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PERIODIC SOLUTIONS OF NONLINEAR SECOND-ORDER DIFFERENTIAL EQUATIONS WITH PARAMETER

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Summary. This paper establishes effective sufficient conditions for existence and uniqueness of periodic solutions of a one-parameter differential equation $y'' - q(t)y = f(t, y, y', \mu)$ vanishing at an arbitrary but fixed point.

Keywords: Periodic solution, nonlinear second-order differential equation with a parameter, Schauder fixed point theorem

AMS classification: 34C25, 34B15

1. INTRODUCTION

In this paper we shall consider the second-order differential equation

$$y'' - q(t)y = f(t, y, y', \mu) \tag{1}$$

with $q \in C^0(\mathbf{R})$, $f \in C^0(\mathbf{R}^3 \times I)$ ω-periodic functions in the variable t, $q(t) > 0$ for $t \in \mathbf{R}$, where $I = \langle a, b\rangle$, $-\infty < a < b < \infty$, containing a parameter μ. Let $t_1 \in \mathbf{R}$ be an arbitrary but fixed number. The problem considered is to determine sufficient conditions on q, f such that it is possible to choose the parameter μ so that there exists an ω-periodic solution y of (1) satisfying

$$y(t_1) = 0. \tag{2}$$

Similarly, the problem of uniqueness of ω-periodic solutions of (1) satisfying (2) is discussed.

2. Notation, preliminary results

Let u, v be solutions of the differential equation

$$(q) \qquad y'' = q(t)y \qquad (q \in C^0(\mathbf{R}), q(t+\omega) = q(t) > 0 \text{ for } t \in \mathbf{R})$$

satisfying the initial conditions $u(t_1) = 0$, $u'(t_1) = 1$, $v(t_1) = 1$, $v'(t_1) = 0$, where $t_1 \in \mathbf{R}$ is an arbitrary but fixed number. Define functions $r\colon \mathbf{R}^2 \to \mathbf{R}$ and $r_1'\colon \mathbf{R}^2 \to \mathbf{R}$ by $r(t,s) := u(t)v(s) - u(s)v(t)$ and $r_1'(t,s) := u'(t)v(s) - u(s)v'(t)$ $(= \frac{\partial r}{\partial t}(t,s))$.

Lemma 1 ([2]). *$r(t,s) > 0$ for $t > s$, $r(t,s) < 0$ for $t < s$, $r_1'(t,s) > 1$ for $t \neq s$ and $r_1'(t,t) = 1$ for $t \in \mathbf{R}$.*

Lemma 2. *Let a function $k\colon \langle t_1, t_1+\omega\rangle \to \mathbf{R}$ be defined by*

$$(3) \qquad k(t) = \frac{r(t_1+\omega, t)}{r(t_1, t_1+\omega)}\left[r_1'(t_1+\omega, t_1) - 1\right] + r_1'(t_1+\omega, t).$$

Then

$$k(t) > 0 \text{ for } t \in \langle t_1, t_1+\omega\rangle.$$

Proof. We may write the function k in the form

$$k(t) = -\frac{1}{u(t_1+\omega)}\big(u'(t_1+\omega) - 1\big)\big(u(t_1+\omega)v(t) - u(t)v(t_1+\omega)\big) + \big(u'(t_1+\omega)v(t) - u(t)v'(t_1+\omega)\big)$$

and then

$$k'(t) = -\frac{1}{u(t_1+\omega)}\big(u'(t_1+\omega) - 1\big)\big(u(t_1+\omega)v'(t) - u'(t)v(t_1+\omega)\big) + \big(u'(t_1+\omega)v'(t) - u'(t)v'(t_1+\omega)\big).$$

Assume to the contrary that $k(\xi) = 0$ for some ξ, $\xi \in (t_1, t_1+\omega)$. If this ξ is unique then $k'(\xi) = 0$ since $k(t_1) = k(t_1+\omega) = 1$. It is easily verified that $k(\xi) = 0$ $(k'(\xi) = 0)$ if and only if

$$\frac{u(\xi)}{v(\xi)} = \frac{u(t_1+\omega)}{v(t_1+\omega) - 1} \qquad \left(\frac{u'(\xi)}{v'(\xi)} = \frac{u(t_1+\omega)}{v(t_1+\omega) - 1}\right).$$

