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*Mathematica Bohemica*, Vol. 125 (2000), No. 3, 331–339

Persistent URL: <http://dml.cz/dmlcz/126124>

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GRAPHS WITH THE SAME PERIPHERAL AND  
CENTER ECCENTRIC VERTICES

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(Received July 27, 1998)

*Abstract.* The eccentricity  $e(v)$  of a vertex  $v$  is the distance from  $v$  to a vertex farthest from  $v$ , and  $u$  is an eccentric vertex for  $v$  if its distance from  $v$  is  $d(u, v) = e(v)$ . A vertex of maximum eccentricity in a graph  $G$  is called peripheral, and the set of all such vertices is the peripherian, denoted  $\text{Peri}(G)$ . We use  $\text{Cep}(G)$  to denote the set of eccentric vertices of vertices in  $C(G)$ . A graph  $G$  is called an S-graph if  $\text{Cep}(G) = \text{Peri}(G)$ . In this paper we characterize S-graphs with diameters less or equal to four, give some constructions of S-graphs and investigate S-graphs with one central vertex. We also correct and generalize some results of F. Gliviak.

*Keywords:* graph, radius, diameter, center

*MSC 1991:* 05C12, 05C20

## 1. INTRODUCTION

We consider nonempty and finite graphs without loops and multiple edges. All terminology as well as notation except that given here is taken from [1].

The set of vertices of a graph  $G$  is denoted by  $V(G)$ , and the set of edges by  $E(G)$ . Let  $G$  be a connected graph with vertices  $u$  and  $v$ . The distance  $d(u, v)$  between  $u$  and  $v$  is the length of a shortest  $u - v$  path in  $G$ . The eccentricity  $e(v)$  of a vertex  $v$  is the distance from  $v$  to a vertex farthest from  $v$ , and  $u$  is an eccentric vertex for  $v$  if  $d(u, v) = e(v)$ . The radius  $r(G)$  of  $G$  is  $\min\{e(v); v \in V(G)\}$ , while the diameter  $d(G)$  of  $G$  is  $\max\{e(v); v \in V(G)\}$ . A diametral path is a path of length  $d(G)$  joining a pair of vertices of the graph  $G$  that are at distance  $d(G)$  from one another. A vertex with minimum eccentricity is called a central vertex and the set of all such vertices is the center of  $G$  denoted by  $C(G)$ . A graph is self-centered if its every vertex is in the center. The neighborhood of a vertex  $v \in V(G)$  is denoted by

$N_G(v)$ . For any nonempty subset  $S$  of vertices in  $G$ , the *induced subgraph* ( $S$ ) is the maximal subgraph of  $G$  with the vertex set  $S$ . A vertex  $u$  is a *center eccentric vertex* of  $G$  if it is an eccentric vertex of some central vertex of  $G$ . A vertex  $v$  is *peripheral* if  $e(v) = d(G)$ , and the set of such vertices is the *peripherian* of  $G$ . We use  $\text{Cep}(G)$  to denote the set of all center eccentric vertices and  $\text{Peri}(G)$  to denote the peripherian of  $G$ . A connected nontrivial graph  $G$  is called an *S-graph* if  $\text{Cep}(G) = \text{Peri}(G)$ .

Buckley and Lewinter [2] proved the existence of an S-graph  $G$  with  $r(G) = a$  and  $d(G) = b$  for every positive integers  $a, b, a \leq b \leq 2a$ , and showed how to embed a graph  $G$  into an S-graph. They also proved that the cartesian product of two graphs is an S-graph if and only if both these graphs are S-graphs.

## 2. MAIN RESULTS

As follows from the definition of an S-graph, any self-centered graph as well as any tree are S-graphs. In particular, the complete graph  $K_n, n \geq 1$  is an S-graph. Further we will investigate only S-graphs that are not self-centered.

