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SOME CONDITIONS FOR A SURFACE IN $\mathbb{E}^4$ TO BE A PART OF THE SPHERE $\mathbb{S}^2$

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Summary. In this paper some properties of an immersion of two-dimensional surface with boundary into $\mathbb{E}^4$ are studied. The main tool is the maximal principle property of a solution of the elliptic system of partial differential equations. Some conditions for a surface to be a part of a 2-dimensional sphere in $\mathbb{E}^4$ are presented.

Keywords: Surfaces with boundary in $\mathbb{E}^4$, maximum principle for elliptic system of PDE

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1. Introduction

In the paper isometric immersion of the two-dimensional oriented Riemannian manifold with boundary into $\mathbb{E}^4$ is studied.

Using a maximal principle property for a solution of an elliptic system of partial differential equations (in [3]) some conditions for such a surface be part of a 2-dimensional sphere in $\mathbb{E}^4$ are given.

2. Surfaces in $\mathbb{E}^4$

Let M be an oriented surface in $\mathbb{E}^4$. Let $(x; e_1, e_2, e_3, e_4)$ be an adapted orthonormal frame field in a domain $U \subset M$ (moving frame in the sense of E. Cartan).

Then we have

$$\begin{aligned} \mathrm{d}x &= \omega^1 e_1 + \omega^2 e_2 \\ \mathrm{d}e_i &= \sum_{j=1}^{4} \omega_i^j e_j \\ \omega_i^j &= -\omega_j^i, \quad i, j = 1, 2, 3, 4. \end{aligned} \tag{1}$$

and

$$d\omega^i = \sum_{j=1}^{4} \omega^j \wedge \omega_i^j$$

(2) $$d\omega_i^j = \sum_{k=1}^{4} \omega_i^k \wedge \omega_k^j.$$

Moreover

$$\omega^3 = 0,\ \omega^4 = 0.$$

From Cartan's lemma we obtain

$$\begin{aligned}
\omega_1^3 &= a_1\omega^1 + a_2\omega^2 \\
\omega_2^3 &= a_2\omega^1 + a_3\omega^2 \\
\omega_1^3 &= b_1\omega^1 + b_2\omega^2 \\
\omega_1^3 &= b_2\omega^1 + b_3\omega^2
\end{aligned}$$

The Gauss curvature K and the mean curvature H on U are given by

$$\mathcal{K} = a_1a_3 - a_2^2 + b_1b_3 - b_2^2$$

(3) $$\mathcal{H} = (a_1 + a_3)^2 + (b_1 + b_3)^2$$

and the mean curvature vector field η on U by

$$\eta = (a_1 + a_3)e_1 + (b_1 + b_3)e_2$$

so we have

$$\|\eta\| = \sqrt{\mathcal{H}}.$$

Let $\Phi = \mathcal{H} - 4\mathcal{K}$.

Then we have on U

$$\Phi = (a_1 - a_3)^2 + 4a_2^2 + (b_1 - b_3)^2 + 4b_2^2$$

Further relations and descriptions can be found in ([1]). The following result is well-known.

Theorem 1. *M is a part of the standard sphere $\mathbb{S}^2$ iff $\Phi \equiv 0$ on M.*

Remark 1. If $\mathcal{H} > 0$ on U, and if we put $e_3 = \frac{\eta}{\|\eta\|}$, then there is a unique e_4 such that (e_1, e_2, e_3, e_4) is coherent with the orientation on M and $\mathbb{E}^4$.

For such a frame we define the 1-form

$$\varphi = \langle de_3, de_4 \rangle$$

on U, which is called the torsion form on M.

3. The maximum principle

For the proofs of the subsequent theorems we need one theorem on the solutions of the system of partial differential equations.

Let D be a bounded domain in $\mathbb{R}^2$ with boundary ∂D, put $\overline{D} = D \cup \partial D$. Let a_{ij}, b_{ij}, c_{ij}, $i, j = 1, 2$ be C^∞ functions on a neighborhood of $\overline{D}$ and

$$\begin{aligned} a_{11} f_x + a_{12} f_y + b_{11} g_x + b_{12} g_y &= c_{11} f + c_{12} g \\ a_{21} f_x + a_{22} f_y + b_{21} g_x + b_{22} g_y &= c_{21} f + c_{22} g \end{aligned} \tag{4}$$

be a system of differential equations for the functions $f(x, y)$, $g(x, y)$.

The system 4 is called elliptic if the quadratic form

$$\Psi = (a_{12} b_{22} - a_{22} b_{12}) \mu^2 + (a_{11} b_{21} - a_{21} b_{11}) \nu^2 - (a_{11} b_{22} - a_{21} b_{12} + a_{12} b_{21} - a_{22} b_{11}) \mu\nu$$

is definite on $\overline{D}$.

Proposition 1. *Let* (4) *be an elliptic system on* $\overline{D}$. *If the functions* f, g *form a solution of* (4) *with* $f \equiv 0$, $g \equiv 0$ *on* ∂D, *then* $f \equiv 0$, $g \equiv 0$ *on* $\overline{D}$.

