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ASYMPTOTIC PERIODICITY OF MARKOV OPERATORS ON SIGNED MEASURES

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Summary. A new criterion of asymptotic periodicity of Markov operators on L^1, established in [3], is extented to the class of Markov operators on signed measures.

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Let Σ be a σ-algebra of subsets of a set X. Let M_Σ be the Banach space of signed measures on Σ with the norm given by the total variations of measures. Let $M \subset M_\Sigma$ be a band (i.e. a Banach lattice such that $\mu \in M$, $\nu \ll \mu \Rightarrow \nu \in M$). By the weak topology on M we understand the topology given by the duality $\langle M_\Sigma, M_\Sigma^* \rangle$. Let D be the subset of nonnegative normalized elements of M. A linear operator $P: M \to M$ is called a Markov operator if

$$P(D) \subset D\,.$$

Definition 1. We say that P is *quasi-constrictive* if there exist a weakly compact set $F \subset M$ and a positive number $\varkappa < 1$ such that

$$\limsup_{n\to\infty} d(P^n\mu, F) \leqq \varkappa \quad \text{for} \quad \mu \in D \tag{1}$$

where $d(\nu, F) = \inf\{\|\nu - \varrho\| : \varrho \in F\}$.

If P is quasi-constrictive for $\varkappa = 0$ then it is called *constrictive*.

Remark. According to [4], a Markov operator P on $L^1(\mu) \approx M_\mu = \{\nu \in M_\Sigma: \nu \ll \mu\}$ (where μ is a σ-finite measure on Σ) is quasi-constrictive if there exist a set $C \in \Sigma(\mu(C) \langle \infty)$ and constants $\varkappa < 1$, $\delta > 0$ such that

$$\limsup \textstyle\int_{B\cap(X-C)} P^n f \,\mathrm{d}\mu \leqq \varkappa \tag{2}$$

for all $f \in D$ and $B \in \Sigma$, $\mu(B) \leqq \delta$.

It is easy to observe that a Markov operator with this property is quasi-constrictive in the sense of our Definition 1. (The converse implication follows from the basic

[1]) A substantial part of this research was done during the leave of the first author from Komensky University in Bratislava.

properties of weakly compact sets in $L^1(\mu)$.) We show that P satisfies (1) with the same $\varkappa$ and $F = \{g \in L^1 : 0 \leqq g \leqq g_0\}$, $g_0 = \delta^{-1} . 1_C$. For any $f \in D$ and $n \in N$ put $g_n = P^n f \wedge g_0$,

$$B_n = \{x \in C : P^n f(x) > \delta^{-1}\} .$$

Obviously

$$\mu(x \in C : P^n f(x) \rangle g_n(x)\} = \mu(B_n) < \delta ,$$
$$\|P^n f - g_n\| = \int_{B_n} (P^n f - \delta^{-1}) \,\mathrm{d}\mu + \int_{X-C} P^n f \,\mathrm{d}\mu \quad \text{for} \quad n \in N .$$

Therefore (2) implies (1).

Definition 2. We say that $\mu \in M$ is *periodic* if there exists a natural number n such that $P^n \mu = \mu$.

We say that a *periodic measure $\mu \in D$ is minimal*, if for any periodic measure $\nu \ll \mu$ there exists a scalar λ such that $\nu = \lambda\mu$.

Theorem 1. *Let P be a quasi-constrictive Markov operator on a band M.*

i) *There exist a finite set F_0 of pairwise orthogonal* periodic elements of D, $F_0 = \{\nu_1, \ldots, \nu_r\}$ and the corresponding continuous linear functionals $\{\lambda_1, \ldots, \lambda_r\}$ on M such that

(3) $$\lim_{n \to \infty} \|P^n(\mu - \sum_{i=1}^{r} \lambda_i(\mu)\, \nu_{\alpha(i)})\| = 0 \quad \text{for any} \quad \mu \in M$$

and

(4) $$P(\nu_i) = \nu_{\alpha(i)} \quad \text{for} \quad i = 1, \ldots, r ,$$

where α is a permutation of the integers $1, \ldots, r$.

ii) The functionals λ_i are nonnegative. Morevoer,

$$\sum_{i=1}^{r} \lambda_i(\nu) = 1 \quad \text{for} \quad \nu \in D$$

and

(5) $$|\lambda_i(\mu)| \leqq \|\mu\|$$

holds for $\mu \in M$.

iii) The measures ν_i, $i = 1, \ldots, r$ are minimal.

iv) The sets $\{\nu_1, \ldots, \nu_r\}$ and $\{\lambda_1, \ldots, \lambda_r\}$ satisfying (3) and (4) are unique.