Let  $G_1, G_2$  be two disjoint connected graphs. Let  $x \in V(G_1), y \in V(G_2)$ . We say that a graph  $G$  arose from  $G_1$  and  $G_2$  by the identification of the vertices  $x, y$  with a new vertex  $t$  ( $t \notin V(G_1), t \notin V(G_2)$ ), if

$$\begin{aligned} V(G) &= V(G_1) \cup V(G_2) - \{x, y\} \cup \{t\}, \\ E(G) &= E(G_1) \cup E(G_2) - \{xu, yu; xu \in E(G_1), yu \in E(G_2)\} \\ &\quad \cup \{tu; xu \in E(G_1) \text{ or } yu \in E(G_2)\}. \end{aligned}$$

Gliviak [4] gave the following construction of S-graphs with one central vertex.

**Construction.** Let  $r \geq 1, n \geq 2$  be natural numbers. Let  $G_i, i = 1, 2, \dots, n$  be vertex disjoint graphs having at least one vertex  $v_i$  of eccentricity  $r$ . Let the graph  $H$  arise from graphs  $G_i$  by identification of all vertices  $v_i$  with one common vertex  $w$ .

He claims (Theorem 2) that a graph  $Q$  is an S-graph of radius one if and only if  $Q$  is either a complete graph  $K_n, n \geq 2$  or can be constructed according to the previous construction, as well as (Theorem 3) that for any  $r \geq 2$  a graph  $Q$  is an S-graph of radius  $r$  with one central vertex if and only if  $Q$  can be constructed according to this construction.

This construction gives S-graphs with one central vertex, but not all such S-graphs. In any S-graph  $G$  which we get by the construction the central vertex is a cut vertex of  $G$  and the diameter of such a graph is equal to  $2r(G)$ . As follows from Fig. 1 there exist S-graphs with one central vertex, which cannot be constructed according

to the above construction. In Fig. 1a there is an S-graph with radius  $r \geq 1$ , diameter  $2r$ , and one central vertex  $v$ . The periphery of this graph consists of all vertices belonging to  $C_{2r}$ . In Fig. 1b there is an S-graph with radius 3, diameter 5 and one central vertex  $v$ . The periphery of this graph is  $\{x, y\}$ .

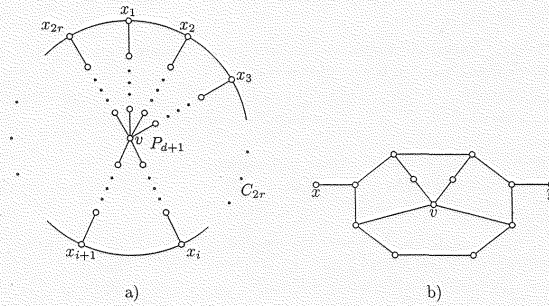


Fig. 1

**Theorem 2.1.** Let  $G$  be a graph with  $r(G) = 1$  and  $d(G) = 2$ . Then  $G$  is an S-graph if and only if  $|C(G)| = 1$ .

*Proof.* If  $C(G) = \{c\}$ , then the eccentricity of any vertex from  $V(G) - \{c\}$  is equal to two. Thus  $\text{Peri}(G) = \text{Cep}(G) = V(G) - \{c\}$ , and  $G$  is an S-graph.

Conversely, let  $G$  be an S-graph. Let  $|C(G)| \geq 2$  and  $x, y \in C(G)$ ,  $x \neq y$ . Since  $x, y \in \text{Cep}(G)$  and no vertex from  $C(G)$  can belong to  $\text{Peri}(G)$ , we have a contradiction. This completes the proof.  $\square$

**Corollary 2.2.** A graph  $G$  with  $r(G) = 1$  is an S-graph if and only if it is a complete graph  $K_p$ ,  $p \geq 2$ , or has one central vertex.