It follows from the equality $\left(\frac{u}{v}\right)' = \frac{1}{v^2}$ that $\frac{u}{v}$ is an increasing function on $\langle t_1, t_1+\omega\rangle$ and, consequently, there exists a unique ξ with above property. Then necessarily

$$\frac{u(\xi)}{v(\xi)} = \frac{u'(\xi)}{v'(\xi)} \qquad \left(= \frac{u(t_1+\omega)}{v(t_1+\omega) - 1}\right),$$

which contradicts $u'v - uv' = 1$. □

Lemma 3. *Let $d \in \mathbf{R}$, $h \in C^0(\mathbf{R})$. Then there exists a unique solution y of the differential equation*

$$y'' - q(t)y = h(t) \tag{4}$$

satisfying the boundary value conditions

$$y(t_1) = y(t_1 + \omega) = d. \tag{5}$$

This solution y can be written in the form

$$\begin{aligned} y(t) = \frac{1}{r(t_1, t_1+\omega)} \Big[d\big(r(t_1, t_1+\omega) - r(t, t_1)\big) \\ + r(t, t_1) \int_{t_1}^{t_1+\omega} r(t_1+\omega, s) h(s) \, \mathrm{d}s \Big] + \int_{t_1}^{t} r(t, s) h(s) \, \mathrm{d}s, \quad t \in \mathbf{R}. \end{aligned} \tag{6}$$

Proof. One can easily and immediately check that the function y defined by (6) is a solution of (4) satisfying (5). The uniqueness follows from the fact that the associated homogeneous boundary value problem: (q), $y(t_1) = y(t_1 + \omega) = 0$ has only the trivial solution. □

Let r_0, r_1 be positive constants, $r_0 > 0$, $r_1 > 0$. Now we shall assume that q, f satisfy some of the following assumptions:

$$\begin{cases} 2\sqrt{r_0}\sqrt{A + r_0 \max_{t \in \mathbf{R}} q(t)} \leqslant r_1, \text{ where } A := \max_{(t, y_1, y_2, \mu) \in D} |f(t, y_1, y_2, \mu)|, \\ D := \langle 0, \omega \rangle \times \langle -r_0, r_0 \rangle \times \langle -r_1, r_1 \rangle \times I; \end{cases} \tag{7}$$

$$|f(t, y_1, y_2, \mu)| \leqslant r_0 q(t) \quad \text{for } (t, y_1, y_2, \mu) \in D; \tag{8}$$

$$\begin{cases} f(t, y_1, y_2, \cdot) \quad \text{is an increasing function on } I \text{ for every} \\ \text{fixed } (t, y_1, y_2) \in \langle 0, \omega \rangle \times \langle -r_0, r_0 \rangle \times \langle -r_1, r_1 \rangle =: D_1; \end{cases} \tag{9}$$

$$f(t, y_1, y_2, a) f(t, y_1, y_2, b) \leqslant 0 \quad \text{for } (t, y_1, y_2) \in D_1. \tag{10}$$

Lemma 4. *Suppose that assumptions (7)–(10) hold for positive constants r_0, r_1. Let $\varphi \in C^1(\mathbf{R})$ be an ω-periodic function, $|\varphi^{(i)}(t)| \leqslant r_i$ for $t \in \mathbf{R}$, $i = 0, 1$. Then there exists a unique μ_0, $\mu_0 \in I$ such that the differential equation*

$$y'' - q(t)y = f\big(t, \varphi(t), \varphi'(t), \mu\big) \tag{11}$$

with $\mu = \mu_0$ has an ω-periodic solution y satisfying (2). This solution y is unique and

$$|y^{(i)}(t)| \leqslant r_i \text{ for } t \in \mathbf{R}, \ i = 0, 1. \tag{12}$$

Proof. If we set $h(t,\mu) := f(t,\varphi(t),\varphi'(t),\mu)$ for $(t,\mu) \in \mathbf{R} \times I$, then h is ω-periodic in t and assumptions (7)–(10) yield $|h(t,\mu)| \leqslant A$ for $(t,\mu) \in \mathbf{R} \times I$, $h(t,\cdot)$ is an increasing function on I for every fixed $t \in \mathbf{R}$ and $h(t,a) \leqslant 0$, $h(t,b) \geqslant 0$ on $\mathbf{R}$. Using the definition of h we can write (11) in the form

$$y'' - q(t)y = h(t,\mu). \tag{13}$$

Let $y(t,\mu)$ be a solution of (13), $y(t_1,\mu) = y(t_1+\omega,\mu) = 0$. Then (by Lemma 3)

$$y(t,\mu) = \frac{r(t,t_1)}{r(t_1,t_1+\omega)} \int_{t_1}^{t_1+\omega} r(t_1+\omega,s)h(s,\mu)\,\mathrm{d}s + \int_{t_1}^{t} r(t,s)h(s,\mu)\,\mathrm{d}s$$