In order to be able to utilize the result of [4], where part i) was proved for the case that $M = M_\nu = \{\mu : \mu \ll \nu\}$ for some $\nu \in M_\Sigma$, we present some auxiliary results.

Lemma 1. *Let $\mu \in D$. Let $\{c_i\}_{i=0}^{\infty}$ be a sequence of positive real numers such that*

$$\sum_{i=0}^{\infty} c_i = 1 .$$

Put

$$(6) \qquad \bar{\mu} = \sum_{i=0}^{\infty} c_i P^i(\mu) .$$

Then $M_{\bar{\mu}} = \{\nu\colon \nu \ll \bar{\mu}\}$ *is the smallest P-invariant band containing* μ.

Proof. $M_{\bar{\mu}}$ is isomorphic to the Banach lattice $L^1(\bar{\mu})$, hence it is a band.
We show that it is P-invariant.
We have

$$P_{\bar{\mu}} = \sum_{i=0}^{\infty} c_i P^{i+1}\mu \ll \bar{\mu} .$$

Moreover, $\nu \in M_{\bar{\mu}}$ is equivalent to

$$\nu = \sup_n \{\nu \wedge n . \bar{\mu}\} .$$

We show that this implies $P\nu \in M_{\bar{\mu}}$. P is a Markov operator, hence $P(\nu \wedge n . \bar{\mu}) \leqq \leqq P\nu \wedge n. P\bar{\mu}$ for any n.
Therefore,

$$0 \leqq P\nu \wedge n . P\bar{\mu} - P(\nu \wedge n . \bar{\mu}) \leqq P\nu - P(\nu \wedge n . \bar{\mu}) =$$
$$= P(\nu - \nu \wedge n . \bar{\mu})$$

and

$$\|P\nu - P\nu \wedge n . P\bar{\mu}\| \leqq \|P(\nu - \nu \wedge n . \bar{\mu})\| = \|\nu - \nu \wedge n . \bar{\mu}\| .$$

Using the Lebesgue bounded convergence theorem we get

$$P\nu = \sup_n \{P\nu \wedge n . P\bar{\mu}\} ,$$

hence

$$P\nu \ll P\bar{\mu} \ll \bar{\mu} .$$

Lemma 2. *Two minimal periodic measures are either identical or orthogonal.*

Proof. Let $\mu, \nu \in D$ be minimal and let n be their common period. We show that $\mu \wedge \nu$ is periodic with period n. We have

$$P^n(\mu \wedge \nu) \leqq P^n\mu \wedge P^n\nu = \mu \wedge \nu , \quad \|P^n(\mu \wedge \nu)\| = \|\mu \wedge \nu\| ,$$

hence

$$P^n(\mu \wedge \nu) = \mu \wedge \nu .$$

Moreover, $\mu \wedge \nu \ll \mu$ and $\mu \wedge \nu \ll \nu$. Thus there exist real numbers λ_1, λ_2 such that

$$\mu \wedge \nu = \lambda_1 \mu = \lambda_2 \nu .$$

If $\mu \wedge \nu \neq 0$ then $\lambda_1 \neq 0$, $\lambda_2 \neq 0$ and $\nu = (\lambda_1/\lambda_2\mu$. But)

$$\|\mu\| = \|\nu\| = 1\,,$$

hence $\lambda_2 = \lambda_1$ and $\mu = \nu$.

Lemma 3. *Let F be a weakly compact subset of M and $\varkappa < 1$. Then the neighbourhood $U(F, \varkappa) = \{\mu\colon \mathrm{d}(\mu, F) < \varkappa\}$ does not contain the infinite number of pairwise orthogonal densities.*

Proof. There exists $\nu \in D$ such that $\varrho \ll \nu$ for any $\varrho \in F$ (cf. [2], Th. IV, 9.2.). Hence there exists $\delta > 0$ such that $\nu(B) < \delta$ implies $\varrho(B) < 1 - \varkappa$ for any $\varrho \in F$. Let $N > \delta^{-1}$ and let $U(F, \varkappa)$ contain N pairwise orthogonal densities $\{\tau_1, \ldots, \tau_N\}$ with supports $B_1, \ldots, B_N$. Let $\varrho_1, \ldots, \varrho_N \in F$ be such that $\mathrm{d}(\varrho_i, \tau_i) < \varkappa$ for $i = 1, \ldots, N$. Then we have

$$\sum_{i=1}^{N} \nu(B_i) = \nu(\bigcup_{i=1}^{N} B_i) \leqq 1\,.$$

Thus there exists $k \in \{1, \ldots, N\}$ such that $\nu(B_k) \leqq N^{-1}$. On the other hand, $\|\tau_k\| = \tau_k(B_k) \leqq \varkappa + \varrho_k(B_k) < 1$.

