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A CERTAIN TYPE OF PARTIAL DIFFERENTIAL EQUATIONS ON TORI

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Summary. The existence of classical solutions for some partial differential equations on tori is shown.

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1. INTRODUCTION

The purpose of this paper is to show the existence of C^2-smooth solutions for the singularly perturbed equation

$$u_{yy} + \varepsilon u_{xx} = \varepsilon f(u, y, x), \tag{1}$$

where u is 2π-periodic in x and y, $f \in C^\infty(\mathbf{R} \times \mathbf{R} \times \mathbf{R}, \mathbf{R})$ is 2π-periodic in x and y, $\varepsilon > 0$ is a small parameter. We shall show that (1) possesses a solution provided f is globally Lipschitz in u uniformly for y, x with a Lipschitz constant $K < 1$ and a certain ordinary differential equation has a 2π-periodic solution. We conclude this paper with a discussion of the equations (1) when f is independent on y. We also show a geometric interpretation of this special case.

Singularly perturbed equations on tori have been studied by several authors [2], [3], [4]. Usually they have used the approach via the Nash-Moser implicit function theorem. We will use only the Banach fixed point theorem.

2. Main results

Theorem 2.1. *If there is a constant K, $0 < K < 1$ such that*

$$|f(u_1,\cdot,\cdot) - f(u_2,\cdot,\cdot)| \leqslant K \cdot |u_1 - u_2| \tag{+}$$

for all $u_1, u_2 \in \mathbf{R}$, then (1) *has a solution u_ε for each small $\varepsilon > 0$ having the form*

$$u_\varepsilon(x,y) = \overline{v}(x) + O(\varepsilon)$$

where $\overline{v}$ is a stable (see $(-)$ in the proof of this theorem) 2π-periodic solution of the equation

$$v'' = \frac{1}{2\pi}\int_0^{2\pi} f(v,s,x)\,\mathrm{d}s. \tag{2}$$

Proof. First of all, we investigate the equation (2). Let

$$H = \left\{ v\colon \mathbf{R} \to \mathbf{R}, v \text{ is } 2\pi\text{-periodic}, \ 2\pi\|v\|^2 = \int_0^{2\pi} v^2(s)\,\mathrm{d}s < \infty \right\}.$$

It is well-known that H is a Hilbert space with the basis

$$\{\sin nt, \cos mt\}_{n\geqslant 1, m\geqslant 0}.$$

Lemma 2.2. *The equation*

$$v'' = g, \quad g \in H, \quad \int_0^{2\pi} g(s)\,\mathrm{d}s = 0$$

has a unique solution $v(g)$ in H such that $\int_0^{2\pi} v = 0$ and $\|v\| \leqslant \|g\|$.

Proof of Lemma 2.2. If $g = \sum_{i=1}^{\infty} a_i \cdot \sin it + b_i \cdot \cos it$ then

$$v = -\sum_{i=1}^{\infty} (a_i \cdot \sin it + b_i \cdot \cos it)/i^2.$$

□

We put $S(g) = v(g)$, $F(g) = \frac{1}{2\pi}\int_0^{2\pi} f(g,s,x)\,\mathrm{d}s$ and $Pg = \frac{1}{2\pi}\int_0^{2\pi} g(s)\,\mathrm{d}s$. Then (2) has the form

$$(3) \qquad \begin{aligned} s &= S(I-P)\cdot F(s+t),\\ 0 &= PF(s+t), \end{aligned}$$

where $s \in \operatorname{Ker} P$, $t \in \operatorname{Im} P \cong \mathbf{R}$. Since f has the property (+) we have

$$\left\|S(I-P)\big(F(s_1+t)-F(s_2+t)\big)\right\| \leqslant K\cdot \|s_1-s_2\|$$

for all $s_1, s_2 \in \operatorname{Ker} P$. Using the Banach fixed point theorem we can solve the first equation of (3) for each t. We insert this solution $s(t)$ into the second equation of (3) obtaining

$$(4) \qquad 0 = PF\big(s(t)+t\big).$$

We see that each solution of (4) determines a unique solution of (2). If a zero of (4) is simple then we say that the solution of (2) determined by this zero is the stable solution of (2) (–).