and

$$\Big(\frac{\partial y}{\partial t}(t,\mu) =: \Big)\quad y'(t,\mu) = \frac{r_1'(t,t_1)}{r(t_1,t_1+\omega)} \int_{t_1}^{t_1+\omega} r(t_1+\omega,s)h(s,\mu)\,\mathrm{d}s + \int_{t_1}^{t} r_1'(t,s)h(s,\mu)\,\mathrm{d}s$$

thus

$$y'(t_1+\omega,\mu) - y'(t_1,\mu) = \frac{1}{r(t_1,t_1+\omega)}(r_1'(t_1+\omega,t_1)-1) \int_{t_1}^{t_1+\omega} r(t_1+\omega,s)h(s,\mu)\,\mathrm{d}s + \int_{t_1}^{t_1+\omega} r_1'(t_1+\omega,s)h(s,\mu)\,\mathrm{d}s = \int_{t_1}^{t_1+\omega} k(s)h(s,\mu)\,\mathrm{d}s,$$

where k is defined by (3). It follows from Lemma 2 that $k(t) > 0$ on $\langle t_1, t_1+\omega\rangle$ and therefore $g(\mu) := y'(t_1+\omega,\mu) - y'(t_1,\mu)$ is increasing on I, $g(a) \leqslant 0$, $g(b) \geqslant 0$. Then there evidently exists a unique μ_0, $\mu_0 \in I$: $g(\mu_0) = 0$. This proves that equation (11) with $\mu = \mu_0$ has solution y satisfying $y^{(i)}(t_1) - y^{(i)}(t_1+\omega) = 0$ $(i = 0, 1)$, that is, y is an ω-periodic solution of (11) with $\mu = \mu_0$.

It remains to prove (12). Since $y(t_1) = y(t_1+\omega) = 0$ there exists a ξ, $\xi \in (t_1, t_1+\omega)$: $|y(t)| \leqslant |y(\xi)|$ for $t \in \langle t_1, t_1+\omega\rangle$. Then $y'(\xi) = 0$ and y has at $t = \xi$ an absolute extreme on $\langle t_1, t_1+\omega\rangle$. Let $|y(\xi)| > r_0$. If $y(\xi) > r_0$ $\big(y(\xi) < -r_0\big)$ we get $y''(\xi) > 0$ $\big(y''(\xi) < 0\big)$ by assumption (8). This, however, contradicts the fact that y has absolute maximum (minimum) at the point $t = \xi$. Hence $|y(\xi)| \leqslant r_0$ and $|y(t) \leqslant r_0$ on $\mathbf{R}$.

Integrating the equality

$$2y''(t)y'(t) = 2q(t)y(t)y'(t) + 2h(t,\mu_0)y'(t), \quad t \in \mathbf{R},$$

from η to T, where η, $T \in \langle t_1, t_1+\omega\rangle$, $y'(\eta) = 0$, $y'(t) \neq 0$ on the open interval J with the end points η and T, we obtain

$$\begin{aligned} y'^2(T) &= 2\int_\eta^T q(t)y(t)y'(t)\,\mathrm{d}t + 2\int_\eta^T h(t,\mu_0)y'(t)\,\mathrm{d}t \\ &= 2\int_\eta^T q(t)y(t)y'(t)\,\mathrm{d}t + 2\int_{y(\eta)}^{y(t)} h(y^{-1}(t),\mu_0)\,\mathrm{d}t, \end{aligned}$$

where y^{-1} denotes the inverse function to y on J. Then

$$\begin{aligned} y'^2(T) &\leqslant 2r_0 \max_{t\in\mathbf{R}} q(t) \left| \int_\eta^T y'(t)\,\mathrm{d}t \right| + 2A \left| \int_{y(\eta)}^{y(T)} \mathrm{d}t \right| \\ &\leqslant 2r_0 \max_{t\in\mathbf{R}} q(t)|y(T) - y(\eta)| + 2A|y(T) - y(\eta)| \\ &\leqslant 4r_0^2 \max_{t\in\mathbf{R}} q(t) + 4Ar_0, \end{aligned}$$

consequently

$$|y'(T)| \leqslant 2\sqrt{r_0}\sqrt{A + r_0 \max_{t\in\mathbf{R}} q(t)} \leqslant r_1$$

and

$$|y'(t)| \leqslant r_1 \text{ for } t \in \mathbf{R}.$$

The uniqueness of the ω-periodic solution y of equation (13) with $\mu = \mu_0$ follows from the fact that the associated homogeneous equation $y'' - q(t)y = 0$ to equation (13) has only the trivial ω-periodic solution satisfying (2). □

3. Results

Theorem 1. *Assume that assumptions* (7)–(10) *hold for positive constants* r_0, r_1. *Then there exists* μ_0, $\mu_0 \in I$ *such that equation* (1) *with* $\mu = \mu_0$ *has an* ω*-periodic solution* y *satisfying* (2) *and* (12).

