and $(a, 0)\,(0,0)\,(0, \varphi_\beta(\varphi_\alpha(a)))\,(a, \varphi_\beta(\varphi_\alpha(a)))$ (condition Q yields the concluding line $(0, 0) \sqcup (a, \varphi_\beta(\varphi_\alpha(a))) = \{(x, \varphi_\gamma(x) \mid x \in S\} \in$ $\in \mathscr{L}_\gamma$ for some $\gamma \in \mathscr{I}^\wedge$). Substituting the coordinates of the point $(a, \varphi_\beta(\varphi_\alpha(a)))$ into $y = \varphi_\gamma(x)$ we obtain $\varphi_\beta(\varphi_\alpha(a)) = \varphi_\gamma(a)$. This means that $\varphi_\alpha \circ \varphi_\beta = \varphi_\gamma$ ($\circ$ is the map composition). Thus $(\Phi, \circ)$ is an associative groupoid with neutral element $\varphi_{\mathrm{III}} = \mathrm{id}_S$ i.e. a monoid.

Let $\alpha, \beta, \gamma \in \mathscr{I}^\wedge$. Then $((\varphi_\alpha + \varphi_\beta) \circ \varphi_\gamma)(x) = \varphi_\gamma((\varphi_\alpha + \varphi_\beta)(x)) =$ $= \varphi_\gamma(\varphi_\alpha(x) + \varphi_\beta(x)) = \varphi_\gamma(\varphi_\alpha(x)) + \varphi_\gamma(\varphi_\beta(x)) = (\varphi_\alpha \circ \varphi_\gamma)(x) + (\varphi_\beta \circ \varphi_\gamma)(x) =$ $= ((\varphi_\alpha \circ \varphi_\gamma) + (\varphi_\beta \circ \varphi_\gamma))(x)$ for all $x \in S$. Thus the distributivity $(\varphi_\alpha + \varphi_\beta) \circ \varphi_\gamma =$ $= (\varphi_\alpha \circ \varphi_\gamma) + (\varphi_\beta \circ \varphi_\gamma)$ is valid. Analogously for the distributivity $\varphi_\alpha \circ (\varphi_\beta + \varphi_\gamma) =$

$= (\varphi_\alpha \circ \varphi_\beta) + (\varphi_\alpha \circ \varphi_\gamma)$. Both distributivities hold trivially also if the index II is permited. Finaly, let $\iota \in \mathscr{I}^\wedge$ and $a \in S \setminus \{0\}$. The quadrilateral $(a, 0)(0, -a) . (0, -\varphi_\iota(a))(\varphi_\iota(a), 0)$ is admissible as $(a, 0) \sqcup (0, -a), (\varphi_\iota(a), 0) \sqcup (0, -\varphi_\iota(a))$ exist and belong to $\mathscr{L}_{\mathrm{III}}$ and also $(a, 0) \sqcup (0, -\varphi_\iota(a))$ exist and belong to $\mathscr{L}_\iota$. Thus we may apply condition Q obtaining the concluding line $(0, -a) \sqcup (\varphi_\iota(a), 0) = \{(x, \varphi_{\bar\iota}(x) - a) \mid x \in S\} \in \mathscr{L}_{\bar\iota}$ for some $\bar\iota \in \mathscr{I}^\wedge$. As the point $(\varphi_\iota(a), 0)$ lies on this line we have $0 = \varphi_{\bar\iota}(\varphi_\iota(a)) - a$ and together with the trivial case $a = 0$ we conclude that $\varphi_\iota \circ \varphi_{\bar\iota} = \mathrm{id}_S = \varphi_{\mathrm{III}}$. We also have $\varphi_{\bar\iota} \circ \varphi_\iota = \mathrm{id}_S$ as $(\varphi_{\bar\iota} \circ \varphi_\iota)(\varphi_\iota(a)) = (\varphi_\iota \circ (\varphi_{\bar\iota} \circ \varphi_\iota))(a) = ((\varphi_\iota \circ \varphi_{\bar\iota}) \circ \varphi_\iota)(a) = \varphi_\iota(\mathrm{id}_S(a)) = \varphi_\iota(a)$. Adding the trivial case with $a = 0$ we also have $\varphi_{\bar\iota} \circ \varphi_\iota = \mathrm{id}_S$. Thus $(\Phi, \circ)$ is a group and $\mathbb{F} = (\Phi, +, \circ)$ is a skew-field, and for all $\iota \in \mathscr{I}^\wedge$, φ_ι is an additive automorphism. This means in other words that the given Q-net is isomorphic to an Ostrom net over $\mathbb{F}$. In fact, the corresponding vector space $\mathscr{V}$ over $\mathbb{F}$ is the additive group $(S, +)$ provided with the operation of scalar multiples $S \times \Phi \to S$, $(a, \alpha) \mapsto \varphi_\alpha(a)$.

