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ON THE CLIQUISH, QUASICONTINUOUS AND MEASURABLE SELECTIONS

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Summary. The purpose of this paper is the investigation of the necessary and sufficient conditions under which a given multifunction admits a cliquish and measurable selection. Our investigation also covers the search for quasicontinuous selections for multifunctions which are continuous with respect to the generalized notion of the semi-quasicontinuity.

Keywords: Quasicontinuity, cliquish and measurable selection.

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There are many theorems in the selection theory which deal with sufficient conditions for the existence of certain selections. Prominent among the selections which have been studied are the measurable selections [7] and the Baire α selections [8]. However, necessary and sufficient conditions have been studied very little. For example, the restriction of a multifunction to each set of the family of all perfect sets characterizes the existence of a Borel 1 selection in [2], and the existence of a measurable and cliquish selection was characterized by certain cluster sets of multifunctions in [9]. In this paper it will be characterized by the so-called $\mathscr{C}$-continuity. The main theorems presented here are based on a few theorems that can be found in [9] and therefore they will not be repeated. Using a slight modification of the proofs from [9] we present some straightforward extentions of results contained in [9]. We hope that this paper will give comprehensive information concerning the necessary and sufficient conditions for the existence of certain selections.

If $F: X \to Y$ is a multifunction from a given topological space X into the space of all non-void subsets of a given topological space Y, then we denote

$$F^-(A) = \{x \in X: F(x) \cap A \neq \emptyset\}$$

$$F^+(A) = \{x \in X: F(x) \subset A\} \quad \text{for any set} \quad A \subset X .$$

A selection for F is any function $f: X \to Y$ such that $f(x) \in F(x)$ for all $x \in X$. By int (A) and cl (A) we denote the interior and the closure of A, respectively.

Starting from early years of modern mathematics, many different types of generalized continuity of multivalued mapping were introduced. One of them has been the concept of a lower (upper) semi-quasicontinuity. For precise definitions as well as further properties see [4], [9], [10] (see also the original definition in [6]). In this paper two more general continuities are presented that can be used to formulate and prove our results for the existence of a quasicontinuous, cliquish and measurable selection. The former introduced and investigated in [9] is as follows.

Definition 1. *A multifunction $F: X \to Y$ is said to be lower (upper) semi-continuous in the Baire sense (briefly 1-$\mathscr{B}$-continuous (u-$\mathscr{B}$-continuous)) at a point $x \in X$ if for any open set $V \subset Y$ such that $V \cap F(x) \neq \emptyset$ $(F(x) \subset V)$ and for any neighbourhood U of x there is a set $B \subset U$ of the second category having the Baire property such that $F(z) \cap V \neq \emptyset$ $(F(z) \subset V)$ for any $z \in B$. F is l-$\mathscr{B}$-continuous (u-$\mathscr{B}$-continuous) if it is l-$\mathscr{B}$-continuous (u-$\mathscr{B}$-continuous) at every point of X.*

Theorem 2. *Let X be a T_1-space, Y a locally compact 2° countable space. If a multifunction $F: X \to Y$ with closed values is u-$\mathscr{B}$-continuous, then F has a quasicontinuous selection.*

Proof. Let $\bar{Y}$ be a one-point compactification of Y. The space $\bar{Y}$ is a separable metric space (see [3, p. 274]) and $F(x)$ is compact in $\bar{Y}$ for all $x \in X$. Since $\bar{F}: X \to \bar{Y}$ defined as $\bar{F}(x) = F(x)$ for all $x \in X$ is u-$\mathscr{B}$-continuous, by [9, Th. 5.3] there *if* a quasicontinuous selection $f: X \to \bar{Y}$ for $\bar{F}$. It is clear that f is a selection for F and s is quasicontinuous as a function from X into Y.

Remark 3. The conditions on a multifunction F from Theorem 3 seem not to be too strong because F does not have to be semi-quasicontinuous. Let us consider $X = Y = \langle 0, 1\rangle$ with the usual topology. Define $F: X \to Y$ as follows: $F(x) = \{0\}$ if x is irrational and $F(x) = \langle 0, 1\rangle$ if x is rational. F is not semi-quasicontinuous and it is u-$\mathscr{B}$-continuous.

If X is a Baire space, then the next definition of continuity generalizes the lower-$\mathscr{B}$-continuity.

Definition 4. *Let $\emptyset \neq \mathscr{C} \subset 2^X$, $\emptyset \notin \mathscr{C}$. A multifunction $F: X \to Y$ is said to bes $\mathscr{C}$-continuous at $x \in X$ if for any neighbourhood V of x there is a point $y \in Y$ such that for any neighbourhood G of y the set $V \cap F^-(G)$ contains a set from $\mathscr{C}$. If F is $\mathscr{C}$-continuous at any $x \in X$, then it is said to be $\mathscr{C}$-continuous. By $\mathscr{C}(F)$ we denote the set of all points at which F is $\mathscr{C}$-continuous.*

Remark 5. It is clear that $\mathscr{C}(F)$ is always closed. Let X be a Baire space and $\mathscr{B} = \{B \subset X$: B is of the second category and B has the Baire property$\}$. It is clear that if F is l-$\mathscr{B}$-continuous on a dense set, then F is $\mathscr{B}$-continuous.

