Mathematica Bohemica

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Mathematica Bohemica, Vol. 117 (1992), No. 4, 425–428

Persistent URL: `http://dml.cz/dmlcz/126056`

117 (1992) MATHEMATICA BOHEMICA No. 4, 425–428

A SHARPENING OF A DISCRETE ANALOG OF WIRTINGER'S AND ISOPERIMETRIC INEQUALITIES

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(Received February 8, 1991)

Summary. A sharpening of a discrete case of Wirtinger's inequality is given. It is then used to sharpen the isoperimetric inequality for polygons.

Keywords: Wirtinger's inequality, trigonometric polynomials

AMS classification: 52A40

Let us recall the sharpening of the continuous Wirtinger's inequality, which was established by Z. Nádeník [3]:

Let $f(x)$ denote a function with period 2π, $f' \in L_2$ and $\int_0^{2\pi} f(x)\,\mathrm{d}x = 0$. Then

$$\int_0^{2\pi} f'(x)^2 \mathrm{d}x \geqslant \int_0^{2\pi} f^2(x)\,\mathrm{d}x + \frac{\pi}{2}\big(f(0) + f(\pi)\big)^2, \tag{1}$$

with the equality holding only for

$$f(x) = A\cos x + B\sin x + C\Big(\frac{2}{\pi} - |\sin x|\Big), \quad A, B, C = \text{const}.$$

We will give a sharpening of a discrete case of Wirtinger's inequality, which is analogous to (1). See also J. Novotná [4]. The main result is as follows:

Theorem 1. *Let $\mathscr{A} = A_0, A_1, \ldots, A_{n-1}$ be a closed n-gon in $\mathbf{R}^N$ with its centroid at the origin of the coordinate system, let n be even, i.e. $n = 2m$. Then for all $p = 0, 1, 2, \ldots, n-1$*

$$\sum_{\nu=0}^{n-1} |A_{\nu+1} - A_\nu|^2 \geqslant 4\sin^2\frac{\pi}{n}\sum_{\nu=0}^{n-1}|A_\nu|^2 + n\sin^2\frac{\pi}{n}\,|A_p + A_{p+m}|^2. \tag{2}$$

Equality in (2) is attained if and only if

$$A_\nu = A\cos\nu\frac{2\pi}{n} + B\sin\nu\frac{2\pi}{n} + C\Big[\frac{2}{n}\operatorname{cotan}\frac{\pi}{n} - \Big|\sin(\nu-p)\frac{2\pi}{n}\Big|\Big],$$

$\nu = 0, 1, \ldots, n-1$, $A, B, C = \text{const.}$

To prove Theorem 1 we need the following lemma:

Lemma. *Let $n = 2m$. Then*

$$\sum_{j=1}^{m-1} \frac{\omega_{2j}^{\nu}}{\sin^2 j\frac{2\pi}{n} - \sin^2\frac{\pi}{n}} = \frac{1}{\sin^2\frac{\pi}{n}} - \frac{n}{\sin\frac{2\pi}{n}}\Big|\sin\nu\frac{2\pi}{n}\Big|,$$

where $\omega_\nu^k = \omega^{\nu k} = \exp(i\nu k\cdot 2\pi/n)$.

Proof. It suffices to prove

$$\Big|\sin\nu\frac{2\pi}{n}\Big| = \frac{1}{n}\sin\frac{2\pi}{n}\sum_{j=0}^{m-1}\frac{\cos 2j\nu\frac{2\pi}{n}}{\sin^2\frac{\pi}{n} - \sin^2 j\frac{2\pi}{n}}. \tag{3}$$

One proves (3) expressing $\left|\sin\nu\frac{2\pi}{n}\right|$ in terms of complex trigonometric polynomials in the same way as $|\sin x|$ is expressed by a Fourier series. □

Proof of Theorem 1. We see that it suffices to prove it for the case $N = 2$. To simplify the proof, we may suppose $p = 0$.