As mentioned above, for any two positive integers  $a$  and  $b$ ,  $a \leq b \leq 2a$ , there exists an S-graph with radius  $a$  and diameter  $b$ . The following theorem shows that there exists no S-graph  $G$  with  $d(G) = r(G) + 1$  and  $|C(G)| = 1$ .

**Theorem 2.3.** Let  $G$  be an S-graph with  $r(G) \neq d(G) \geq 3$  and  $|C(G)| = 1$ . Then  $r(G) < d(G) - 1$ .

*Proof.* Let  $G$  be an S-graph with  $C(G) = \{c\}$  and  $d(G) \geq 3$ . Let  $r(G) = d(G) - 1$ . The eccentricity of any vertex  $t \in N_G(c)$  is greater than  $r(G)$ , but  $e_G(t) \leq$

$d(G) = r(G) + 1$ . Therefore  $e_G(t) = d(G)$  and  $t \in \text{Peri}(G) = \text{Cep}(G)$ , which is a contradiction with  $d_G(t, c) = 1 < r(G)$ . This completes the proof.  $\square$

**Lemma 2.4.** *Let  $G$  be an S-graph,  $r(G) \geq 2$  and  $r(G) \neq d(G)$ . Then the distance of any two vertices from  $C(G)$  in  $G$  is less than  $r(G)$  and  $d(\langle V(G) - C(G) \rangle) \geq d(G)$ .*

**Proof.** The distance of no two vertices from  $C(G)$  can be equal to  $r(G)$ , since no vertex from  $C(G)$  can belong to  $\text{Peri}(G)$ .

If  $e_{\langle V(G) - C(G) \rangle}(x) < d(G)$  for any vertex  $x \in V(G) - C(G)$ , then the eccentricity of any vertex from  $V(G)$  in  $G$  is less than  $d(G)$ , which is a contradiction.  $\square$

**Theorem 2.5.** *Let  $G$  be a graph with  $d(G) = r(G) + 1$  and  $r(G) \geq 2$ . Then  $G$  is an S-graph if and only if any vertex from  $V(G) - C(G)$  is a center eccentric vertex of  $G$ .*

**Proof.** Let  $G$  be an S-graph. Then  $\text{Peri}(G) = V(G) - C(G)$ , because the eccentricity of any vertex from  $V(G) - C(G)$  is equal to  $d(G)$ . Since  $G$  is an S-graph, every vertex from  $\text{Peri}(G)$  is eccentric for some vertex from  $C(G)$ .

Conversely, let any vertex of  $V(G) - C(G)$  be a center eccentric vertex of  $G$ . Since  $\text{Peri}(G) = V(G) - C(G)$ , the graph  $G$  is an S-graph. This completes the proof.  $\square$

**Corollary 2.6.** *Let  $G$  be a graph with  $d(G) = 3$  and  $r(G) = 2$ . Then  $G$  is an S-graph if and only if no vertex from  $V(G) - C(G)$  is joined to all central vertices of  $G$ .*

**Theorem 2.7.** *Let  $G$  be a graph with diameter four and radius two. Let  $Q = \langle V(G) - C(G) \rangle$ , where  $C(G)$  is the center of  $G$ . Then  $G$  is an S-graph if and only if*

- 1)  $\langle C(G) \rangle$  is a complete graph and
- 2) the cardinality of the set  $T = \{x \in V(Q); N_G(x) \cap C(G) = \emptyset\}$  is at least two, and for every vertex  $x \in T$  at least one vertex  $y \in V(Q)$  such that  $d_Q(x, y) \geq 4$  belongs to  $T$ .