Let Y be a regular 2° countable space, X a T_1-Baire space, and $\mathscr{G} = \{G \subset X$: G is non-empty and open$\}$. If F is u-$\mathscr{B}$-continuous, then F is lower semi-quasicontinuous except for a set of the first category (see [9, Th. 2.1]); that means F is $\mathscr{G}$-continuous.

Our definition of $\mathscr{C}$-continuity and the cluster points of multifunctions are very close as follows from

Lemma 6. *Let Y be a compact space. A multifunction $F: X \to Y$ is $\mathscr{C}$-continuous at a point x if and only if there is a point $y \in Y$ such that for any open sets V, G with $x \in V$, $y \in G$ there is a set $C \in \mathscr{C}$ such that $C \subset V \cap F^{-}(G)$.*

We omit the trivial proof.

Definition 7. *A multifunction $F: X \to Y$ is measurable if $F^{-}(G)$ has the Baire property for any open set $G \subset Y$.*

A function $f: X \to Y$ (Y a metric space with a metric d) is said to be cliquish at a point $x \in X$ if for each $\varepsilon > 0$ and each neighbourhood U of x there is an open non-empty set $H \subset U$ such that for each two points $x_1, x_2 \in H$ the inequality $d(f(x_1), f(x_2)) < \varepsilon$ holds. A function f is said to be cliquish if it is cliquish at each point $x \in X$ (see [1], [5]).

Lemma 8. (see [5]). *Let X be a Baire space, Y a metric one, and let $f: X \to Y$ be an arbitrary function. The function f is cliquish if and only if the set of its points of discontinuity is of the first category.*

The $\mathscr{C}$-continuity for $\mathscr{C} = \mathscr{G} = \{G \subset X$: G is nom-empty and open$\}$ is equivalent to the existence of a cliquish selection.

Theorem 9. *Let X be a T_1-Baire space, Y a locally compact separable metric space. Let $F: X \to Y$ be a closed-valued multifunction. The following conditions are equivalent.*

(a) *F is $\mathscr{G}$-continuous,*

(b) *F has a cliquish selection.*

Proof. Let F be $\mathscr{G}$-continuous at x. By Lemma 6 there is a point $y \in \bar{Y}$ ($\bar{Y}$ as in the proof of Theorem 2) such that $\operatorname{int}(V \cap F^{-}(G)) \neq \emptyset$ for any open sets V, G with $x \in V$, $y \in G$. By [9, Th. 5.4)] $\bar{F}: X \to \bar{Y}$ defined by $\bar{F}(x) = F(x)$ for all $x \in X$ has a selection f such that the set of all points at which f is not quasicontinuous is of the first category. By [4, Th. 2] the set of all points at which f is discontinuous is of the first category and by Lemma 8, f is cliquish.

If F has a cliquish selection f, then f is continuous except for a set S of the first category. It is clear that F is $\mathscr{G}$-continuous at x for any $x \in X \setminus S$. Since $\mathscr{G}(F)$ is closed and $X \setminus S$ is dense, F is $\mathscr{G}$-continuous.

Corollary 10. *Under the same conditions on X, Y and F as in Theorem 9 we have:*

(a) *If F is lower semi-quasicontinuous on a dense set, then F has a cliquish selection.*

(b) *If F is single-valued, then F is cliquish if and only if F is $\mathscr{G}$-continuous.*

Theorem 11. *Under the same conditions on X, Y and F as in Theorem 9, F has a measurable selection if and only if F is $\mathscr{B}$-continuous, where $\mathscr{B} = \{B \subset X: B$ is a set of the second category having the Baire property$\}$.*

Proof. Let f be a measurable selection for F. By [9, Th. 3.3] there is a dense set S such that for any $x \in S$, and for any open sets G, V with $x \in V$, $f(x) \in G$, the set $f^{-1}(G) \cap V$ contains a set $B \in \mathscr{B}$. By Lemma 6, F is $\mathscr{B}$-continuous at $x \in S$. Since S is dense, F is $\mathscr{B}$-continuous.

Let F be $\mathscr{B}$-continuous. Let $\bar{F}: X \to \bar{Y}$, $\bar{F}(x) = F(x)$ for all $x \in X$ ($\bar{Y}$ as in the proof of Theorem 2). Since $\bar{F}$ is $\mathscr{B}$-continuous, for any $x \in X$ there is a point $y \in \bar{Y}$ such that $F^{-}(G) \cap V$ contains a set from $\mathscr{B}$ for any open sets G, V with $x \in V$, $y \in G$ (see Lemma 6). By [9, Th. 5.5], $\bar{F}$ has a measurable selection $f: X \to \bar{Y}$. It is clear that f is measurable as a function from X into Y.

Corollary 12. *Under the same conditions on X, Y and F as in Theorem 9 we have:*

(a) *If F is measurable, then F has a measurable selection (this is the consequence of Theorem 3.3 from [9] and Theorem 11).*

(b) *A function $f: X \to Y$ is measurable if and only if f is $\mathscr{B}$-continuous.*

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Súhrn

O KĽUKATÝCH, KVÁZISPOJITÝCH A MERATEĽNÝCH SELEKTOROCH

V článku sú skúmané nutné a postačujúce podmienky existencie kľukatých a merateľných selektorov a postačujúce podmienky pre existenciu kvázispojitých selektorov. Pomocou zovšeobecného pojmu kvázispojitosti je charakterizovaná merateľnosť a kľukatosť funkcií.

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