Hajrudin Fejzić; Jan Mařík; Clifford E. Weil
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Mathematica Bohemica, Vol. 119 (1994), No. 4, 387–406

Persistent URL: http://dml.cz/dmlcz/126113

119 (1994) MATHEMATICA BOHEMICA No. 4, 387–406

EXTENDING PEANO DERIVATIVES

Hajrudin Fejzić, San Bernardino,
Jan Mařík, Clifford E. Weil, East Lansing

(Received July 29, 1993)

Summary. Let $H \subset [0,1]$ be a closed set, k a positive integer and f a function defined on H so that the k-th Peano derivative relative to H exists. The major result of this paper is that if H has finite Denjoy index, then f has an extension, F, to $[0,1]$ which is k times Peano differentiable on $[0,1]$ with $f_i = F_i$ on H for $i = 1, 2, \ldots, k$.

Keywords: Peano derivative, Denjoy index

AMS classification: 26A24

1. Introduction

Throughout this paper $\mathbb{R}$ will denote the real line and $\mathbb{N}$, the positive integers. The letter k will denote a element of $\mathbb{N}$ and H will denote a closed subset of $[0,1]$ with $H^c = [0,1] \setminus H$. Let f be a function defined on H having a k-th derivative relative to H. In this paper we consider the problem of extending f to the entire interval $[0,1]$ so that the new function, F, is k times differentiable on $[0,1]$ with $F^{(i)} = f^{(i)}$ on H for $i = 0, 1, \ldots, k$. A solution for the case $k = 1$ was given in [9] as well as in [10] and [8] with the added assumption that H is perfect. In the second section we provide a new proof of the result in [9]. For the rest of the paper we consider the case $k \geqslant 2$. In Section 3 we recall an example (given by the second author) of a set, H, and a function, f, defined on H which is twice differentiable there but having no twice differentiable extension to $[0,1]$. The nature of this example demonstrates the futility of pursuing the problem for the ordinary k-th derivative. So we consider the k-th Peano derivative instead and prove a theorem giving conditions on the *function* that assure its extendability to a function having a k-th Peano derivative rather than

Professor Jan Mařík died on January 6, 1994.

a k-th ordinary derivative. (The pertinent definition is given in that section.) In the fourth section we turn our attention to the underlying set H. There is a perfect set H, and a function, f, defined on H having a second Peano derivative, f_2, relative to H but for which there is no twice Peano differentiable function, F, on $[0,1]$ whose Peano derivatives agree with those of f on H. (See Buczolich [1].) We will show that the desired extension does exist if H is a perfect set having finite Denjoy index (defined in the section). In particular there is an extension for any such function defined on the Cantor set since its Denjoy index is 3 as can be easily verified.

2. Extending first derivatives

In this section we will give a new proof of the result proved by Mařík, in [9]; namely Theorem 2.1 below. For the case H perfect, the theorem was proved in [10] and [8]. Although Mařík's proof is elementary in the sense that it uses no advanced theorems, it is very complicated. Here we give a relatively simple proof.

Theorem 2.1. *Let $H \subset [0,1]$ be closed and $f\colon H \to \mathbb{R}$. Suppose that f', computed relative to H, exists at every point $x \in H$. (In case x is an isolated point of H, the value of $f'(x)$ is arbitrary.) Then there is a function F differentiable on $[0,1]$ so that $F = f$ and $F' = f'$ on H.*

Proof. We describe how to define an initial extension, G, of the function f to each component interval of H^c—the complement of H. If H^c has a component interval of the form $[0,b)$, set $G(x) = f(b) + (x-b)f'(b)$ for each $x \in [0,b)$. Proceed similarly if H^c has a component interval of the form $(a,1]$. Let (a,b) be a component interval of H^c with $a,b \in [0,1]$. For the left endpoint, a, we distinguish two cases. If a is an isolated point of H, then there is a component interval of H^c of the form (c,a) (unless $a=0$). Choose $d_a \in (a, \frac{a+b}{2})$ so that

$$d_a - a \leqslant \min\left\{\frac{(b-a)^2}{1+|f'(a)|}, \frac{(a-c)^2}{1+|f'(a)|}\right\}$$

($d_a - a \leqslant \frac{(b-a)^2}{1+|f'(a)|}$ if $a=0$). If a is not an isolated point, then there is a strictly increasing sequence $\{x_n\}$ in H converging to a with $a - x_1 < \frac{b-a}{2}$. In this case for each $n \in \mathbb{N}$ we let $x'_n = 2a - x_n$; that is, the point symmetric to x_n in a. Similarly either b is an isolated point of H in which case there is a component interval (b,d) of H^c (unless $b=1$) and we choose $d_b \in (\frac{a+b}{2}, b)$ so that

$$b - d_b \leqslant \min\left\{\frac{(b-a)^2}{1+|f'(b)|}, \frac{(d-b)^2}{1+|f'(b)|}\right\}$$

(with the obvious modification if $b = 1$) or there is a strictly decreasing sequence $\{y_n\}$ in H converging to b with $y_1 - b < \frac{1}{2}(b-a)$ in which case we set $y'_n = 2b - y_n$. Define G on (a,b) as follows. First let

$$G(x) = \begin{cases} 2f(a) - f(x_n) & \text{if } x = x'_n \text{ for some } n \in \mathbb{N} \\ 2f(b) - f(y_n) & \text{if } x = y'_n \text{ for some } n \in \mathbb{N} \\ f(a) + (x-a)f'(a) & \text{if } x \in (a, d_a) \\ f(b) + (x-b)f'(b) & \text{if } x \in (d_b, b). \end{cases}$$

Note that the set where G is not yet defined consists of open subintervals of (a,b). On each of these intervals define G to be linear on the corresponding closed interval.

Having defined G above on H^c we define $G(x) = f(x)$ for $x \in H$. We will show that G is differentiable everywhere on $[0,1]$ except possibly at points in a component interval, (a,b), of H^c of the form x'_n, y'_n, d_a or d_b and that $G' = f'$ on H. To this end let $w \in H$ and $\varepsilon > 0$. Set $M = \max\{7, 6+4|f'(w)|\}$. By the differentiability of f and by the definition of G, there is a $\delta_1 > 0$ so that $|G(y) - G(w) - (y-w)f'(w)| \leqslant \frac{\varepsilon}{M}|y-w|$ whenever $y \in H$ and $|y - w| < \delta_1$. One can choose $0 < \delta \leqslant \frac{\varepsilon}{M}$ so small that if (a,b) is a component interval of H^c so that if 1) $w \leqslant a < b$ with $b - w < \delta$, then $b + \frac{b-a}{2} - w < \delta_1$, or if 2) $a < b \leqslant w$ with $w - a < \delta$, then $w - (a - \frac{a+b}{2}) < \delta_1$. Let $x \in H^c$ and assume $w < x$. (The case $x < w$ is similar.) Let (a,b) be the component interval of H^c containing x. First suppose $x = x'_n$ for some $n \in \mathbb{N}$. Then

$$\begin{aligned} |G(x) - G(w) - (x-w)f'(w)| &= |2(f(a) - f(w) - (a-w)f'(w)) \\ &\quad - (f(x_n) - f(w) - (x_n - w)f'(w))| \\ &\leqslant 2\frac{\varepsilon}{M}|a-w| + \frac{\varepsilon}{M}|x_n - w| \\ &< 3\frac{\varepsilon}{M}|x-w| < \varepsilon|x-w|. \end{aligned}$$

Next suppose $x = y'_n$ for some $n \in \mathbb{N}$. Then similarly

$$\begin{aligned} |G(x) - G(w) - (x-w)f'(w)| &\leqslant 2\frac{\varepsilon}{M}|b-w| + \frac{\varepsilon}{M}|y_n - w| \\ &= 2\frac{\varepsilon}{M}|b-w| + \frac{\varepsilon}{M}(|b-w| + |b-x|) \\ &\leqslant 4\frac{\varepsilon}{M}|b-x| + 3\frac{\varepsilon}{M}|x-w| \\ &< 7\frac{\varepsilon}{M}|x-w| \leqslant \varepsilon|x-w|. \end{aligned}$$