**Proof.** Let  $G$  be an S-graph. As follows from Lemma 2.4,  $\langle C(G) \rangle$  is a complete graph, and  $d(Q) \geq 4$ . Any vertex  $t \in V(Q) - T$  has eccentricity  $e_G(t) = 3$ . Thus any vertex with eccentricity four belongs to  $T$ , i.e.  $\text{Peri}(G)$  is a subset of  $T$ . Since  $G$  is an S-graph,  $\text{Peri}(G) = \text{Cep}(G)$ . But  $T = \text{Cep}(G)$ , because  $T$  consists of all center eccentric vertices of  $G$ . Obviously,  $|\text{Cep}(G)| \geq 2$ . Let  $x \in T$ . If there is no vertex  $y \in V(Q)$  such that  $d_Q(x, y) \geq 4$  belongs to  $T$ , then  $e_G(x) \leq 3$ , which is a contradiction with  $x \in \text{Peri}(G)$ .

Conversely, let 1) and 2) hold. As follows from 2),  $d(Q) \geq 4$  and  $T$  consists of all center eccentric vertices of  $G$ . Thus  $\text{Cep}(G) = T$ . Let  $x \in T$  and  $y \in T$  be such that  $d_Q(x, y) \geq 4$ . Then  $e_G(x) = 4$ , and  $T$  consists of all diametral vertices of  $G$ . Thus  $\text{Peri}(G) = T$ .

This completes the proof.  $\square$

There exists no self-centered S-graph with one central vertex. As follows from Theorem 2.3, there exists no S-graph  $G$  with one central vertex and  $r(G) = d(G) - 1$ . Next we prove that for all other possible cases of radius and diameter there does exist an S-graph with one central vertex.

**Theorem 2.8.** For each pair of positive integers  $d$  and  $r$ ,  $r \geq 2$ ,  $r + 2 \leq d \leq 2r$ , there exists an S-graph  $G$  with  $r(G) = r$ ,  $d(G) = d$  and  $|C(G)| = 1$ .

*Proof.* If  $d = 2r$ , then we can use the graph formed by  $P_{d+1}$ .

Let  $d = r + k$ ,  $2 \leq k < r$ . To construct an S-graph  $G$  with the required properties we can proceed as follows.

Let  $T_1, T_2, \dots, T_m$ , where  $m = 4k - 2$ , be disjoint copies of the trees from Fig. 2, such that  $T_1$  and  $T_{2k}$  are copies of the tree  $H$ ,  $T_k$  and  $T_{3k-1}$  are copies of  $S$  and all the other  $T_i$ ,  $1 \leq i \leq m$  are copies of  $Q$ .

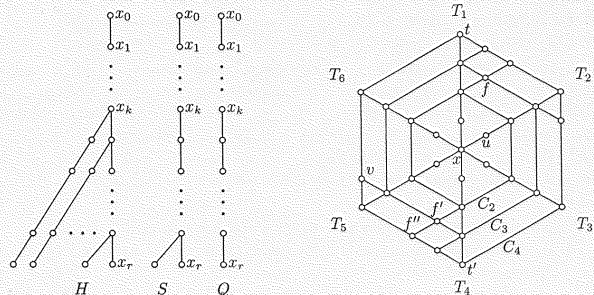


Fig. 2

Let the tree  $T$  arise from  $T_1, T_2, \dots, T_m$  by the identification of all the corresponding copies of the vertex  $x_0$  from  $T_1, T_2, \dots, T_m$  in this order, with one common vertex  $x$ .

Let  $G$  be the graph which we get by forming a cycle  $C_j$  on all vertices with the distance  $j$  from  $x$ , for  $j = k, k+1, \dots, r$ . (The corresponding graph  $G$  with the cycles  $C_2, C_3, C_4$  for  $r = 4$  and  $d = 6$  is in Fig. 2). There are  $l = r - k + 1$  such cycles  $C_j$  with lengths  $2d - 2(r - j + 1) = 2(k + j - 1)$  for  $j = k, k + 1, \dots, r - 1$ , and  $2d$  for  $j = r$ .

It is easy to check that the eccentricity of any vertex  $t$  of  $C_{2d}$  is equal to  $d$  and this eccentricity is attained for the vertex  $t'$  of  $C_{2d}$  whose distance in  $C_{2d}$  from  $t$  is  $d$  (Fig. 2).