Proof. Let X be the Banach space of ω-periodic C^1-functions on $\mathbf{R}$ with the norm $\|y\| = \max_{t\in\mathbf{R}}(|y(t)| + |y'(t)|)$ for $y \in X$ and let $K := \{y\colon y \in X\colon y(t_1) = 0, |y^{(i)}(t)| \leqslant r_i \text{ for } t \in \mathbf{R}, i = 0,1\}$. K is a closed bounded convex subset of X, $K \subset X$. Let $\varphi \in K$. By Lemma 4 there exists a unique μ_0, $\mu_0 \in I$ such that equation (11) with $\mu = \mu_0$ has a unique ω-periodic solution y satisfying (2) and (12), and thus $y \in K$. We may write this solution y in the form

$$y(t) = \frac{r(t,t_1)}{r(t_1,t_1+\omega)} \int_{t_1}^{t_1+\omega} r(t_1+\omega,s) f\big(s,\varphi(s),\varphi'(s),\mu_0\big)\,ds + \int_{t_1}^{t} r(t,s) f\big(s,\varphi(s),\varphi'(s),\mu_0\big)\,ds, \quad t \in \mathbf{R},$$

by Lemma 3. Setting $T(\varphi) = y$ we obtain an operator $T\colon K \to K$. We will prove that T is a completely continuous operator.

Let $\{y_n\}$, $y_n \in K$ be a convergent sequence, $\lim_{n\to\infty} y_n = y$ and $z_n = T(y_n)$, $z = T(y)$. Then there exists $\{\mu_n\}$, $\mu_n \in I$ and $\mu_0 \in I$ such that

$$(14) \qquad z_n(t) = \frac{r(t,t_1)}{r(t_1,t_1+\omega)} \int_{t_1}^{t_1+\omega} r(t_1+\omega,s) f\big(s,y_n(s),y_n'(s),\mu_n\big)\,ds + \int_{t_1}^{t} r(t,s) f\big(s,y_n(s),y_n'(s),\mu_n\big)\,ds, \quad t \in \mathbf{R},$$

and

$$z(t) = \frac{r(t,t_1)}{r(t_1,t_1+\omega)} \int_{t_1}^{t_1+\omega} r(t_1+\omega,s) f\big(s,y(s),y'(s)\ \mu_0\big)\,ds + \int_{t_1}^{t} r(t,s) f\big(s,y(s),y'(s),\mu_0\big)\,ds, \quad t \in \mathbf{R}.$$

Differentiating (14) we get

$$(15) \qquad z_n'(t) = \frac{r_1'(t,t_1)}{r(t_1,t_1+\omega)} \int_{t_1}^{t_1+\omega} r(t_1+\omega,s) f(s,y_n(s),y_n'(s),\mu_n)\,ds + \int_{t_1}^{t} r_1'(t,s) f(s,y_n(s),y_n'(s),\mu_n)\,ds, \quad t \in \mathbf{R}.$$

Suppose that $\{\mu_n\}$ is not convergent. Then there exist convergent subsequences $\{\mu_{k_n}\}$, $\{\mu_{r_n}\}$, $\lim_{n\to\infty} \mu_{k_n} = \lambda_1$, $\lim_{n\to\infty} \mu_{r_n} = \lambda_2$, $\lambda_1 < \lambda_2$. Inserting k_n and r_n instead of n in (15) and taking limits on both sides of these equalities, we obtain

$$(16) \qquad \lim_{n\to\infty} z_{k_n}'(t) = \frac{r_1'(t,t_1)}{r(t_1,t_1+\omega)} \int_{t_1}^{t_1+\omega} r(t_1+\omega,s) f(s,y(s),y'(s),\lambda_1)\,ds + \int_{t_1}^{t} r_1'(t,s) f(s,y(s),y'(s),\lambda_1)\,ds, \quad t \in \mathbf{R},$$

and

$$(17)\qquad \lim_{n\to\infty} z'_{r_n}(t) = \frac{r'_1(t,t_1)}{r(t_1,t_1+\omega)} \int\limits_{t_1}^{t_1+\omega} r(t_1+\omega,s) f(s,y(s),y'(s),\lambda_2)\,\mathrm{d}s + \int_{t_1}^{t} r'_1(t,s) f(s,y(s),y'(s),\lambda_2)\,\mathrm{d}s, \quad t\in\mathbf{R},$$