Now suppose that $x \in [a, d_a]$. Then

$$\begin{aligned}
&|G(x) - G(w) - (x-w)f'(w)| \\
&\qquad = |f(a) - f(w) - (a-w)f'(w) - (x-a)(f'(w) - f'(a))| \\
&\qquad\quad < \frac{\varepsilon}{M}|a-w| + |x-a|(|f'(a)| + |f'(w)|) \\
&\qquad\quad \leqslant \frac{\varepsilon}{M}|x-w| + |d_a - a|(|f'(a)| + |f'(w)|) \\
&\qquad\quad < \frac{\varepsilon}{M}|x-w| + |a-w|^2(1 + |f'(w)|) \\
&\qquad\quad \leqslant \frac{\varepsilon}{M}|x-w| + |x-w|^2(1 + |f'(w)|) \\
&\qquad\quad < \frac{\varepsilon}{M}|x-w| + \frac{\varepsilon}{M}|x-w|(1 + |f'(w)|) \\
&\qquad\quad < \frac{\varepsilon}{M}|x-w|(2 + |f'(w)|) < \varepsilon|x-w|.
\end{aligned}$$

Next suppose that $x \in [d_b, b]$. Then as above

$$|G(x) - G(w) - (x-w)f'(w)| < \frac{\varepsilon}{M}|b-w| + |b-w|^2(1 + |f'(w)|).$$

Since $|b-w| \leqslant |b-a| + |a-w| \leqslant 2|x-a| + |a-w| < 2|x-w|$,

$$\begin{aligned}
|G(x) - G(w) - (x-w)f'(w)| &< 2\frac{\varepsilon}{M}|x-w| + 4|x-w|^2(1 + |f'(w)|) \\
&< 2\frac{\varepsilon}{M}|x-w| + 4\frac{\varepsilon}{M}|x-w|(1 + |f'(w)|) \\
&< \frac{\varepsilon}{M}|x-w|(6 + 4|f'(w)|) \leqslant \varepsilon|x-w|.
\end{aligned}$$

Consequently in all four cases,

$$|G(x) - G(w) - (x-w)f'(w)| < \varepsilon|x-w|. \tag{1}$$

Finally any $x \in (a, b)$ not covered by one of the four cases above is in an interval (c, d) where both c and d are one of the four types discussed above. Then there are $\alpha, \beta \in [0, 1]$ with $\alpha + \beta = 1$ such that $x = \alpha c + \beta d$ and $G(x) = \alpha G(c) + \beta G(d)$. Thus we have

$$\begin{aligned}
|G(x) - G(w) - (x-w)f'(w)| &= \alpha|G(c) - G(w) - (c-w)f'(w)| \\
&\quad + \beta|G(d) - G(w) - (d-w)f'(w)| \\
&< \alpha\varepsilon(c-w) + \beta\varepsilon(d-w) = \varepsilon|x-w|.
\end{aligned}$$

Therefore G is differentiable on H with $G' = f'$ on H.

In fact G is differentiable everywhere except possibly at the points of the first four types discussed above since these are the points which are simultaneous the endpoints of two intervals on which G is linear. The objective now is to redefine G in small neighborhoods of such points so that the new function, F, is differentiable on $[0,1]$. To accomplish this goal let c be such a point and let (a,b) be the component interval of H^c with $c \in (a,b)$. Let z_1 be the midpoint of the interval to the left of c on which G is linear and z_2 the midpoint of the interval to the right on which G is linear. There is a function $\tilde{G}$ differentiable on $[z_1, z_2]$ so that $\tilde{G}(z_1) = G(z_1)$, $\tilde{G}(z_2) = G(z_2)$, $\tilde{G}'(z_1) = G'(z_1)$, $\tilde{G}'(z_2) = G'(z_2)$ and the graph of $\tilde{G}$ lies in the triangle with vertices $(z_1, G(z_1))$, $(c, G(c))$ and $(z_2, G(z_2))$. Now define a function F on $[0,1]$ to be this function, $\tilde{G}$ on each of the intervals, $[z_1, z_2]$ and to be G otherwise. Clearly F is differentiable on H^c. So it remains only to check that F is differentiable on H with $F' = f'$ on H. For this purpose let $w \in H$ and $\varepsilon > 0$. There is a $\delta > 0$ so that $|x - w| < \delta$ implies

$$|G(x) - G(w) - (x - w)f'(w)| < \varepsilon|x - w|. \tag{2}$$

If x belongs to one of the intervals $[z_1, z_2]$, (Employing the notation of the previous paragraph, $c \in [z_1, z_2]$ denotes a common endpoint of two intervals on which G is linear.) then there are $\alpha, \beta, \gamma \in [0,1]$ with $\alpha + \beta + \gamma = 1$ such that $x = \alpha z_1 + \beta c + \gamma z_2$ and $F(x) = \alpha G(z_1) + \beta G(c) + \gamma G(z_2)$. Thus

$$\begin{aligned}
|F(x) - F(w) - (x - w)f'(w)| &= |\alpha G(z_1) + \beta G(c) + \gamma G(z_2) - (\alpha + \beta + \gamma)G(w) \\
&\quad - (\alpha z_1 + \beta c + \gamma z_2 - (\alpha + \beta + \gamma)w)f'(w)| \\
&\leqslant \alpha|G(z_1) - G(w) - (z_1 - w)f'(w)| \\
&\quad + \beta|G(c) - G(w) - (c - w)f'(w)| \\
&\quad + \gamma|G(z_2) - G(w) - (z_2 - w)f'(w)| \\
&\leqslant \varepsilon(\alpha(z_1 - w) + \beta(c - w) + \gamma(z_3 - w)) = \varepsilon(x - w).
\end{aligned}$$

Since ε was arbitrary, we see that the function F satisfies the assertion of the theorem. $\square$

3. Extending k-th derivatives

We became interested in the problem of extending higher order derivatives through a question posed by Professor Richard O'Malley. He asked if a function which is defined and say twice differentiable on a perfect set, H, relative to H can be extended to a function defined on $[0,1]$ which is twice differentiable on $[0,1]$. We begin this section by discussing an example given in [9] showing that in general the answer to O'Malley's question is no. In [9] Mařík started with any sequence $\{x_n\}$ converging monotonically to 0 and constructed a sequence $\{[x_n, y_n]\}$ of disjoint, closed subintervals of $[0,1]$ and put $H = \bigcup_{n=1}^{\infty} [x_n, y_n] \cup \{0\}$. Clearly H is a perfect set. He defined f on H by $f(0) = 0$ and for each $n \in \mathbb{N}$ and each $x \in [x_n, y_n]$ by $f(x) = x_n^2$. It is obvious that for each $n \in \mathbb{N}$ and each $x \in [x_n, y_n]$ $f'(x) = 0$. Since the graph of f lies betweeny $= x^2$ and $y = 0$, it is easy to see that $f'(0) = 0$. Consequently, for each $i \in \mathbb{N}$ (in particular for $i = 2$) $f^{(i)} = 0$ on H. In [9] it is shown that f can not be extended to a twice differentiable function on $[0,1]$ by showing that any such extension would have a first derivative that is unbounded on any neighborhood of 0. What this example illustrates, besides its intended purpose, is that for functions whose domains are not connected, the usual notion of differentiation is not the correct one to use for higher order differentiation. One which is considerably better is the notion of the k-th Peano derivative whose definition we recall next.