Let  $f$  be a vertex from  $C_j$ ,  $k \leq j \leq r - 1$  and let  $f'$  be the vertex from  $C_j$  with distance  $d - r + j - 1 = k + j - 1$  from  $f$  in  $C_j$ . Let  $f''$  be a nearest vertex from  $C_{2d}$  for the vertex  $f'$  in  $G$ . Then  $d_G(f, f'') = d - 1 = e_G(f)$ .

Let  $u$  be a vertex with a distance  $i \leq k - 1$  from  $x$ , and let  $u$  belong to  $T_n$ ,  $1 \leq n \leq m$ . Let  $v$  be a vertex from  $C_{2d}$  belonging to  $T_{(n+2k-1) \bmod (4k-2)}$ . Then  $d_G(u, v) = r + i = e_G(u)$ .

Thus  $\text{Peri}(G)$  consists of all vertices of  $C_j$ ,  $\text{Cep}(G) = \text{Peri}(G)$  and  $C(G) = \{x\}$ , which completes the proof.  $\square$

For any vertex  $v$  of a connected graph  $G$  there exists a spanning tree  $T$  that is distance preserving from  $v$ , i.e.  $d_T(v, u) = d_G(v, u)$  for any vertex  $u \in V(G)$ . If  $G$  is an S-graph with one central vertex  $x$ , then its distance preserving spanning tree  $T_x$  from  $x$  is also an S-graph with  $d(T_x) = 2r(G)$ ,  $C(T_x) = C(G) = \{x\}$  and  $\text{Peri}(T_x) = \text{Cep}(T_x) = \text{Peri}(G) = \text{Cep}(G)$ . Therefore any S-graph  $G$  with one central vertex can be constructed by adding new edges to a proper tree of radius  $r(G)$  and diameter  $2r(G)$ . On the other hand, not every tree  $T$  with radius  $r$  and one central vertex, is a distance preserving spanning tree of an S-graph with radius  $r$ , diameter  $d(G) < d(T)$  and one central vertex. For example,  $P_7$  cannot be supplemented (by adding new edges) to an S-graph  $G$  with one central vertex, of diameter 5 and radius 3. As follows from the proof of Theorem 2.8 for the case  $r(G) = 3$  and  $d(G) = 5$  there exists such a distance preserving spanning tree given by the construction of  $T$  as in Fig. 2.

Given an S-graph  $G$  we will say that an edge  $e \in E(G)$  is *superfluous* in  $G$ , if  $G - e$  is also an S-graph with the same radius, diameter and peripherian as in  $G$ . An S-graph  $G$  is said to be *critical*, if it has no superfluous edge. For example,  $G$  is a critical S-graph with one central vertex, radius  $r(G)$  and diameter  $d(G) = 2r(G)$  if and only if it is a tree with one central vertex, radius  $r(G)$  and diameter  $2r(G)$ . We suggest to investigate critical S-graphs with one central vertex.

Buckley and Lewinter [2] studied two other interesting classes of graphs called F-graphs and L'-graphs. A connected graph  $G$  is an F-graph if it has at least two central vertices, and for each pair of central vertices  $u$  and  $v$ ,  $d(u, v) = r(G)$  holds.

They showed that the only graph that is both an F-graph and an S-graph is  $K_n$ ,  $n \geq 2$ .

If no diametral path of a connected graph  $G$  contains a central vertex,  $G$  is called an L'-graph.

Let  $G$  be an S-graph with one central vertex  $c$  and  $d(G) < 2r$ . Since  $G$  is an S-graph,  $\text{Peri}(G) = \text{Cep}(G)$  and the distance between  $c$  and any diametral vertex is equal to  $r(G)$ . Since no diametral path in  $G$  can contain the central vertex  $c$ ,  $G$  is an L'-graph. The opposite is not true as there are L'-graphs that are not S-graphs. One such graph  $G$  with diameter 5 and radius 3 is shown in Fig. 3. The center of this graph is  $C(G) = \{t\}$ ,  $\text{Cep}(G) = \{x, v, q, y\}$  and  $\text{Peri}(G) = \{x, y\}$ .