Without loss of generality we can assume that $\overline{v} \equiv 0$, i.e. $t=0$, $s(0)=0$. We denote

$$X = \Big\{u\colon \mathbf{R}\times\mathbf{R}\to\mathbf{R},\ u \text{ is } 2\pi\text{-periodic in } x,y,\\ \|u\| = \frac{1}{2\pi}\sqrt{\int_0^{2\pi}\int_0^{2\pi} u^2(x,y)\,\mathrm{d}x\,\mathrm{d}y} < \infty\Big\}.$$

X is a Hilbert space with the basis

$$\{\sin mt\cdot\sin i\tilde{t}, \sin mt\cdot\cos i\tilde{t}, \cos mt\cdot\cos i\tilde{t}, \cos mt\cdot\sin i\tilde{t}\}$$

Lemma 2.3. *The equation*

$$w_{yy}+\varepsilon w_{xx} = g, \quad g \in X, \quad \int_0^{2\pi} g(x,\cdot)\,\mathrm{d}x = 0$$

has a unique solution $w_g \in X$ satisfying $\int_0^{2\pi} w_g(x,\cdot)\,\mathrm{d}x = 0$. Moreover,

$$\|w_g\| \leqslant \|g\|\cdot\frac{1}{\varepsilon}.$$

Proof. The proof is the same as that of Lemma 2.2. □

We put

$$T_\varepsilon(g) = w_g, \quad \tilde{R}g = \frac{1}{2\pi}\int_0^{2\pi} g(x,y)\,\mathrm{d}x, \quad G(g) = f(g,\cdot,\cdot).$$

Then (1) has the form

$$\begin{aligned}
w &= \varepsilon \cdot T_\varepsilon \cdot (I - \tilde{R}) \cdot G(w + v + t), \\
v &= \varepsilon \cdot S \cdot (I - P) \cdot \tilde{R} \cdot G(w + v + t), \\
0 &= P \cdot \tilde{R} \cdot G(w + v + t),
\end{aligned} \tag{5}$$

where $w \in \operatorname{Ker} \tilde{R}$, $v \in \operatorname{Im} \tilde{R} \cap \operatorname{Ker} P$, $t \in \operatorname{Im} P \cong R$. We note that v is independent on x since $\operatorname{Im} \tilde{R} \subset H$. By (+), Lemma 2.2, Lemma 2.3 we see that the mapping

$$(w,v) \to \big(\varepsilon \cdot T_\varepsilon \cdot (I - \tilde{R}) \cdot G(w + v + t), \varepsilon \cdot S \cdot (I - P)\tilde{R} \cdot G(w + v + t)\big)$$

defined on $\operatorname{Ker} \tilde{R} \times \operatorname{Ker} P$ with the norm $\|.\| + \|.\|$ is Lipschitz with a Lipschitz constant K_1, $K < K_1 < 1$ for $\varepsilon > 0$ small, $t \in \mathbf{R}$.

Thus the first two equations of (5) have unique solutions $w_\varepsilon(t)$, $v_\varepsilon(t)$ for each $t \in \mathbf{R}$, $\varepsilon > 0$ small, and $\|w_\varepsilon(t)\|$, $\|v_\varepsilon(t)\|$ are bounded on each bounded subset of $\mathbf{R}$. Using these estimates and the Sobolev imbedding theorem we see that $w_\varepsilon(t)$, $v_\varepsilon(t) \in C^3$ and $|w_\varepsilon(t)|_{C^3}$, $|v_\varepsilon(t)|_{C^3}$, are uniformly bounded for $\varepsilon > 0$ small, $|t| \leqslant 1$. We take a sequence $\varepsilon_i \to 0$, $\varepsilon_i > 0$, $t_i \to t$, $|t_i| \leqslant 1$, ε_i small. Then by the Arzela-Ascoli theorem, $\{w_{\varepsilon_i}(t_i), v_{\varepsilon_i}(t_i)\}_0^\infty$ has a subsequence tending to $(\overline{w}, \overline{v})$ in C^2.