Definition 3.1. Let H $\subset [0,1]$ be closed, let $k \in \mathbb{N}$, let $f \colon H \to \mathbb{R}$ and let $x \in H$. Then f is k times Peano differentiable at x relative to H means there are numbers $f_1(x,H), \ldots, f_k(x,H)$ so that $y \in H$ implies

$$f(y) = f(x) + (y-x)f_1(x,H) + \cdots + \frac{(y-x)^k}{k!}(f_k(x,H) + \varepsilon_k(y))$$

where $\lim_{y \in H, y \to x} \varepsilon_k(y) = 0$. The number $f_k(x,H)$ is called the k-th Peano derivative of f at x relative to H. It will be convenient to denote $f(x)$ by $f_0(x,H)$. When H is an interval we simply write $f_k(x,H) = f_k(x)$. If f is k times Peano differentiable at each $x \in H$, we say that f is k times Peano differentiable on H relative to H. At an isolated point $x \in H$ the choice of the numbers, $f_1(x,H), \ldots, f_k(x,H)$, is completely arbitrary.

For more information about the theory of k-th Peano derivative, the reader is referred to [4] and [11]. Here we only note that it follows from the classical form of Taylor's Theorem that if f is defined on a neighborhood of x and is k times differentiable at x in the usual sense, then f is k times Peano differentiable at x and $f^{(k)}(x) = f_k(x)$. For this equality to hold it is essential that the domain of f contains a neighborhood of x as is demonstrated by the example presented above. It is not hard to show that if, in that example, one selects $x_n = \frac{1}{n+1}$, then $f_2(0,H) = 2$. This observation gives another way to conclude that no extension of f is possible. For if F is such an extension, then since $f''(0) = 0$, we must have $F''(0) = 0$. But since $f_2(0,H) = 2$, we must have $F_2(0) = 2$. However by Taylor's theorem, $F''(0)$ and $F_2(0)$ must be equal. By a similar argument it can be seen that a necessary condition for extendability of a k-th ordinary derivative is the existence of the first k Peano derivatives and their agreement with the first k ordinary derivatives. This condition is not sufficient for extendability to a k times differentiable function as will be shown, but is sufficient for extendability to a k times Peano differentiable function.

Lemma 3.2. *Let $H \subset [0,1]$ be closed, let $k \in \mathbb{N}$ and let $g\colon H \to \mathbb{R}$ be k times Peano differentiable on H relative to H. Suppose that for each $i \in \mathbb{N}$ with $i \leqslant k$, $g_i = 0$ on H. Then there is a function $G\colon [0,1] \to \mathbb{R}$ which is k times Peano differentiable on $[0,1]$ such that $G|_H = g$.*

Proof. The assumption simply states that for each $x \in H$ we have $g(y) = g(x) + (y-x)^k \varepsilon(y)$ for $y \in H$ where $\lim_{y \in H, y \to x} \varepsilon(y) = 0$. (The $k!$ is absorbed into the function ε.) To define the extension let (a,b) be a component interval of H^c. There is a unique polynomial, p, of degree $2k+1$ defined on $[a,b]$ such that $p(a) = g(a)$, $p(b) = g(b)$, and for each $i \in \mathbb{N}$ with $i \leqslant k$, $p^{(i)}(a) = p^{(i)}(b) = 0$. (If the component interval is of the form $[0,b)$, then simply set $p(x) = g(b) + (b-x)^{k+1}$. Proceed similarly if the component interval is of the form $(a,1]$.) Let $G = p$ on (a,b) for each component interval (a,b) of H^c and let $G = g$ on H. Clearly G is k times (and hence k times Peano) differentiable on H^c. It remains to show that for each $x \in [0,1]$, $G_i(x)$ exists and $G_i(x) = 0$ for $i = 1, 2, \ldots, k$. To do so we must first investigate the polynomial p more closely. Since the first k derivatives of p are 0 at both a and b, it follows that for each $y \in (a,b)$, $p'(y) = A(y-a)^k(y-b)^k$ where A is a number to be determined. So

$$\begin{aligned} p'(y) &= A(y-a)^k(y-a+a-b)^k \\ &= A(y-a)^k \sum_{j=0}^{k} \binom{k}{j} (y-a)^j (a-b)^{k-j} \\ &= A \sum_{j=0}^{k} \binom{k}{j} (a-b)^{k-j} (y-a)^{k+j}. \end{aligned}$$

Integrating once gives

$$p(y) = A \sum_{j=0}^{k} \binom{k}{j} \frac{(a-b)^{k-j}}{k+j+1} (y-a)^{k+j+1} + B.$$

Since $p(a) = g(a)$, $B = g(a)$, and since $p(b) = g(b)$,

$$g(b) - g(a) = A\sum_{j=0}^{k} \binom{k}{j} \frac{(a-b)^{k-j}}{k+j+1}(b-a)^{k+j+1}$$
$$= A(b-a)^{2k+1}\sum_{j=0}^{k} \binom{k}{j} \frac{(-1)^{k-j}}{k+j+1}.$$

Thus $A = \frac{g(b)-g(a)}{(b-a)^{2k+1}} C_k$ where $C_k = \sum_{j=0}^{k} \binom{k}{j} \frac{(-1)^{k-j}}{k+j+1}$ depends only on k. Hence

$$G(y) = p(y) = \frac{g(b)-g(a)}{(b-a)^{2k+1}} C_k \sum_{j=0}^{k} \binom{k}{j} \frac{(a-b)^{k-j}}{k+j+1}(y-a)^{k+j+1} + g(a).$$

Let $x \in H$. It will be shown that $G_1(x) = G_2(x) = \cdots = G_k(x) = 0$; that is, $\lim\limits_{y\to x} \frac{G(y)-G(x)}{(y-x)^k} = 0$. Since $g_1(x) = g_2(x) = \cdots = g_k(x) = 0$, we need only consider $y \in H^c$.

Suppose (a, b) is a component interval of H^c with $x < a < b$. The case of $a < b < x$ is dealt with in a similar fashion. Then

$$\left|\frac{G(y)-G(x)}{(y-x)^k}\right| \leqslant \left|\frac{G(y)-G(a)}{(y-x)^k}\right| + \left|\frac{g(a)-g(x)}{(a-x)^k}\right| \left|\frac{(a-x)^k}{(y-x)^k}\right|.$$

Since $a - x \leqslant y - x$, and since $g_1(x) = g_2(x) = \cdots = g_k(x) = 0$, the second term tends to 0 as $y \to x$. So we estimate the first term.

$$\left|\frac{G(y)-G(a)}{(y-x)^k}\right| = \left|\frac{g(b)-g(a)}{(y-x)^k(b-a)^{2k+1}} C_k \sum_{j=0}^{k} \binom{k}{j} \frac{(a-b)^{k-j}}{k+j+1}(y-a)^{k+j+1}\right|$$
$$= \left|\frac{(b-x)^k\varepsilon(b) + (a-x)^k\varepsilon(a)}{(y-x)^k(b-a)^{2k+1}} C_k \sum_{j=0}^{k} \binom{k}{j} \frac{(a-b)^{k-j}}{k+j+1}(y-a)^{k+j+1}\right|$$
$$\leqslant \Bigg(\frac{(b-x)^k(y-a)^k(b-a)^{k+1}|\varepsilon(b)|}{(y-x)^k(b-a)^{2k+1}}$$
$$+ \frac{(a-x)^k(b-a)^{2k+1}|\varepsilon(a)|}{(y-x)^k(b-a)^{2k+1}}\Bigg) C_k \sum_{j=0}^{k} \binom{k}{j} \frac{1}{k+j+1}.$$

Since $a - x \leqslant y - x$, the second term in the parentheses above tends to 0 as $y \to x$. So we consider the first term. There are two cases. First assume that $x - a \leqslant b - a$. Then $b - x = b - a + a - x \leqslant 2(b - a)$. Since $y - a \leqslant y - x$, the first term is no more than $\frac{(y-a)^k}{(y-x)^k} \frac{(b-x)^k}{(b-a)^k}|\varepsilon(b)| \leqslant 2^k|\varepsilon(b)|$. The other case is $b - a \leqslant x - a$. Then $b - x = b - a + a - x \leqslant 2(a - x) < 2(y - x)$ and so in this case since $y - a \leqslant b - a$, the first term is no more than $\frac{(b-x)^k}{(y-x)^k}|\varepsilon(b)| \leqslant 2^k|\varepsilon(b)|$ and hence the first term also tends to 0 as $y \to x$, which completes the proof of the lemma. □