uniformly on $\mathbf{R}$, respectively. Relations (16) and (17) yield

$$\lim_{n\to\infty} \left(z'_{k_n}(t_1+\omega) - z'_{k_n}(t_1)\right) = \frac{r'_1(t+\omega,t_1)-1}{r(t_1,t_1+\omega)} \int\limits_{t_1}^{t_1+\omega} r(t_1+\omega,s) f(s,y(s),y'(s),\lambda_1)\,\mathrm{d}s + \int_{t_1}^{t} r'_1(t_1+\omega,s) f(s,y(s),y'(s),\lambda_1)\,\mathrm{d}s,$$

$$\lim_{n\to\infty} \left(z'_{r_n}(t_1+\omega) - z'_{r_n}(t_1)\right) = \frac{r'_1(t+\omega,t_1)-1}{r(t_1,t_1+\omega)} \int\limits_{t_1}^{t_1+\omega} r(t_1+\omega,s) f(s,y(s),y'(s),\lambda_2)\,\mathrm{d}s + \int_{t_1}^{t} r'_1(t_1+\omega,s) f(s,y(s),y'(s),\lambda_2)\,\mathrm{d}s.$$

Since the function z_n is ω-periodic for all $n\in N$, we have $z'_n(t_1+\omega) - z'_n(t_1) = 0$ and thus

$$\begin{aligned}
0 &= \frac{r'_1(t+\omega,t_1)-1}{r(t_1,t_1+\omega)} \int\limits_{t_1}^{t_1+\omega} r(t_1+\omega,s)\big(f(s,y(s),y'(s),\lambda_1) - f(s,y(s),y'(s),\lambda_2)\big)\,\mathrm{d}s \\
&\quad + \int_{t_1}^{t} r'_1(t_1+\omega,s)\Big(f\big(s,y(s),y'(s),\lambda_1\big) - f\big(s,y(s),y'(s),\lambda_2\big)\Big)\,\mathrm{d}s \\
&= \int\limits_{t_1}^{t_1+\omega} k(s)\Big(f\big(s,y(s),y'(s),\lambda_1\big) - f\big(s,y(s),y'(s),\lambda_2\big)\Big)\,\mathrm{d}s,
\end{aligned}$$

where k is the function defined by (3). This, however, contradicts the facts that $k(t) > 0$ (by Lemma 2) and $f\big(t,y(t),y'(t),\lambda_1\big) - f\big(t,y(t),y'(t),\lambda_2\big) < 0$ (by assumption (9)) for $t \in \langle t_1, t_1+\omega\rangle$. Consequently, $\{\mu_n\}$ is a convergent sequence and $\lim\limits_{n\to\infty} \mu_n = \mu^*$. If we take limits for $n\to\infty$ in (14) and (15) we get

$$(z^*(t) :=)\lim_{n\to\infty} z_n(t) = \frac{r(t,t_1)}{r(t_1,t_1+\omega)} \int\limits_{t_1}^{t_1+\omega} r(t_1+\omega,s) f\big(s,y(s),y'(s),\mu^*\big)\,\mathrm{d}s + \int_{t_1}^{t} r(t,s) f\big(s,y(s),y'(s),\mu^*\big)\,\mathrm{d}s$$

and

$$\lim_{n\to\infty} z_n'(t) = \frac{r_1'(t,t_1)}{r(t_1,t_1+\omega)} \int_{t_1}^{t_1+\omega} r(t_1+\omega,s) f(s,y(s),y'(s),\mu^*)\,\mathrm{d}s$$
$$+\int_{t_1}^{t} r_1'(t,s) f(s,y(s),y'(s),\mu^*)\,\mathrm{d}s \quad (= z^{*\prime}(t))$$

uniformly on $\mathbf{R}$. Then z^* is a (then necessarily unique) ω-periodic solution of the equation

$$x'' - q(t)x = f(t,y(t),y'(t),\mu^*)$$

satisfying (2) and $z^* \in K$. Consequently, Lemma 4 implies $z = z^*$ and $\mu_0 = \mu^*$. Since $\lim\limits_{n\to\infty} z_n'(t) = z'(t)$ uniformly on $\mathbf{R}$ we obtain $\lim\limits_{n\to\infty} z_n = z$ and therefore T is a continuous operator on K.