4. Further characterizations of surfaces in $\mathbb{E}^4$

Using notation from the preceding parts, we get the following results:

Theorem 2. *Let* D *be a bounded domain in* $\mathbb{R}^2$, ∂D *its boundary and let* $x \colon \overline{D} \to \mathbb{E}^4$ *be a surface satisfying*

i) $\Phi \equiv 0$ *on* ∂D,

ii) $\mathcal{H} > 0$ *on* $\overline{D}$,

iii) $\varphi \equiv 0$ *on* $\overline{D}$.

Then there is a subspace $\mathbb{E}^3 \subset \mathbb{E}^4$ *such that* $x(\overline{D}) \subset \mathbb{E}^3$.

Proof. Take a coordinate system (u, v) in a neighborhood V of $\overline{D}$ in such a way that the riemannian metric g on $x(D)$ has the expression

$$g = r^2 \, \mathrm{d}u^2 + s^2 \, \mathrm{d}v^2 .$$

and take e_3, e_4 as in 1. Then we obtain (from the equality $\varphi \equiv 0$) a system of equations for b_1, b_2 $(b_3 = b_1)$

$$\begin{aligned} s b_{1u} + r b_{2v} + 2 b_2 r_v + 2 b_1 s_u &= 0, \\ -r b_{1v} + s b_{2u} + 2 b_2 s_u - 2 b_1 r_v &= 0, \end{aligned} \tag{5}$$

which is elliptic. Boundary condition $b_1 = b_2 = 0$ on ∂D and Proposition 1 imply $b_1 = b_2 = b_3 = 0$ on D, $de_4 = \mathbf{o}$, so that e_4 is constant. □

Using some characterization of the sphere in $\mathbb{E}^3$ we obtain e.g.

Theorem 3. *Let x be as in Theorem 1. Further let one of the following conditions be satisfied:*

(i) *H is constant,*

(ii) *K is constant > 0,*

(iii) *there exist functions $R_i\colon D \to \mathbb{R}^1$, $i = 1,2,3,4$ such that*

$$R_1 d\mathcal{H} + R_2 d\mathcal{K} + R_3 * d\mathcal{H} + R_4 * d\mathcal{K} = 0$$

and

$$R_1^2 + R_3^2 + 4\mathcal{H}(R_1R_2 + R_3R_4) + 4\mathcal{K}(R_2^2 + R_4^2) > 0$$

where $*$ is a star operator on M.

Then $x(D)$ is a part of the sphere $\mathbb{S}^2$ in $\mathbb{E}^4$.

Proof. Theorem 2 follows immediately from the Thm. 1 and the results from [2]. □

Theorem 4. : *Let $x\colon \overline{D} \to \mathbb{E}^4$ be a non-flat surface with $\mathcal{H} > 0$ and a parallel second fundamental form. If $\Phi = 0$ on ∂D then $x(\overline{D})$ is a part of the sphere.*

Proof. The second fundamental form of x has the form:

$$\begin{aligned}\Omega = {} & [a_1(\omega^1)^2 + 2a_2\omega^1\omega^2 + a_3(\omega_2)^2]e_3 \\ & + [b_1(\omega^1)^2 + 2b_2\omega^1\omega^2 + b_3(\omega_2)^2]e_4\end{aligned}$$

on V. Ω is parallel iff $\alpha^i = 0$, $\beta^i = 0$, $i = 1,2,3,4$. If $\mathcal{H} > 0$, we have $\varphi = 0$ and we get a system of differential equations

$$\begin{aligned} & s(a_1 - a_3)_u + 2ra_{2v} + 2(a_1 - a_3)s_u + 4a_2r_v = 0 \\ (6) \qquad & -r(a_1 - a_3)_v + 2sa_{2u} + 2(a_1 - a_3)r_v + 4a_2s_u = 0 \end{aligned}$$

which is elliptic and Proposition 1 implies $\Phi \equiv 0$ on ∂D. □

Theorem 5. *Let $x\colon \overline{D} \to \mathbb{E}^4$ be a non-flat surface with $\mathcal{H} > 0$ and a parallel mean curvature vector. If $\Phi = 0$ on ∂D then $x(\overline{D})$ is a part of the sphere.*

Proof. The condition that η is parallel is equivalent to the systems (5) and (6) of differential equations (where the torsion form φ is zero). □

Theorem 6. *Let $x\colon \overline{D} \to \mathbb{E}^4$ be a non-flat surface with $\Phi = 0$ on ∂D. If $x(\overline{D})$ is pseudoumbilical and one of the conditions* i) *or* ii) *holds, where*

i) *the torsion form φ of x is zero, and*

ii) *H is constant,*

then $x(\overline{D})$ is a part of the sphere.

Proof. A surface is pseudoumbilical since $\mathcal{H} > 0$. The second fundamental form with respect to $e_3 = \frac{\eta}{\sqrt{\mathcal{H}}}$ has the form $k_1 \sum_i (\omega^i)^2$. For the coordinate system in the proof of Thm. 1 and Remark 1 as in Thm. 3 we now have $\varphi = 0$. Thus from (i) or (ii) we obtain the system of differential equations (6) and (7) on ∂D, which yields the result. $\square$

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