But this contradicts the assumption $\tau_k \in D$.

Proof of Theorem 1. First we give the proof for the case that $M = M_\nu = \{\mu \ll \nu\}$. Part

i) of the theorem was proved in [3] and [4].

ii) In [4] it was shown that the functionals λ_i can be expressed in the form

$$\lambda_i(f) = \int_X k_i(x) f(x)\,\mathrm{d}\nu(x)$$

for some nonnegative bounded functions k_i, which implies positivity of λ_i. From (3) and (4) we get $\lambda_i(\nu_i) = 1$ and $\lambda_j(\nu_i) = 0$ for $i \neq j$, hence $\|\lambda_i\| \geqq 1$.

Let $\mu \in D$. We have

$$1 = \lim_{n\to\infty} \|P^n\mu\| = \lim_{n\to\infty} \|\sum_{i=1}^{r} \lambda_i(\mu)\, \nu_{\alpha^n(i)}\| = \sum_{i=1}^{r} \lambda_i(\mu)\,.$$

Therefore $0 \leqq \lambda_i(\mu) \leqq 1$ for $\mu \in D$, which obviously implies that $\|\lambda_i\| \leqq 1$.

iii) Let $\mu \in D$ be periodic and let $\mu \ll \nu_i$ for some $i \in \{1, \ldots, r\}$. For $j \neq i$ we have

$$0 \leqq \lambda_j(\mu) \leqq \lim_{k\to\infty} k\, \lambda_j(\nu_i) = 0\,.$$

From (3) we get that the norms of differences

$$\|P^n(\mu) - \lambda_i(\mu)\, \nu_{\alpha^n(i)}\| \quad \text{converge to zero for} \quad n \to \infty\,.$$

But they form a periodic function of n, hence they are equal to zero for all n. Therefore $\nu = \lambda_i(\nu)\, \nu_i$.

iv) The uniqueness of the set $\{v_1, \ldots, v_r\}$ follows from their minimality via Lemma 2.

Let $\{\lambda_1^1, \ldots, \lambda_r^1\}$ be another set of functionals that satisfy (3). Then for any $\mu \in M$

$$\left\| \sum_{i=1}^{r} \lambda_i(\mu)\, v_{\alpha^n(i)} - \sum_{i=1}^{r} \lambda_i^1(\mu)\, v_{\alpha^n(i)} \right\| = \sum_{i=1}^{r} |\lambda_i(\mu) - \lambda_i^1(\mu)| = 0 .$$

Therefore

$$\lambda_i^1(\mu) = \lambda_i(\mu) \quad \text{for} \quad i = 1, \ldots, r .$$

Now we relinquish the assumption $M = M_v$ for some $v \in M$.

We say that $v \in D$ is admissible if $Pv \ll v$. Let us denote the set of all admissible densities by D_a. Using the same arguments as in Lemma 1 we get that for $v \in D_a$

$$M_v = \{\mu : \mu \ll v\} \quad \text{is a } P\text{-invariant band} .$$

We show that the restriction P_v of P to M_v is quasiconstrictive. For any $\mu \in M$ we can write $\mu = \mu_v + \mu_v^1$, where $\mu_v \ll v$ and $\mu_v^1 \perp v$.