On the other hand, (5) implies

$$\begin{aligned}
v_{yy} &= \frac{\varepsilon}{2\pi}\int_0^{2\pi} f(w + v + t, y, x)\,\mathrm{d}x, \\
w_{yy} + \varepsilon w_{xx} &= \varepsilon\Big(f(w + v + t, y, x) - \frac{1}{2\pi}\int_0^{2\pi} f(w + v + t, y, x)\,\mathrm{d}x\Big).
\end{aligned}$$

It follows that $\overline{v} \equiv 0$, $\overline{w}$ is independent on y, $\overline{w} = \overline{w}(x)$ satisfies

$$\overline{w}'' = \frac{1}{2\pi}\int_0^{2\pi} f(\overline{w} + t, y, x)\,\mathrm{d}y - \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} f(\overline{w} + t, y, x)\,\mathrm{d}y\,\mathrm{d}x.$$

However, this equation is precisely the first equation of (3) and thus $\overline{w} \approx s(t)$. This implies

$$\lim_{\varepsilon \to 0_+} w_\varepsilon(t) = s(t), \quad \lim_{\varepsilon \to 0_+} v_\varepsilon(t) = 0$$

in the space C^2. Hence for $\varepsilon > 0$ small the last equation of (5) is C^1-close to the equation

$$0 = P \cdot \tilde{R} \cdot G(\overline{w} + t) = P \cdot F\big(s(t) + t\big)$$

on the interval $\langle -\frac{1}{2}, \frac{1}{2} \rangle \subset \langle -1, 1 \rangle$. But we know that $P \cdot F\big(s(0) + 0\big) = 0$ and this root is simple. Thus the equation

$$0 = P \cdot \tilde{R} \cdot G\big(w_\varepsilon(t) + v_\varepsilon(t) + t\big)$$

has a solution on $\langle -\frac{1}{2}, \frac{1}{2} \rangle$ for $\varepsilon > 0$ small tending to 0 as $\varepsilon \to 0$. This completes the proof. □

It is clear that we can repeat the above proof if f depends smoothly also on ε, i.e. $f = f(u, y, x, \varepsilon)$.

Remark 2.4. Since a small smooth perturbation of a function having a simple root also has a simple root, it is not difficult to see that each stable solution $\overline{v}$ of (2) has the following property: Each 2π-periodic (smooth) perturbation of (2) possesses a 2π-periodic solution near $\overline{v}$.

Finally, let $f(u, y, x, 0) = g(u)$ and $g(c) = 0$, $g'(c) \neq 0$, $|g'(.)| < 1$. Then the equation (3) has the form

$$s'' = g(s + t) - \frac{1}{2\pi} \int_0^{2\pi} g\big(s(u) + t\big)\, du,$$
$$0 = \frac{1}{2\pi} \int_0^{2\pi} g\big(s(u) + t\big)\, du.$$

We see that the first equation has a unique solution $s \equiv 0$ for each $t \in \mathbf{R}$ and thus (4) has the form

$$0 = g(t).$$

Since $g(c) = 0$, $g'(c) \neq 0$, the trivial solution $u \equiv c$ of $u'' = g(u)$ is stable.

3. A special case

In this section we assume that f is independent on y, i.e. we investigate the equation

$$\frac{1}{\varepsilon} u_{yy} + u_{xx} = f(u, x) \tag{6}$$

on the torus $S^1 \times S^1$.