Theorem 3.3. *Let $H \subset [0,1]$ be closed, let $k \in \mathbb{N}$ and let $f\colon H \to \mathbb{R}$ be k times Peano differentiable on H relative to H. Suppose in addition that f is k times differentiable in the usual sense on H relative to H and that for each $i, j \in \mathbb{N} \cup \{0\}$ with $i + j \leqslant k$ $f^{(i)}$ is j times Peano differentiable on H relative to H and $\left(f^{(i)}\right)_j = f^{(i+j)}$. Then there is a function $F\colon [0,1] \to \mathbb{R}$ which is k times Peano differentiable on $[0,1]$ such that $F|_H = f$.*

Proof. We proceed by induction on k. For $k = 1$ the assertion is just that of Theorem 2.1. So suppose the assertion is true for $k - 1$ and let f satisfy the hypotheses for k. Then f' satisfies the hypotheses for $k - 1$. Consequently there is a function $S\colon [0,1] \to \mathbb{R}$ which is $k - 1$ times Peano differentiable on $[0,1]$ such that $S|_H = f'$. Let T be any antiderivative of S. Then for each $i = 1, 2, \cdots k$ on H we have $(f - T)_i = f_i - T_i = f_i - (T')_{i-1} = f_i - S_{i-1} = f_i - (f')_{i-1} = f_i - f_i = 0$. Thus the function $f - T$ satisfies the assumptions of Lemma 3.2. Hence there is a function $G\colon [0,1] \to \mathbb{R}$ k times Peano differentiable on $[0,1]$ such that $G|_H = f - T$. Let $F = T + G$. Then $F|_H = f$ and the proof is complete. □

We close this section with an example showing that the extension whose existence was just proved in general need not be k times differentiable for $k \geqslant 2$. In fact the example is for $k = 2$.

For each $n \in \mathbb{N}$ let $b_n = \frac{1}{n+1}$ and let $a_n = b_n - b_n^3$. It is routine to show that $b_{n+1} < a_n$, and that the line joining $(a_n, 0)$ and (b_n, b_n^3) has slope 1. For each $n \in \mathbb{N}$ let $d_n = \frac{a_n - b_{n+1}}{3}$. Let $H = \{0\} \cup \bigcup\limits_{n=1}^{\infty} [b_{n+1}, b_{n+1} + d_n] \cup [a_n - d_n, a_n]$. Define f on H by $f(0) = 0$ and for each $n \in \mathbb{N}$ $f = b_{n+1}^3$ on $[b_{n+1}, b_{n+1} + d_n]$ while $f = 0$ on $[a_n - d_n, a_n]$. Since the graph of f lies between $y = x^3$ and $y = 0$, $f_1(0) = f_2(0) = 0$. Moreover for each $n \in \mathbb{N}$ $f' = f'' = 0$ on $[b_{n+1}, b_{n+1} + d_n] \cup [a_n - d_n, a_n]$. Consequently $f'(0) = f''(0) = 0$. Since for each $n \in \mathbb{N}$, the slope of the line joining $(a_n, 0)$ and (b_n, b_n^3) is 1, any differentiable extension of f to $[0,1]$ will have at least one point in each $[a_n, b_n]$ where the derivative is 1. Hence the derivative can't even be continuous at 0 let alone differentiable there.

4. Extending Peano derivatives

In this section we present a sufficient condition on the *set* H under which every function $f\colon H \to \mathbb{R}$ which is k times Peano differentiable on H relative to H can be extended to a function $F\colon [0,1] \to \mathbb{R}$ which is k times Peano differentiable on $[0,1]$ so that $F_i(x) = f_i(x, H)$ on H for $i = 0, 1, \cdots k$. The family of closed sets with this property will be denoted by P_k. For $k = 1$ Theorem 2.1 states that $P_1 = \{H \subset [0,1]\colon H \text{ is closed}\}$. For the case $k \geqslant 2$ not every perfect set is in P_k. For the case $k = 2$, Buczolich provided an example of a perfect set, H, that is not in P_2. (See [1].) In [3] Denjoy gave an example of a perfect set, H, and a function, f, defined on H with $f_1(x, H) = f_2(x, H) = 0$ and $f_3(x, H) = 1$. The set, H, doesn't belong to P_3, for if so there would be an extension, F, of f so that F is three times Peano differentiable on $[0, 1]$. By a theorem in [5] there is a decomposition of H into a countable collection of closed sets, $\{A_n\}$, so that $H = \bigcup_n A_n$ and $F_3(x) = F_2'(x)$ on A_n. Since F is an extension of f, $F_2(x) = 0$ and $F_3(x) = 1$ on H which is contrary to the choice of the sets A_n. The function f from Denjoy's example can easily be modified so that H is not in P_k, $k \geqslant 3$ odd. One common property of the Buczolich and the Denjoy examples is that both are perfect sets which are extremely rare at each point of the set in the sense that the symmetric porosity of each set at each of its points is 1. The condition that we require could be stated in terms of porosity, but we use the notion of the Denjoy index instead. The Denjoy index, α, of a set, H, at one of its points is related to the symmetric porosity, p, of that set at the same point by the formula $\alpha = \frac{1}{1-p}$. The concept of index of a perfect set was introduced by Denjoy in [3, page 285]. The reader can learn more about the relationship between the index of a perfect set at a point of the set and the corresponding porosity in [2]. The condition we require is essentially that the set have finite Denjoy index at each point, but with some uniformity added.

Definition 4.1. Let $H \subset [0, 1]$ be closed. Then the Denjoy index of H is

$$\begin{aligned}\inf\{\lambda \geqslant 1\colon & \text{ for some } \theta > 0 \text{ and for each } x \in H \text{ there is a sequence } \{k_n\}_{n\in\mathbb{N}} \\ & \text{with } \lim_{n\to\infty} k_n = 0 \text{ and } |k_1| \geqslant \theta \text{ such that } x + k_n \in H \\ & \text{and } 1 < |k_n|/|k_{n+1}| \leqslant \lambda \text{ for each } n \in \mathbb{N}\}.\end{aligned}$$

It is possible that there is no λ satisfying the definition in which case we invoke the convention that the infimum of $\emptyset$ is ∞. This is clearly the case if H has an isolated point. The uniform θ implies that the first term of the sequence $\{x + k_n\}$ is at least a fixed distance, θ, from x. The uniform λ guarantees that $\{x + k_n\}$ converges to x no faster than λ^{-n+1} converges to 0. Since the θ depends on λ, the Denjoy index itself need not be a λ. However it may be. It is not hard to see that if H is the Cantor set, then for $\lambda = 3$ there is a corresponding θ; namely $\theta = 1/3$, but not for any $\lambda < 3$. So the Denjoy index of the Cantor set is 3.

The purpose of this section is to prove that a closed set with finite Denjoy index is in P_k for every k. Note that such a set must be perfect. As noted above, our result applies to the Cantor set. We use the ideas of the proof of Theorem 2.1, but with substantial modification. There were two main ideas in the proof of Theorem 2.1. First, to extend a function, f, to the entire interval $[0, 1]$, so that the extension, G, is differentiable on H. It is here that finite Denjoy index will be used for the general case. The second step was to modify the extension so as to be differentiable everywhere. This step generalizes to the case $k \geqslant 2$ with no restriction on H. (See Lemma 4.6 below.) Concerning the first step, suppose (a, b) is a component interval of H^c. Recall that in the proof of Theorem 2.1 we selected two sequences from H at random one converging to each endpoint of (a, b). Then we reflected the terms of each of these sequences in the endpoint to which the sequence converged and defined the extension to be $2f(a) - f(x_n)$ for the endpoint a and $2f(b) - f(y_n)$ for the endpoint b. Here we use the assumption of finite Denjoy index to select the two sequences. These sequences are then reflected as before, but to define the function between two reflected points we use a certain weighted average of the values of the function at the endpoint and at k of the points of the original sequence. We begin with two lemmas which give rise to the weights used for the extension.