Let $y \in K$ and $z = T(y)$. Then $z''(t) = q(t)z(t) + f(t,z(t),z'(t),\mu_0)$ for $t \in \mathbf{R}$, where $\mu_0 \in I$ is an appropriate number, and therefore $|z''(t)| \leqslant r_0 \max\limits_{t\in\mathbf{R}} q(t) + A =: B$ on $\mathbf{R}$ and $T(K) \subset L := \{y;\, y \in C^2(\mathbf{R}) \cap K, |y''(t)| \leqslant B \text{ for } t \in \mathbf{R}\} \subset K$. Since L is a compact subset of X, $T(K)$ is a relative compact subset of X. By Schauder's fixed point theorem there exists y, $y \in K$ such that $T(y) = y$, that is, there exists μ_0, $\mu_0 \in I$ such that y is an ω-periodic solution of (1) with $\mu = \mu_0$ satisfying (2) and (12). This completes the proof. □

Corollary 1. *Assume that assumption* (9) *and* (10) *are satisfied for positive constants r_0, r_1. Let A be defined as in* (7) *and let $2r_0\sqrt{\max\limits_{t\in\mathbf{R}} q(t)} < r_1$. Then there is ∂, $\partial > 0$ such that for each ε, $0 < \varepsilon \leqslant \partial$ there exists μ_ε, $\mu_\varepsilon \in I$ such that the equation $y'' - q(t)y = \varepsilon f(t,y,y',\mu)$ with $\mu = \mu_\varepsilon$ has an ω-periodic solution y satisfying* (2) *and* (12).

Proof. Let $\partial = \min\big\{\frac{r_0}{A}\min\limits_{t\in\mathbf{R}} q(t), \frac{1}{A}\big(\frac{r_1^2}{4r_0} - r_0\max\limits_{t\in\mathbf{R}} q(t)\big)\big\}$. Then εf satisfies for $0 < \varepsilon \leqslant \partial$ the same assumptions as f in Theorem 1 and thus Corollary 1 follows immediately from Theorem 1. □

Lemma 5. *Let r_0, r_1 be positive constants and let S be the set of ω-periodic functions y, $y \in C^2(\mathbf{R})$, $y(t_1) = 0$, $|y^{(i)}(t)| \leqslant r_i$ for $t \in \mathbf{R}$, $i = 0, 1$. Assume that*

(18) $$|f(t,y_1,y_2,\mu) - f(t,z_1,z_2,\mu)| \leqslant h_1(t)|y_1 - z_1| + h_2(t)|y_2 - y_2|$$
for $(t,y_1,y_2,\mu),\ (t,z_1,z_2,\mu) \in \mathbf{R} \times \langle -r_0, r_0\rangle \times \langle -r_1, r_1\rangle \times I,$

where h_1, $h_2 \in C^0(\mathbf{R})$ are ω-periodic functions, and let at least one of the following four conditions

$$\int_{t_1}^{t_1+\omega} \Big[\Big(\exp\int_{t_1}^{s} h_2(\tau)\,\mathrm{d}\tau\Big)\int_{t_1}^{s} (q(\tau)+h_1(\tau))\,\mathrm{d}\tau\Big]\,\mathrm{d}s \leqslant 1, \tag{19}$$

$$\int_{t_1}^{t_1+\omega} \Big[(q(s)+h_1(s))(s-t_1)+h_2(s)\Big]\,\mathrm{d}s \leqslant 1, \tag{20}$$

$$\int_{t_1}^{t_1+\omega} \Big[\Big(\exp\int_{s}^{t_1+\omega} h_2(\tau)\,\mathrm{d}\tau\Big)\int_{s}^{t_1+\omega} (q(\tau)+h_1(\tau))\,\mathrm{d}\tau\Big]\,\mathrm{d}s \leqslant 1, \tag{21}$$

$$\int_{t_1}^{t_1+\omega} \Big[(q(s)+h_1(s))(t_1+\omega-s)+h_2(s)\Big]\,\mathrm{d}s \leqslant 1, \tag{22}$$

holds. Then equation (1) has at most one solution y in the set S for every μ, $\mu \in I$.

Proof. The method of the proof is very similar to that of the proof of Lemma 6 ([2]). Assume that $y_1, y_2 \in S$, $y_1 \neq y_2$ are solutions of (1) with some $\mu = \mu_0$, $\mu_0 \in I$ and define $w := y_1 - y_2$. Since $w(t_1) = w(t_1+\omega) = 0$ there exists a $\xi \in (t_1, t_1+\omega)$ such that $|w(t)| \leqslant |w(\xi)|$ for $t \in \langle t_1, t_1+\omega\rangle$, and $w'(\xi) = 0$.