We suppose that there is a $K > 0$ satisfying

$$\left|\frac{\partial f}{\partial u}(\cdot,\cdot)\right| < K.$$

The operator $A_\varepsilon : \mathrm{Dom}(A_\varepsilon) \subset X \to X$,

$$A_\varepsilon u = \frac{1}{\varepsilon} u_{yy} + u_{xx},$$

has the invariant subspace

$$H_1 = \mathrm{span}\{\sin mx, \cos mx\}.$$

Further,

$$H_1 \oplus H_2 = X,$$
$$H_2 = \mathrm{span}\{\sin my \cdot \cos jx, \sin my \cdot \sin jx, \cos my \cdot \cos jx, \cos my \cdot \sin jx\}_j^{m \geqslant 1}.$$

Hence the spectrum of A_ε/H_2 is

$$\left\{-\frac{1}{\varepsilon} m^2 - j^2\right\}_{m \geqslant 1} = \sigma(A_\varepsilon/H_2).$$

On the other hand, if $F(u) = f(u, \cdot)$ then

$$F(H_1) \subset H_1, \quad \|F(u_1) - F(u_2)\| \leqslant K \cdot \|u_1 - u_2\|.$$

Summing up we obtain $\sigma(A_\varepsilon/H_2) \cap (-K, K) = \emptyset$ for $\varepsilon > 0$ small.

Thus applying Theorem 2 from [1] we obtain

Theorem 3.5. *For $\varepsilon > 0$ small each 2π-periodic solution of (6) is independent on y.*

Finally, Theorem 3.5 has the following simple geometric interpretation: Consider the equation

$$u_{yy} + u_{xx} = f(u, x) \tag{7}$$

on the torus $M_\varepsilon = S^1 \times \{z \in \mathbf{R}^2, |z| = \varepsilon\}$ $(x \in S^1)$. Then by using a suitable scaling of variables (7) can be transformed into (6). Hence for $\varepsilon > 0$ small the equation (7) has only C^2-solutions on M_ε which are independent on y. Of course, provided they exist.

Remark 3.6. Similarly we can study the following problem: Let us consider the system of equations

$$E_x^p u_p + \varepsilon E_y^p u_p = \varepsilon f_p(x, y, u_1, \ldots, u_m), \quad p = 1, \ldots, m$$

where $(x, y) \in T^{\overline{m}} \times T^{\overline{m}}$, $u_p = u_p(x, y) \in \mathbf{R}$, ε is a small nonnegative parameter, $E_x^p = E^p$, $E_y^p = E^p$, E^p is a strongly elliptic operator on the $\overline{m}$-dimensional torus $T^{\overline{m}} = S^1 \times \ldots \times S^1$, i.e.

$$E^p u = \sum_{i,j} \frac{\partial}{\partial z_i}\Big(a_{i,j}^p(z)\frac{\partial}{\partial z_j}u\Big),$$

where $a_{i,j}^p$ are 2π-periodic in all coordinates of z and the matrices $\{a_{i,j}^p(.)\}$ are symmetric positive definite. Further, f_p are 2π-periodic in (x, y) and globally Lipschitz in $u = (u_1, \ldots, u_m)$ with a Lipschitz constant K_p i.e.

$$|f_p(\cdot,\cdot,u_1^1,\ldots,u_m^1) - f_p(\cdot,\cdot,u_1^2,\ldots,u_m^2)| \leqslant K_p\sqrt{(u_1^1 - u_1^2)^2 + \ldots + (u_m^1 - u_m^2)^2}.$$

Let Λ_p be the first nonzero eigenvalue of E^p. We assume

$$\sum_{p=1}^{m}(K_p/\Lambda_p)^2 < 1.$$

Then following the above procedure we obtain: The above mentioned equation has a solution u in the form $u_p = v_p + O(\varepsilon)$, $p = 1, \ldots, m$, for each $\varepsilon \geqslant 0$ small where $v = (v_1, \ldots, v_m)$ is a stable solution of

$$E_y^p v_p = \frac{1}{(2\pi)^m}\int_0^{2\pi}\cdots\int_0^{2\pi} f_p(x, y, v_1, \ldots, v_m)\,\mathrm{d}x.$$

The stability of v means that under a small perturbation of the right hand side of this equation there always exists a unique solution near v.

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Souhrn

URČITÝ TYP PARCIÁLNYCH DIFERENCIÁLNYCH ROVNÍC NA TÓROCH

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V práci sa študujú špeciálne parciálne diferenciálne rovnice na tóroch, pričom sa dokazuje existencia ich klasického riešenia.

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