13. It remain's to prove that every Ostrom net is necessarily a Q-net. This will be done in this concluding section. First we restate one known property of Ostrom nets: Every Ostrom net having the index set $\mathscr{I}$ is ι-transitive[1]) for every $\iota \in \mathscr{I}$. Indeed, let $\mathscr{N}$ be a given Ostrom net over a skew-field $\mathbb{F} = (F, +, \cdot)$ with the full point set $\mathscr{V} \times \mathscr{V}$ where $\mathscr{V}$ is a non-trivial left vector space over $\mathbb{F}$. Let $(\mathbf{x}_1, \mathbf{y}_1), (\mathbf{x}_2, \mathbf{y}_2)$ be points with $\mathbf{x}_1 \neq \mathbf{x}_2$ on the line $\{(\mathbf{x}, u\mathbf{x} + \mathbf{v}) \mid \mathbf{x} \in \mathscr{V}\}$ with slope u and intercept $\mathbf{v}$. So $\mathbf{y}_1 = u\mathbf{x}_1 + \mathbf{v}$, $\mathbf{y}_2 = u\mathbf{x}_2 + \mathbf{v}$. Now the map $\mathscr{V} \times \mathscr{V} \to \mathscr{V} \times \mathscr{V}$, $(\mathbf{x}, \mathbf{y}) \mapsto (\mathbf{x} + u(\mathbf{x}_2 - \mathbf{x}_1), \mathbf{y} + u(\mathbf{y}_2 - \mathbf{y}_1))$ it the requested u-translation sending $(\mathbf{x}_1, \mathbf{y}_1)$ to $(\mathbf{x}_2, \mathbf{y}_2)$. The case of two points $(\mathbf{c}, \mathbf{y}_1), (\mathbf{c}, \mathbf{y}_2)$ (i.e. points lying on a vertical line $\{(\mathbf{c}, \mathbf{y}) \mid \mathbf{y} \in \mathscr{V}\}$) leads to an ∞-translation (the slope and simultaneously the index of the pencil of vertical lines) $\mathscr{V} \times \mathscr{V} \to \mathscr{V} \times \mathscr{V}$, $(\mathbf{c}, \mathbf{y}) \mapsto (\mathbf{c}, \mathbf{y} + (\mathbf{y}_2 - \mathbf{y}_1))$.

We proceed to the proof of validity of condition Q in a given Ostrom net $\mathscr{N}$: Choose mutually distinct indexes $\alpha, \beta, \gamma \in F$ and put $l_1 = [(0, 0), \alpha]$, $l_2 = [(0, 0), \beta]$, $l_3 = [(0, 0), \gamma]$. Further, choose a point $(\mathbf{a}, \mathbf{y}_0)$ with $\mathbf{a} \neq 0$. Let l_4 and l_5 be lines from $\mathscr{L}_\varepsilon$ and $\mathscr{L}_\delta$, respectively ($\varepsilon, \delta \in F$ are further indexes), such that $(\mathbf{a}, \mathbf{y}_0) \in l_4, l_5, l_2$. Put $l_3 \sqcap l_4 = (\mathbf{x}_1, \mathbf{y}_1)$, $l_1 \sqcap l_5 = (\mathbf{x}_2, \mathbf{y}_2)$. We have $l_2 = \{(\mathbf{x}, \beta\mathbf{x}) \mid \mathbf{x} \in \mathscr{V}\}$, $l_3 = \{(\mathbf{x}, \gamma\mathbf{x}) \mid \mathbf{x} \in \mathscr{V}\}$, $l_4 = \{(\mathbf{x}, \varepsilon\mathbf{x} + \mathbf{c}) \mid \mathbf{x} \in \mathscr{V}\}$ for some $\mathbf{c} \in \mathscr{V}$. The relation $(\mathbf{a}, \beta\mathbf{a}) \in l_4$ implies $\beta\mathbf{a} = \varepsilon\mathbf{a} + \mathbf{c}$ and consequently $\mathbf{c} = (\beta - \varepsilon)\mathbf{a}$. Thus $l_4 = \{(\mathbf{x}, \varepsilon\mathbf{x} + (\beta - \varepsilon)\mathbf{a}) \mid \mathbf{x} \in \mathscr{V}\}$. From this we calculate $\mathbf{x}_1$ and $\mathbf{y}_1$: $\gamma\mathbf{x} = \varepsilon\mathbf{x} + (\beta - \varepsilon)\mathbf{a}$, $(\gamma - \varepsilon)\mathbf{x} = (\beta - \varepsilon)\mathbf{a}$, $\mathbf{x}_1 = (\gamma - \varepsilon)^{-1}(\beta - \varepsilon)\mathbf{a}$, $\mathbf{y}_1 = \gamma(\gamma - \varepsilon)^{-1} . (\beta - \varepsilon)\mathbf{a}$. Analogously $l_1 = \{(\mathbf{x}, \alpha\mathbf{x}) \mid \mathbf{x} \in \mathscr{V}\}$, $l_5 = \{(\mathbf{x}, \delta\mathbf{x} + \mathbf{d}) \mid \mathbf{x} \in \mathscr{V}\}$ for some $\mathbf{d} \in \mathscr{V}$ where $\beta\mathbf{a} = \delta\mathbf{a} + \mathbf{d}$, $\mathbf{d} = (\beta - \delta)\mathbf{a}$. Thus $l_5 = \{(\mathbf{x}, \delta\mathbf{x} + (\beta - \delta)\mathbf{a}) \mid \mathbf{x} \in \mathscr{V}\}$. Further we have $\alpha\mathbf{x} = \delta\mathbf{x} + (\beta - \delta)\mathbf{a}$, $(\alpha - \delta)\mathbf{x} = (\beta - \delta)\mathbf{a}$, $\mathbf{x}_2 = (\alpha - \delta)^{-1}(\beta - \delta)\mathbf{a}$, $\mathbf{y}_2 = \alpha(\alpha - \delta)^{-1}(\beta - \delta)\mathbf{a}$. Thus we have already four points as vertices of the quadrilateral, and five of its edges. The remaining edge