Lemma 4.2. *Let $x_1, x_2, \ldots, x_k$, $a, x \in \mathbb{R}$ be distinct. Then the system*

$$(x-a)^j = \sum_{i=1}^{k} \alpha_i (x_i - a)^j \quad j = 1, \ldots, k$$

has a solution for $(\alpha_1, \ldots, \alpha_k)$. Moreover

$$\alpha_i = \frac{(x-a) \prod\limits_{j \neq i} (x - x_j)}{(x_i - a) \prod\limits_{j \neq i} (x_i - x_j)}.$$

Proof. The assertion follows from the fact that if $\beta_1, \ldots, \beta_k \in \mathbb{R}$, then the determinant of the matrix

$$\begin{bmatrix} \beta_1 & \cdots\cdots & \beta_k \\ \beta_1^2 & \cdots\cdots & \beta_k^2 \\ \vdots & & \vdots \\ \beta_1^k & \cdots\cdots & \beta_k^k \end{bmatrix}$$

is $\prod\limits_{k \geqslant s \geqslant 1} \beta_s \prod\limits_{k \geqslant s > j \geqslant 1} (\beta_s - \beta_j)$. □

The next lemma shows that $(x-w)^j$ can be represented in terms of the numbers $\alpha_1, \ldots, \alpha_k$ of the above lemma for any $w \in \mathbb{R}$; not just a.

Lemma 4.3. *Let $x_1, x_2, \ldots, x_k$, $a, x \in \mathbb{R}$ be distinct and let $\alpha_1, \ldots, \alpha_k$ be as in the conclusion of Lemma 4.2. (Thus $(x-a)^j = \sum\limits_{i=1}^{k} \alpha_i (x_i - a)^j$ for $j = 1, \ldots, k$.) Then for any $w \in \mathbb{R}$*

$$(x-w)^j = \sum_{i=1}^{k} \alpha_i (x_i - w)^j + \left(1 - \sum_{i=1}^{k} \alpha_i\right)(a-w)^j \qquad \text{for } j = 1, \ldots, k.$$

Proof. Let $w \in \mathbb{R}$ and let $j \in \{1, 2, \cdots, k\}$. Then

$$
\begin{aligned}
(x-w)^j &= (x-a+a-w)^j \\
&= \sum_{s=0}^{j} \binom{j}{s} (x-a)^s (a-w)^{j-s} \\
&= (a-w)^j + \sum_{s=1}^{j} \binom{j}{s} \left(\sum_{i=1}^{k} \alpha_i (x_i - a)^s \right) (a-w)^{j-s} \\
&= (a-w)^j + \sum_{i=1}^{k} \alpha_i \sum_{s=1}^{j} \binom{j}{s} (x_i - a)^s (a-w)^{j-s} \\
&= (a-w)^j + \sum_{i=1}^{k} \alpha_i \left(\sum_{s=0}^{j} \binom{j}{s} (x_i - a)^s (a-w)^{j-s} - (a-w)^j \right) \\
&= (a-w)^j + \sum_{i=1}^{k} \alpha_i \left((x_i - w)^j - (a-w)^j \right) \\
&= \sum_{i=1}^{k} \alpha_i (x_i - w)^j + (a-w)^j \left(1 - \sum_{i=1}^{k} \alpha_i \right).
\end{aligned}
$$

□

Since the numbers, α_i, of the previous two lemmas depend on the $k+2$ numbers, $x_1, x_2, \ldots, x_k, a, x$, in the next lemma we denote them by $\alpha_i(x_1, x_2, \cdots, x_k, a, x)$. That lemma selects the sequences that will be reflected into the component intervals of H^c.

Lemma 4.4. *Let $H \subset [0,1]$ be a perfect set with finite Denjoy index and let (a,b) be a component interval of H^c. Then there are a strictly increasing sequence $\{x_n\}$ in H converging to a, a strictly decreasing sequence $\{y_n\}$ in H converging to b and a constant, K, (depending only on the choice of λ and θ from the definition of Denjoy index of H) so that for each $n \in \mathbb{N}$ and $i = 1, \ldots, k$ we have $|\alpha_i(x_{n+1}, \ldots, x_{n+k}, a, x)| \leqslant K$ for $|x-a| \leqslant |x_n - a|$ and $|\alpha_i(y_{n+1}, \ldots, y_{n+k}, b, y)| \leqslant K$ for $|y-b| \leqslant |y_n - b|$. Moreover for $i = 1, \ldots, k$ we have $|\alpha_i(x_1, x_2, \ldots, x_k, a, x)| \leqslant K$ for $|x-a| \leqslant \frac{b-a}{2}$ and $|\alpha_i(y_1, y_2, \ldots, y_k, b, y)| \leqslant K$ for $|y-b| \leqslant \frac{b-a}{2}$. (In case the component interval is of the form $[0,b)$ we assert only the existence of the sequence $\{y_n\}$ with a similar adjustment in the case $(a,1]$.)*

Proof. Since H has finite Denjoy index, there are $\lambda \geqslant 1$ and $\theta > 0$ satisfying Definition 4.1. If $\theta < \frac{b-a}{2}$ (which is true for at most finitely many of the component intervals), let $m = 0$. Otherwise let $m \in \mathbb{N}$ satisfy $\frac{\theta}{\lambda^m} < \frac{b-a}{2} \leqslant \frac{\theta}{\lambda^{m-1}}$. It follows from

Definition 4.1 that there is an increasing sequence, $\{x_n\}$, in H converging to a so that $\frac{\theta}{\lambda^{m+2n+1}} < a - x_{n+1} \leqslant \frac{\theta}{\lambda^{m+2n}}$. Suppose $|x-a| \leqslant a-x_n$. Then for $j = n+1, \ldots, n+k$

$$|x - x_j| \leqslant |x - a| + a - x_j \leqslant |x - a| + a - x_{n+1} \leqslant \frac{\theta}{\lambda^{m+2n-2}} + \frac{\theta}{\lambda^{m+2n}} = \frac{\theta(\lambda^2 + 1)}{\lambda^{m+2n}}.$$

Also $a - x_j \geqslant a - x_{n+k} \geqslant \frac{\theta}{\lambda^{m+2n+2k-1}}$ while for $l = n+1, \ldots, n+k$, $l \neq j$, $|x_j - x_l| \geqslant x_{n+k} - x_{n+k-1} \geqslant \frac{\theta}{\lambda^{m+2n+2k-3}} - \frac{\theta}{\lambda^{m+2n+2k-2}} = \frac{\theta(\lambda-1)}{\lambda^{m+2n+2k-2}}$. Let $i \in \{1, \ldots, k\}$. By Lemma 4.2

$$\begin{aligned}
|\alpha_i(x_{n+1}, \ldots, x_{n+k}, a, x)| &\leqslant \frac{(x-a)(a-x_{n+1})^{k-1}}{(a - x_{n+k}) \left(\frac{\theta(\lambda-1)}{\lambda^{m+2n+2k-2}} \right)^{k-1}} \\
&\leqslant \frac{\frac{\theta}{\lambda^{m+2n+2k-1}} \left(\frac{\theta(\lambda^2+1)}{\lambda^{m+2n}} \right)^{k-1}}{\left(\frac{\theta}{\lambda^{m+2n+2k-1}} \right) \left(\frac{\theta(\lambda-1)}{\lambda^{m+2n+2k-2}} \right)^{k-1}} \\
&= \lambda^{2k+1} \left(\frac{(\lambda^2+1)\lambda^{2k-2}}{\lambda - 1} \right)^{k-1} \\
&= \lambda^2 \left(\frac{(\lambda^2+1)\lambda^{2k}}{\lambda - 1} \right)^{k-1} = K_1
\end{aligned}$$

Now suppose $|x - a| \leqslant \frac{b-a}{2}$. If in addition $\theta \geqslant \frac{b-a}{2}$, then by an argument similar to the one above we get

$$|\alpha_i(x_1, x_2, \ldots, x_k, a, x)| \leqslant \lambda^2 \left(\frac{\lambda+1}{\lambda-1} \lambda^{2k} \right)^{k-1} = K_2.$$

However, if $\theta < \frac{b-a}{2}$, then a similar argument gives

$$|\alpha_i(x_1, x_2, \ldots, x_k, a, x)| \leqslant \frac{1}{2\theta} 2^{k-1} \lambda^{2k-1} \left(\frac{\lambda^{2k-2}}{\lambda - 1} \right)^{k-1} = K_3.$$

Hence it is enough to take $K = \max\{K_1, K_2, K_3\}$.