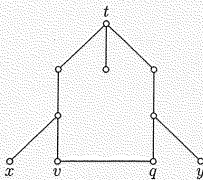


Fig. 3

Let  $H$  and  $Q$  be two graphs and let  $s$  be an arbitrary vertex from  $V(H)$ . We say that the graph  $G$  arose from  $H$  by the substitution of  $s$  by  $Q$  if

$$V(G) = V(H) \cup V(Q) - \{s\}$$

and

$$E(G) = E(H) \cup E(Q) \cup \{xy; x \in V(Q) \text{ and } sy \in E(H)\} - \{sy; sy \in E(H)\}.$$

Gliviak [4] used this substitution to prove the existence of an S-graph  $G$  with a prescribed radius  $a$  and diameter  $b$  containing a given graph  $Q$  as an induced subgraph of  $G$ . To prove this assertion for the case  $a \geq 2$  and  $b = 2a - 1$  it is suggested to construct such an S-graph by the substitution of any central vertices of  $P_{2a}$  by  $G$  and for  $a \geq 2$  and  $b = 2a$  by the substitution of the central vertex of  $P_{2a+1}$  by  $G$ . If  $a = 2$  and  $d(G) \geq 2$  then the resulting graph is not an S-graph. This can be corrected by substituting not a central but a diametral vertex of  $P_{2a}$  and  $P_{2a-1}$ , respectively.

Next we formulate a more general theorem based on the substitution of a vertex of an S-graph.



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**Proof.** For every  $i = 1, 2, \dots, n$  the eccentricity  $e_G(u_i) = r + r(H)$  and the eccentricity of any vertex  $z \in V(G)$ ,  $z \neq u_i$  for  $i = 1, 2, \dots, n$  is  $e_G(z) > r + r(H)$ . Thus  $r(G) = r + r(H)$  and  $C(G) = \{u_1, u_2, \dots, u_n\}$ .

The distance of any two vertices  $x, y \in V(G)$  is  $d_G(x, y) \leq 2r + r(H)$ . The equality holds only if  $x$  and  $y$  are center eccentric vertices of the graphs  $G_i$  and  $G_j$ , respectively, for which  $d_G(u_i, u_j) = r(H)$ . Thus  $\text{Peri}(G) = \bigcup_{i=1}^n \text{Peri}(G_i)$ .

Similarly, the distance of a vertex  $x \in V(G)$  from  $u_i$ ,  $i \in \langle 1, n \rangle$  is  $d_G(x, u_i) = r + r(H)$  only if  $x$  is a center eccentric vertex of  $G_j$  for which  $d_G(u_i, u_j) = r + r(H)$ . Therefore  $\text{Cep}(G) = \bigcup_{i=1}^n \text{Cep}(G_i)$ .  $\square$

In the conclusion we give an estimate of the number of edges of an S-graph.

**Theorem 2.12.** *Let  $G$  be an S-graph with  $p$  vertices and  $q$  edges. Let  $d(G) = d \geq 3$ . Then  $p - 1 \leq q \leq d + 1/2(p - d + 1)(p - d + 4)$ .*

**Proof.** Ore [5] proved this upper bound for any connected graph with diameter  $d \geq 3$ . As follows from the example in Fig. 4, this upper bound is attained also for S-graphs. The lower bound is obvious and it is attained for  $P_{d+1}$ .  $\square$

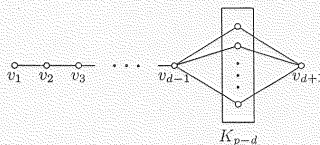


Fig. 4

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