[1]) ι-transitive means: for every couple A, A' of distinct collinear points with $A \sqcup A' \in \mathscr{L}_\iota$ there is a non-identical ι-translation sending A to A'.

exists if and only if there is a $\lambda \in F$ such that $\boldsymbol{y}_1 - \lambda \boldsymbol{x}_1 = \boldsymbol{y}_2 - \lambda \boldsymbol{x}_2$. After substituing the values of $\boldsymbol{x}_1, \boldsymbol{y}_1, \boldsymbol{x}_2, \boldsymbol{y}_2$ we obtain $\gamma(\gamma - \varepsilon)^{-1} (\beta - \varepsilon) \boldsymbol{a} - \lambda(\gamma - \varepsilon)^{-1} (\beta - \varepsilon) \boldsymbol{a} = \alpha(\alpha - \delta)^{-1} (\beta - \delta) \boldsymbol{a} - \lambda(\alpha - \delta)^{-1} (\beta - \delta) \boldsymbol{a}$ and further $(\gamma(\gamma - \varepsilon)^{-1} (\beta - \varepsilon) - \alpha(\alpha - \delta)^{-1} (\beta - \delta)) \boldsymbol{a} = \lambda((\gamma - \varepsilon)^{-1} (\beta - \varepsilon) - (\alpha - \delta)^{-1} (\beta - \delta)) \boldsymbol{a}$. Consequently $\lambda((\gamma - \varepsilon)^{-1} (\beta - \varepsilon) - (\alpha - \delta)^{-1} (\beta - \delta)) = \gamma(\gamma - \varepsilon)^{-1} (\beta - \varepsilon) - \alpha(\alpha - \delta)^{-1} (\beta - \delta)$. If $(\gamma - \varepsilon)^{-1} (\beta - \varepsilon) - (\alpha - \delta)^{-1} (\beta - \delta) \neq 0$ then $\lambda = (\gamma(\gamma - \varepsilon)^{-1} (\beta - \varepsilon) - \alpha(\alpha - \delta)^{-1} (\beta - \delta)) ((\gamma - \varepsilon)^{-1} (\beta - \varepsilon) - (\alpha - \delta)^{-1} . (\beta - \delta))^{-1}$, whereas by $(\gamma - \varepsilon)^{-1} (\beta - \varepsilon) - (\alpha - \delta)^{-1} (\beta - \delta) = 0$ we obtain $(\gamma - \varepsilon)^{-1} (\beta - \varepsilon) = (\alpha - \delta)^{-1} (\beta - \delta)$ i.e. $\boldsymbol{x}_1 = \boldsymbol{x}_2$ and the sixth join line is vertical. The cases in which also the slope ∞ appears can be investigated similarly.

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Souhrn

KONFIGURAČNÍ CHARAKTERIZACE OSTROMOVÝCH TKÁNÍ

JAROMÍR BAŠTINEC

V práci jsou studovány tkáně splňující universální čtyřúhelníkovou podmínku: Jestliže libovolné čtyři různé body tkáně, z nichž žádné tři neleží na jedné přímce, jsou po dvou spojitelné pěticí přímek, potom existuje i jednoznačně určená šestá přímka, spojující zbývající dvojici bodů.

Je dokázáno, že každá tkáň splňující podmínku je Ostromovou tkání (tj. tkání nad tělesem). Naopak, každá Ostromova tkáň splňuje uvedenou podmínku.

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