Proceeding in an analogous fashion one can find a sequence $\{y_n\}$ that satisfies the assertion of the lemma. □

We observe that in the proof of the preceding lemma we used the finite Denjoy index condition only for the endpoints of the component intervals of H^c. Indeed the condition is not needed for the other points in H.

The following theorem is the major step toward accomplishing the goal of this section.

Theorem 4.5. *Let $H \subset [0,1]$ be a perfect set with finite Denjoy index and let $f\colon H \to \mathbf{R}$ so that $f_k(x,H)$ exists at every point $x \in H$. Then there is a function, $F\colon [0,1] \to \mathbf{R}$, so that $F_k(x)$ exists for every $x \in H$ and $F_i(x) = f_i(x,H)$ for $x \in H$ and $i = 0,1,\ldots,k$. (Recall that $f_0(x,H) = f(x)$.)*

Proof. Let (a,b) be a component interval of H^c and let K, $\{x_n\}$ and $\{y_n\}$ be as in Lemma 4.4. For $x \in (a,b)$ and for $1 \leqslant i \leqslant k$ define $\alpha_i(x)$ by

$$\alpha_i(x) = \begin{cases} \alpha_i(x_{n+1},\ldots,x_{n+k},a,x) & \text{for } x \in (2a - x_{n+1}, 2a - x_n] \text{ and } n \in \mathbb{N} \\ \alpha_i(y_{n+1},\ldots,y_{n+k},b,x) & \text{for } x \in [2b - y_n, 2b - y_{n+1}) \text{ and } n \in \mathbb{N} \\ \alpha_i(x_1,x_2,\ldots,x_k,a,x) & \text{for } x \in (2a - x_1, \frac{a+b}{2}] \\ \alpha_i(y_1,y_2,\ldots,y_k,b,x) & \text{for } x \in (\frac{a+b}{2}, 2b - y_1). \end{cases}$$

By Lemma 4.4, $|\alpha_i(x)| \leqslant K$ for all $x \in (a,b)$.

Define the function F on (a,b) as follows:

$$F(x) = \begin{cases} \sum\limits_{i=1}^{k} \alpha_i(x) f(x_{n+i}) + \Big(1 - \sum\limits_{i=1}^{k} \alpha_i(x)\Big) f(a) & \text{for } x \in (2a - x_{n+1}, 2a - x_n] \\ & \text{and } n \in \mathbb{N} \\ \sum\limits_{i=1}^{k} \alpha_i(x) f(y_{n+i}) + \Big(1 - \sum\limits_{i=1}^{k} \alpha_i(x)\Big) f(b) & \text{for } x \in [2b - y_n, 2b - y_{n+1}) \\ & \text{and } n \in \mathbb{N} \\ \sum\limits_{i=1}^{k} \alpha_i(x) f(x_i) + \Big(1 - \sum\limits_{i=1}^{k} \alpha_i(x)\Big) f(a) & \text{for } x \in (2a - x_1, \frac{a+b}{2}] \\ \sum\limits_{i=1}^{k} \alpha_i(x) f(y_i) + \Big(1 - \sum\limits_{i=1}^{k} \alpha_i(x)\Big) f(b) & \text{for } x \in (\frac{a+b}{2}, 2b - y_1). \end{cases}$$

(In case the component interval is of the form $[0,b)$ eliminate the first and third conditions and in the fourth condition replace $(\frac{a+b}{2}, 2b - y_1)$ by $[0, 2b - y_1)$. Make a similar adjustment for the case $(a,1]$.) Having defined F on each component interval of H^c, we set $F = f$ on H and now have defined F on $[0,1]$.

We will prove that F satisfies the assertion of the theorem. The first step will be to show that for each component interval, (a,b) of H^c, the Peano derivatives of F at the endpoints computed from within (a,b) agree with those of f at the endpoints. The details will be given only for the endpoint a. Let $\varepsilon > 0$. There is a $\delta_1 > 0$ such that $|x - a| < \delta_1$ with $x \in H$ implies

$$\left| f(x) - \sum_{j=0}^{k} \frac{(x-a)^j}{j!} f_j(a,H) \right| < \varepsilon |x - a|^k.$$

Let $\delta = \min\{\delta_1, \frac{b-a}{2}\}$. Suppose $0 < x - a < \delta$. Then there is $n \in \mathbb{N}$ with $x \in (2a - x_{n+1}, 2a - x_n]$. By the definition of F and by Lemma 4.2,

$$\begin{aligned}
&\left|F(x) - F(a) - \sum_{j=1}^{k} \frac{(x-a)^j}{j!} f_j(w, H)\right| \\
&= \left|\sum_{i=1}^{k} \alpha_i(x) f(x_{n+i}) + \left(1 - \sum_{i=1}^{k} \alpha_i(x)\right) f(a) - f(a) - \sum_{j=1}^{k} \frac{(x-a)^j}{j!} f_j(a, H)\right| \\
&= \left|\sum_{i=1}^{k} \alpha_i(x) f(x_{n+i}) - \left(\sum_{i=1}^{k} \alpha_i(x)\right) f(a) - \sum_{j=1}^{k} \frac{\sum_{i=1}^{k} \alpha_i(x)(x_{n+i} - a)^j}{j!} f_j(a, H)\right| \\
&= \left|\sum_{i=1}^{k} \alpha_i(x)\big(f(x_{n+i}) - f(a) - \sum_{j=1}^{k} \frac{(x_{n+i} - a)^j}{j!} f_j(a, H)\big)\right| \\
&\leqslant \sum_{i=1}^{k} |\alpha_i(x)| \varepsilon |x_{n+i} - a|^k \leqslant \sum_{i=1}^{k} |\alpha_i(x)| \varepsilon |x - a| \leqslant kK\varepsilon |x - a|^k.
\end{aligned}$$

Now let $w \in H$. We will consider only approach to w from the right. So we may assume that w is not the left endpoint of a component interval of H^c. We quickly dispose of the situation where there is a $w' > w$ with $[w, w'] \subset H$. So assume that w is the limit from the right of a sequence of component intervals of H^c. Let $\varepsilon > 0$. There is a $\delta_1 > 0$ so that $|x - w| < \delta_1$ with $x \in H$ implies

$$\left|f(x) - \sum_{j=0}^{k} \frac{(x-w)^j}{j!} f_j(w, H)\right| < \varepsilon |x - w|^k. \tag{3}$$