Let assumption (19) be satisfied. Using Gronwall's lemma we obtain from the inequality

$$|w'(t)| \leqslant \left|\int_{\xi}^{t} [(q(s)+h_1(s))|w(s)| + h_2(s)|w'(s)|]\,\mathrm{d}s\right|, \quad t \in \langle t_1, t_1+\omega\rangle, \tag{23}$$

the estimate

$$|w'(t)| \leqslant \Big(\exp\int_{\xi}^{t} h_2(s)\,\mathrm{d}s\Big)\int_{\xi}^{t} (q(s)+h_1(s))|w(s)|\,\mathrm{d}s, \quad t \in \langle \xi, t_1+\omega\rangle,$$

and thus

$$\begin{aligned} |w(\xi)| &= |w(t_1+\omega) - w(\xi)| = \left|\int_{\xi}^{t_1+\omega} w'(s)\,\mathrm{d}s\right| \\ &\leqslant \int_{\xi}^{t_1+\omega} \Big[\Big(\exp\int_{\xi}^{s} h_2(\tau)\,\mathrm{d}\tau\Big)\int_{\xi}^{s} (q(\tau)+h_1(\tau))|w(\tau)|\,\mathrm{d}\tau\Big]\,\mathrm{d}s \\ &< |w(\xi| \int_{t_1}^{t_1+\omega} \Big[\Big(\exp\int_{t_1}^{s} h_2(\tau)\,\mathrm{d}\tau\Big)\int_{t_1}^{s} (q(\tau)+h_1(\tau))\,\mathrm{d}\tau\Big]\,\mathrm{d}s. \end{aligned}$$

Then (since $w(\xi) \neq 0$)

$$1 < \int_{t_1}^{t_1+\omega} \left[\left(\exp \int_{t_1}^{s} h_2(\tau)\,\mathrm{d}\tau \right) \int_{t_1}^{s} (q(\tau) + h_1(\tau))\,\mathrm{d}\tau \right] \mathrm{d}s,$$

which contradicts assumption (19).

Let assumption (20) be satisfied. From (23) and the inequality $|w(t)| \leqslant \int_{t_1}^{t} |w'(s)|\,\mathrm{d}s$ for $t \in \langle t_1, t_1 + \omega \rangle$ we obtain

$$|w'(t)| \leqslant \int_{t_1}^{t} \left[(q(s) + h_1(s)) \int_{t_1}^{s} |w'(\tau)|\,\mathrm{d}\tau + h_2(s)|w'(s)| \right] \mathrm{d}s, \quad t \in \langle t_1, t_1 + \omega \rangle.$$

If we put $X(t) := \max_{t_1 \leqslant s \leqslant t} |w'(s)|$ for $t \in \langle t_1, t_1 + \omega \rangle$, then if $X(t_1 + \omega) > 0$ we get

$$|w'(t)| < X(t_1 + \omega) \int_{t_1}^{t_1+\omega} \left[(q(s) + h_1(s))(s - t_1) + h_2(s) \right] \mathrm{d}s, \quad t \in \langle t_1, t_1 + \omega \rangle.$$

Consequently

$$X(t_1 + \omega) < X(t_1 + \omega) \int_{t_1}^{t_1+\omega} \left[(q(s) + h_1(s))(s - t_1) + h_2(s) \right] \mathrm{d}s$$

and

$$1 < \int_{t_1}^{t_1+\omega} \left[(q(s) + h_1(s))(s - t_1) + h_2(s) \right] \mathrm{d}s,$$

which contradicts (20). Therefore $X(t_1 + \omega) = 0$, that is, w is a constant function on the interval $\langle t_1, t_1 + \omega \rangle$ and since $w(t_1) = 0$ we obtain $w(t) = 0$ for $t \in \langle t_1, t_1 + \omega \rangle$ which is a contradiction again.

If assumption (21) or (22) is satisfied, the proof is very similar to the above and therefore is omitted. □

Lemma 6. *Assume that assumption* (9) *is satisfied with positive constants* r_0, r_1, *the functions* $\frac{\partial f}{\partial y_1}(t, y_1, y_2, \mu)$, $\frac{\partial f}{\partial y_2}(t, y_1, y_2, \mu)$ *are continuous on* D $(= \langle 0, \omega \rangle \times \langle -r_0, r_0 \rangle \times \langle -r_1, r_1 \rangle \times I)$ *and*

$$(24) \qquad q(t) + \frac{\partial f}{\partial y_1}(t, y_1, y_2, \mu) \geqslant 0 \text{ for } (t, y_1, y_2, \mu) \in D.$$

Let the set S be defined as in Lemma 5.