The mapping $\Pi_v : \mu \to \mu_v$ is linear and continuous, because of

$$|\mu| = |\mu_v| + |\mu_v^1| \geqq |\mu_v| .$$

Therefore, the image $\Pi_v(U(F, \varkappa))$ of the set $U(F, \varkappa)$ from Lemma 3 is contained in the neighbourhood $U(\Pi_v(F), \varkappa)$ of the weak compact $\Pi_v(F)$. Moreover, for any $\mu \in M$ and $\varrho \in F$ we have

$$|\mu - \varrho| = |\mu - \varrho_v| + |\varrho_v^1| , \quad \text{hence} \quad \|\mu - \Pi_v(\varrho)\| \leqq \|\mu - \varrho\| .$$

Therefore

$$\limsup_n \mathrm{d}(P^n\mu, \Pi_v(F)) \leqq \limsup_n \mathrm{d}(P^n\mu, F) < \varkappa \quad \text{for} \quad \mu \in M_v \cap D .$$

Hence P_v is a quasi-constrictive Markov operator on M_v. Using the validity of Theorem 1 for P_v on M_v we conclude that there exists a finite set of pairwise orthogonal measures in $D \cap M_v$ that are minimal and periodic with respect to P_v, hence periodic with respect to P.

The set D_0 of minimal periodic elements of D is nonempty. The fact that P is quasi-constrictive yields that $D_0 \subset U(F, \varkappa)$. According to Lemma 2 and Lemma 3 can write $D_0 = \{v_1, \ldots, v_r\}$. This set of measures is P-invariant, hence there exists a permutation α such that (4) holds on D_0.

We say that $v \in D_a$ is complete if the band M_v contains the set D_0. Let us denote by D_c the set of all complete elements of D. It is obvious that $v_0 = (v + v_1 + \ldots \ldots + v_r)/(r + 1) \in D_C$ for $v \in D_a$.

Combining this fact with Lemma 1 we conclude that for every $\mu \in M$ there exists $v \in D_C$ such that $\mu \in M_v$.

In other words

$$M = \bigcup \{M_v : v \in D_C\} .$$

Now we define continuous functionals $\lambda_1, \ldots, \lambda_r$ on M by first defining them on every band M_ν for $\nu \in D_C$ and then showing that they coincide on intersections $M_{\nu_1} \cap M_{\nu_2}$ for $\nu_1, \nu_2 \in D_C$.

For every $\nu \in D_C$ we can use the validity of Theorem 1 on M_Σ that ensures the existence of continuous linear functions $\lambda_1, \ldots, \lambda_r$ on M_ν such that (3) and (5) hold for $\mu \in M_\nu$.

Let $\nu_1, \nu_2 \in D_C$. Let λ_i^j, $i = 1, \ldots, r$, $j = 1, 2$ be the corresponding families of linear functionals defined on the bands M_{ν_j}. Let $\mu \in M_{\nu_1} \cap M_{\nu_2}$. Since (3) holds in M_{ν_1} as well as in M_{ν_2} we have

$$0 = \lim_{n\to\infty} \left\|P^n f - \sum_{i=1}^{r} \lambda_i^1(\mu)\, \nu_{\alpha^n(i)}\right\| + \lim_{n\to\infty} \left\|P^n \mu - \sum_{i=1}^{r} \lambda_i^2(\mu)\, \nu_{\alpha^n(i)}\right\| \geqq$$

$$\geqq \lim_{n\to\infty} \sum_{i=1}^{r} \left\|\lambda_i^1(\mu)\, \nu_{\alpha^n(i)} - \lambda_i^2(\mu)\, \nu_{\alpha^n(i)}\right\| = \sum_{i=1}^{r} \left|\lambda^1(\mu) - \lambda_i^2(\mu)\right|.$$

Hence $\lambda_i^1(\mu) = \lambda_i^2(\mu)$ for $i = 1, \ldots, r$.

Therefore, the real functions $\lambda_1, \ldots, \lambda_r$ are well defined on M.

Their linearity follows from the fact that $\mu_i \in M_{\nu_i}$ for $i = 1, 2$ implies $\mu_1, \mu_2 \in M_\nu$ for $\nu = (\nu_1 + \nu_2)/2$. Finally, (5) holds on M because it holds on M_ν, $\nu \in D_c$. Therefore $\lambda_1, \ldots, \lambda_r$ are continuous functionals on M. The rest of the proof obviously follows from the fact that Theorem 1 is satisfied on M_ν for $\nu \in D_C$.

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Súhrn

ASYMPTOTICKÁ PERIODIČNOSŤ MARKOVOVÝCH OPERÁTOROV NA ZOVŠEOBECNENÝCH MIERACH

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Článok vychádza z niektorých nových výsledkov, udávajúcich postačujúce podmienky asymptotickej preiodičnosti Markovových operátorov na priestoroch L^1 a zovšeobecňuje ich na triedu Markovových operátorov na priestoroch znamienkových mier.

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