Let (c, d) be a component interval of H^c so that $w \leqslant c < d < w + \frac{1}{2}\delta_1$. (Our assumption guarantees that such an interval exists.) Set $\delta = d - w$. Let $x \in (w, w+\delta)$. If $x \in H$, then $|x - w| < \delta_1$. So by equation (3) $|f(x) - \sum_{j=0}^{k} \frac{(x-w)^j}{j!} f_j(w, H)| < \varepsilon |x - w|^k$. Therefore suppose that there is a component interval (a, b) of H^c so that $x \in (a, b)$. By the choice of δ we have that $|x_n - w| < \delta_1$ and $|y_n - w| < \delta_1$ for every $n \in \mathbb{N}$ where $\{x_n\}$ and $\{y_n\}$ are the sequences from Lemma 4.4 that correspond to the interval (a, b). Assume first that $z \in (2a - x_{n+1}, 2a - x_n]$. By the definition of

F and by Lemma 4.3 we have

$$\begin{aligned}
&\left|F(x) - F(w) - \sum_{j=1}^{k} \frac{(x-w)^j}{j!} f_j(w,H)\right| \\
&\quad = \left| \sum_{i=1}^{k} \alpha_i(x) f(x_{n+i}) + \left(1 - \sum_{i=1}^{k} \alpha_i(x)\right) f(a) - f(w) \right. \\
&\qquad \left. - \sum_{j=1}^{k} \frac{\sum_{i=1}^{k} \alpha_i(x)(x-w)^j + \left(1 - \sum_{i=1}^{k} \alpha_i(x)\right)(a-w)^j}{j!} f_j(w,H) \right| \\
&\quad = \left| \sum_{i=1}^{k} \alpha_i(x) \left(f(x_{n+i}) - f(w) - \sum_{j=1}^{k} \frac{(x_{n+i}-w)^j}{j!} f_j(w,H) \right) \right. \\
&\qquad \left. + \left(1 - \sum_{i=1}^{k} \alpha_i(x)\right) \left(f(a) - f(w) - \sum_{j=1}^{k} \frac{(a-w)^j}{j!} f_j(w,H) \right) \right| \\
&\quad \leqslant \sum_{i=1}^{k} |\alpha_i(x)| \varepsilon |x_{n+i} - w|^k + \left(1 + \sum_{i=1}^{k} |\alpha_i(x)|\right) \varepsilon |a-w|^k \\
&\quad \leqslant K\varepsilon \sum_{i=1}^{k} |x_{n+i} - w|^k + (1+kK)\varepsilon |a-w|^k \leqslant (1+2kK)\varepsilon |x-w|^k
\end{aligned}$$

by Lemma 4.4 and since $|x_{n+i} - w| \leqslant |x - w|$.

If $x \in (2a - x_1, \frac{a+b}{2})$, then by essentially the same argument as above, we arrive at the same estimate,

$$\left|F(x) - F(w) - \sum_{j=1}^{k} \frac{(x-w)^j}{j!} f_j(w,H)\right| \leqslant (1+2kK)\varepsilon |x-w|^k.$$

If $x \in [2b - y_n, 2b - y_{n+1})$, then as above but with a replaced by b and with $f(a)$ replaced by $f(b)$, we get

$$|F(x) - F(w) - \sum_{j=1}^{k} \frac{(x-w)^j}{j!} f_j(w,H)| \leqslant K\varepsilon \sum_{i=1}^{k} |y_{n+i} - w|^k + (1+kK)\varepsilon |b-w|^k.$$

Now $|b-w| \leqslant |b-x| + |x-w| \leqslant \frac{b-a}{2} + |x-w| \leqslant 2|x-w|$ and $|y_{n+i} - w| \leqslant |y_{n+i} - b| + |b-w| \leqslant \frac{b-a}{2} + 2|x-w| \leqslant 3|x-w|$. Thus we arrive at the estimate.

$$\left|F(x) - F(w) - \sum_{j=1}^{k} \frac{(x-w)^j}{j!} f_j(w,H)\right| \leqslant \left(1 + kK(3^k + 2^k)\right)\varepsilon |x-w|^k.$$

We get exactly the same estimate in the final case, $x \in (\frac{a+b}{2}, 2b - y_1)$. Hence $F_k(w)$ exists and equals $f(w,H)$ which completes the proof. □

The final step toward proving the main theorem of this section is the next lemma.

Lemma 4.6. *Let $H \subset [0,1]$ be a closed set and F be a function defined on $[0,1]$ so that $F_k(x)$ exists for every $x \in H$. Then there is a function, G, which is k times Peano differentiable at every point $x \in [0,1]$ so that $G_i = F_i$ on H for $i = 0, 1, \ldots, k$.*

Proof. First we replace the given function, F, with one having the same Peano derivatives on H. To this end let (a,b) be a component interval of H^c and let $x_0 = y_0 = \frac{a+b}{2}$. There are sequences $\{x_n\}$ decreasing to a and $\{y_n\}$ increasing to b such that $x_{n-1} - x_n = (x_n - a)^k$ and $y_n - y_{n-1} = (b - y_{n-1})^k$ for each $n \in \mathbb{N}$. (If the component interval is of the form $[0,b)$, then let $y_0 = 0$ and select only the sequence $\{y_n\}$ with a similar adjustment in case the component interval is of the form $(a,1]$.) For each $n \in \mathbb{N}$ let $R(x_n) = F(x_n)$ and $R(y_n) = F(y_n)$. Extend R to be linear on the intervals $[x_n, x_{n-1}]$ and $[y_{n-1}, y_n]$. Define R to be F on H. Now we show that R is k-times Peano differentiable on H with $R_i = F_i$ on H for $i = 0, 1, \ldots, k$. Let $w \in H$ and let $\varepsilon > 0$. There is a $0 < \delta \leqslant \varepsilon$ such that $|x - w| < \delta$ implies $|F(x) - F(w) - \sum_{j=1}^{k} \frac{(w-x)^j}{j!} F_j(w)| < \varepsilon (x-w)^k$. Let $x \in H^c$ with $|x - w| < \delta$. (We need not consider the case $x \in H$.) Then x lies in one of the component intervals, (a,b), of H. Furthermore for some $n \in \mathbb{N}$ x lies in either $[x_n, x_{n-1}]$ or $[y_{n-1}, y_n]$. Let $[c,d]$ denote that interval containing x and note that $|d-c| \leqslant |c-a|^k$ and $|d-c| \leqslant |d-b|^k$. So $|d-w| \leqslant |d-x| + |x-w| \leqslant |d-c| + |x-w| \leqslant |x-w|^k + |x-w| \leqslant 2|x-w|$ and similarly $|c-w| \leqslant 2|x-w|$. For the sake of simplicity set $\alpha = \frac{d-x}{d-c}$ and $\beta = \frac{x-c}{d-c}$. Then $\alpha + \beta = 1$ and $R(x) = \alpha F(c) + \beta F(d)$. Thus

$$\begin{aligned}
\left| R(x) - R(w) - \sum_{j=1}^{k} \frac{(x-w)^j}{j!} F_j(w) \right| &= \left| \alpha \left(F(c) - F(w) - \sum_{j=1}^{k} \frac{(c-w)^j}{j!} F_j(w) \right) \right. \\
&\quad + \beta \left(F(d) - F(w) - \sum_{j=1}^{k} \frac{(d-w)^j}{j!} F_j(w) \right) \\
&\quad \left. + \sum_{j=2}^{k} \frac{\alpha (c-w)^j + \beta (d-w)^j - (x-w)^j}{j!} F_j(w) \right| \\
&\leqslant \alpha \varepsilon |c-w|^k + \beta \varepsilon |d-w|^k + \left| \sum_{j=2}^{k} \frac{\alpha (c-w)^j + \beta (d-w)^j - (x-w)^j}{j!} F_j(w) \right| \\
&\leqslant \alpha \varepsilon 2^k |x-w|^k + \beta \varepsilon 2^k |x-w|^k + \left| \sum_{j=2}^{k} \frac{\alpha (c-w)^j + \beta (d-w)^j - (x-w)^j}{j!} F_j(w) \right|.
\end{aligned}$$