Then there exists at most one μ_0, $\mu_0 \in I$ such that equation (1) *with $\mu = \mu_0$ has a solution y, $y \in S$. In this case the solution y is unique.*

Proof. Let y_1 and y_2 be solutions of (1) with $\mu = \mu_1$ and $\mu = \mu_2$, respectively, $\mu_1, \mu_2 \in I$, $\mu_1 \leqslant \mu_2$; $y_1, y_2 \in S$, $y_1 \neq y_2$. Using assumptions (9), (24) and Taylor's formula we get

$$\begin{aligned} &f(t, y_1(t), y_1'(t), \mu_1) - f(t, y_2(t), y_2'(t), \mu_2) \\ &\leqslant g(t)\big(y_1(t) - y_2(t)\big) + h(t)\big(y_1(t) - y_2(t)\big)', \quad t \in \mathbf{R}, \end{aligned} \tag{25}$$

where g, h are ω-periodic continuous functions, $q(t) + g(t) \geqslant 0$ on $\mathbf{R}$ and if $\mu_1 < \mu_2$ ($\mu_1 = \mu_2$) then (25) holds with the strict inequality (equality). For $w := y_1 - y_2$ we then obtain the inequality

$$w''(t) \leqslant \big(q(t) + g(t)\big)w(t) + h(t)w'(t), \quad t \in \mathbf{R}, \tag{26}$$

$w(t_1) = w(t_1 + \omega) = 0$.

Let $\mu_1 < \mu_2$. If $w'(t_1) \leqslant 0$ then, using (26) and Tschaplygin's lemma ([1]), we get $w(t) < 0$ on $(t_1, t_1 + \omega\rangle$ which contradicts $w(t_1 + \omega) = 0$. If $w'(t_1) > 0$ then there exists η, $\eta \in (t_1, t_1 + \omega)$ such that $w(t) > 0$ for $t \in (t_1, \eta)$, $w(\eta) = 0$ and $w'(\eta) \leqslant 0$. Therefore $w(t) < 0$ on $(\eta, t_1 + \omega\rangle$ which again contradicts $w(t_1 + \omega)$.

Let $\mu_1 = \mu_2$. Since $q(t) + g(t) \leqslant 0$ on $\mathbf{R}$, the equation $y'' = \big(q(t) + g(t)\big)y + h(t)y'$ is disconjugate on $\mathbf{R}$ which contradicts $w(t_1) = w(t_1 + \omega) = 0$. □

Theorem 2. *Assume that assumptions* (7)–(10) *are satisfied for positive constants r_0, r_1. Let $\frac{\partial f}{\partial y_1}, \frac{\partial f}{\partial y_2} \in C^0(D)$ and let assumption* (24) *be satisfied.*

Then there exists a unique μ_0, $\mu_0 \in I$, such that equation (1) *with $\mu = \mu_0$ has an ω-periodic solution y satisfying* (2) *and* (12). *This solution y is unique.*

The proof follows from Theorem 1 and Lemma 6.

Example 1. Consider the equation

$$y'' - 3\big(\exp(2 + \sin t)\big)y = \sin t \cos y' \mathrm{e}^{y^3} + \mu, \tag{27}$$

where $\mu \in I_1 := \langle -\mathrm{e}, \mathrm{e}\rangle$. Let $t_1 \in \mathbf{R}$ be a number. Assumptions (7)–(10) are satisfied with $r_0 = 1$, $r_1 = 2\sqrt{\mathrm{e}}\sqrt{2 + 3\mathrm{e}^2}$ and

$$3\exp(2 + \sin t) + \frac{\partial}{\partial y_1}(\sin t \cos y_2 \mathrm{e}^{y_1^3} + \mu) \geqslant 0$$

for $(t, y_1, y_2, \mu) \in \mathbf{R} \times \langle -1, 1\rangle \times \big\langle -2\sqrt{\mathrm{e}}\sqrt{2 + 3\mathrm{e}^2}, 2\sqrt{\mathrm{e}}\sqrt{2 + 3\mathrm{e}^2}\big\rangle \times I_1$. By Theorem 2 there exists a unique μ_0, $\mu_0 \in I_1$ such that equation (27) with $\mu = \mu_0$ has an ω-periodic solution y satisfying $y(t_1) = 0$, $|y(t)| \leqslant 1$ and $|y'(t)| \leqslant 2\sqrt{\mathrm{e}}\sqrt{2 + 3\mathrm{e}^2}$ for $t \in \mathbf{R}$. This solution y is unique.

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