To estimate the last term we note that from the conditions, $|d-c| \leqslant |c-a|^k$ and $|d-c| \leqslant |d-b|^k$, it follows easily that $|d-c| \leqslant |c-w|^k$ and $|d-c| \leqslant |d-w|^k$. Thus

$$\left|\sum_{j=2}^{k} \frac{\alpha(c-w)^j + \beta(d-w)^j - (x-w)^j}{j!} F_j(w)\right|$$

$$= \left|\sum_{j=2}^{k} \frac{\alpha \sum_{i=0}^{j} \binom{j}{i}(c-x)^i(x-w)^{j-i} + \beta \sum_{i=0}^{j} \binom{j}{i}(d-x)^i(x-w)^{j-i} - (x-w)^j}{j!} F_j(w)\right|$$

$$\left|\sum_{j=2}^{k} \frac{\sum_{i=1}^{j} \binom{j}{i}(x-w)^{j-i}\left(\alpha(c-x)^i + \beta(d-x)^i\right)}{j!} F_j(w)\right|$$

$$\leqslant \sum_{j=2}^{k} \frac{\sum_{i=1}^{j} \binom{j}{i}|x-w|^{j-i}\left(\alpha|x-w|^{ik} + \beta|x-w|^{ik}\right)}{j!} |F_j(w)|$$

$$= \sum_{j=2}^{k} \frac{\sum_{i=1}^{j} \binom{j}{i}|x-w|^{j-i+ik}}{j!} |F_j(w)| \leqslant \sum_{j=2}^{k} |x-w|^{k+1} \frac{\sum_{i=1}^{j} \binom{j}{i}}{j!} |F_j(w)|$$

$$\leqslant \sum_{j=2}^{k} \frac{2^j}{j!} |x-w|^{k+1} |F_j(w)|.$$

Therefore

$$\left|R(x) - R(w) - \sum_{j=1}^{k} \frac{(x-w)^j}{j!} F_j(w)\right| \leqslant \left(\varepsilon 2^k + \sum_{j=2}^{k} \frac{2^j}{j!}\varepsilon\right) |x-w|^k.$$

From this estimate it is obvious that R is k-times Peano differentiable on H with $R_i = F_i$ on H for $i = 0, 1, \ldots, k$. In fact R is k times Peano differentiable everywhere except possibly at the endpoints of the subintervals $[x_n, x_{n-1}]$ and $[y_{n-1}, y_n]$ of a component interval, (a, b), of H^c. Let $[c, d]$ and $[d, e]$ be two such adjacent intervals. Let $z_1 = d - \frac{1}{2}\min\{d-c, e-d\}$ and $z_2 = d + \frac{1}{2}\min\{d-c, e-d\}$. If the graph of R on $[c, e]$ is not a line segment, then there is a function $\tilde{G}$ defined on $[z_1, z_2]$ so that $\tilde{G}$ is infinitely differentiable on $[z_1, z_2]$, $\tilde{G}^{(i)}(z_1) = R_i(z_1)$ and $\tilde{G}^{(i)}(z_2) = R_i(z_2)$ for $i = 0, 1, \ldots, k$ and such that the graph of $\tilde{G}$ lies in the triangle with vertices $(z_1, R(z_1))$, $(d, R(d))$ and $(z_2, R(z_2))$. If the graph of R on $[c, e]$ is a line segment, then R is already infinitely differentiable so in this case let $\tilde{G} = R$. Now define G on the entire interval (a, b) to be $\tilde{G}$ on each of the intervals of the form $[z_1, z_2]$ and to be R elsewhere on (a, b). Having now defined G on H^c we define G to be R on H.

We now show that G satisfies the conclusion of the theorem. By definition G is k-times differentiable in the usual sense at each $w \in H^c$. So let $w \in H$. We need only consider an x in an interval of the form $[z_1, z_2]$ in one of the component intervals, (a, b), of H^c. First note that $|z_2 - z_1| \leqslant |z_1 - a|^k$ and $|z_2 - z_1| \leqslant |z_2 - b|^k$. For the estimate to come, we need the following inequalities. Let $z \in [z_1, z_2]$. Then proceeding as above $|z - w| \leqslant 2|x - w|$ and $|z - x| \leqslant (z_2 - z_1) \leqslant |x - w|^k$. As in the proof of Theorem 2.1, there are $\alpha, \beta, \gamma \in [0, 1]$ with $\alpha + \beta + \gamma = 1$ such that $x = \alpha z_1 + \beta d + \gamma z_2$ and $G(x) = \alpha R(z_1) + \beta R(d) + \gamma R(z_2)$. Thus employing estimates similar to those used in the previous argument

$$
\begin{aligned}
&|G(x) - G(w) - \sum_{j=1}^{k} \frac{(z-w)^j}{j!} R_j(w)| \leqslant \alpha \left| R(z_1) - R(w) - \sum_{j=1}^{k} \frac{(z_1 - w)^j}{j!} R_j(w) \right| \\
&+\beta \left| R(d) - R(w) - \sum_{j=1}^{k} \frac{(d-w)^j}{j!} R_j(w) \right| + \gamma \left| R(z_2) - R(w) - \sum_{j=1}^{k} \frac{(z_2 - w)^j}{j!} R_j(w) \right| \\
&+\left| \sum_{j=2}^{k} \frac{\sum_{i=1}^{j} \binom{j}{i} (z-w)^{j-i} \left(\alpha (z_1 - x)^i + \beta (d-x)^i + \gamma (z_2 - x)^i \right)}{j!} R_j(w) \right| \\
&\leqslant \alpha \left| R(z_1) - R(w) - \sum_{j=1}^{k} \frac{(z_1 - w)^j}{j!} R_j(w) \right| + \beta \left| R(d) - R(w) - \sum_{j=1}^{k} \frac{(d-w)^j}{j!} R_j(w) \right| \\
&+\gamma \left| R(z_2) - R(w) - \sum_{j=1}^{k} \frac{(z_2 - w)^j}{j!} R_j(w) \right| + \sum_{j=2}^{k} \frac{2^j}{j!} |x - w|^{k+1} |R_j(w)|.
\end{aligned}
$$

From this and the inequalities mentioned above it is obvious that $G_i(w)$ exists and equals $R_i(w)$ for $i = 0, 1, \ldots, k$ which completes the proof. □

Combining Theorem 4.5 and Lemma 4.6 we get the main theorem of this section.

Theorem 4.7. *Let $H \subset [0, 1]$ be a perfect set having finite Denjoy index, Let $k \in \mathbb{N} \setminus \{1\}$ let f be a function defined on H so that $f_k(x, H)$ exists for every $x \in H$. Then there is a k-times Peano differentiable function, $F\colon [0, 1] \to \mathbb{R}$, so that $F_i(x) = f_i(x, H)$ for every $x \in H$ and $i = 0, 1, \ldots, k$. In other words $H \in P_k$.*

We end this article with an application of Theorem 4.7.

Corollary 4.8. *Suppose that the assumptions of Theorem 4.7 hold. Let $S \subset H$ be closed. Then there is an interval I so that $\emptyset \neq S \cap I$ and that for every $0 \leqslant s \leqslant k$ $(f_s)_{(p-s)}(x, S \cap I)$ exists for every $x \in S \cap I$ and $(f_s)_{(p-s)}(x, S \cap I) = f_k(x, H)$ for $p = 0, 1, \ldots, k - s$.*

Proof. The assertion follows directly from Theorem 4.7 and a generalization of Theorem 2 in [5] which is Theorem 1.1.20 and can be found in the Ph. D. dissertation of the first autho □

Corollary 4.8 is a generalization of a result due to Denjoy. (See Theorem 2 in [3].)

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Authors' addresses: *Hajrudin Fejzić*, Mathematics Department, California State University, San Bernardino, CA 92407-2397, USA; *Jan Mařík*, *Clifford E. Weil*, Mathematics Department, Michigan State University, East Lansing, MI 48824-1027, USA.