We shall express vertices $A_0, A_1, \ldots, A_{n-1}$ of the n-gon $\mathscr{A}$ in the form of complex trigonometric polynomials (I. J. Schoenberg [5] called them Fourier polynomials). There exist numbers $\vartheta_0, \vartheta_1, \ldots, \vartheta_{n-1}$ such that $A_\nu = \sum_{k=0}^{n-1}\vartheta_k\omega_\nu^k$, $\nu = 0, 1, \ldots, n-1$. A discrete analog of Parseval's relation of completeness gives

$$\sum_{\nu=0}^{n-1}|A_\nu|^2 = n\sum_{k=0}^{n-1}|\vartheta_k|^2, \quad \sum_{\nu=0}^{n-1}|A_{\nu+1} - A_\nu|^2 = n\sum_{k=0}^{n-1}|\vartheta_k|^2|\omega_k - 1|^2,$$

$|A_0 + A_m|^2 = 4\Big|\sum_{k=0}^{m-1}\vartheta_{2k}\Big|^2$. The condition $\sum_{\nu=0}^{n-1}A_\nu = 0$ implies $\vartheta_0 = 0$. Instead of (2) we may write

$$\sum_{k=1}^{n-1}|\vartheta_k|^2\Big(\sin^2 k\frac{\pi}{n} - \sin^2\frac{\pi}{n}\Big) \geqslant \sin^2\frac{\pi}{n}\Big|\sum_{k=1}^{m-1}\vartheta_{2k}\Big|^2.$$

We will show that even

$$(4)\qquad \sum_{k=1}^{m-1} |\vartheta_{2k}|^2\left(\sin^2 k\frac{2\pi}{n} - \sin^2\frac{\pi}{n}\right) \geqslant \sin^2\frac{\pi}{n}\left|\sum_{k=1}^{n-1}\vartheta_{2k}\right|^2$$

holds.

In order to prove (4) we start with the inequality

$$(5)\qquad \sum_{j,k=1}^{m-1} \frac{1}{S_j}\cdot\frac{1}{S_k}|S_j\vartheta_{2j} - S_k\vartheta_{2k}|^2 \geqslant 0,$$

where

$$S_r = \sin^2 r\frac{2\pi}{n} - \sin^2\frac{\pi}{n}.$$

In view of the equality

$$\sum_{j=1}^{m-1}\frac{1}{S_j} = \frac{1}{\sin^2\frac{\pi}{n}},$$

(5) implies the inequality (4).

The sign of equality occurs in (2) if and only if $\vartheta_k = 0$ for $k = 3, 5, \ldots, n-3$ and, according to (5),

$$\nu_{2k} = \frac{\sin^2\frac{2\pi}{n} - \sin^2\frac{\pi}{n}}{\sin^2 k\frac{2\pi}{n} - \sin^2\frac{\pi}{n}}\vartheta_2.$$

By virtue of the lemma we have

$$A_\nu = \vartheta_1\omega_1^\nu + \vartheta_{n-1}\omega_{n-1}^\nu + \vartheta_2\frac{n\left(\sin^2\frac{2\pi}{n} - \sin^2\frac{\pi}{n}\right)}{\sin\frac{2\pi}{n}}\left[\frac{2}{n}\operatorname{cotan}\frac{\pi}{n} - \left|\sin\nu\frac{2\pi}{n}\right|\right].$$

Separating the real and imaginary parts of the complex numbers we get our statement. □

Remark. By the Mean Value Theorem it is easy to show that (2) is a discrete analog of (1). See K. Fan, O. Taussky, J. Todd [2].

Now we will establish a sharpening of the isoperimetric inequality for polygons. We will prove

Theorem 2. *Let $\mathscr{A} = A_0, A_1, \ldots, A_{n-1}$ denote a plane closed n-gon of area F and perimeter L, let $n = 2m$. Then for all $p = 0, 1, \ldots, n-1$,*

$$(6)\qquad \sum_{\nu=0}^{n-1}|A_{\nu+1} - A_\nu|^2 - 4\tan\frac{\pi}{n}F \geqslant \frac{n}{2}\tan^2\frac{\pi}{n}|A_p + A_{p+m}|^2,$$

with the equality holding only for the regular n-gon.

Proof. Denote $A_\nu = [x_\nu, y_\nu]$, $\nu = 0, 1, \ldots, n-1$. We may suppose that $\sum_{\nu=0}^{n-1} A_\nu = 0$ and $4F = \sum_{\nu=0}^{n-1} [(x_{\nu+1} + x_\nu)(y_{\nu+1} - y_\nu) - (x_{\nu+1} - x_\nu)(y_{\nu+1} + y_\nu)]$. In virtue of (2) we have

$$\sum_{\nu=0}^{n-1} |A_{\nu+1} - A_\nu|^2 - 4\tan\frac{\pi}{n} F = \frac{1}{2}\sum_{\nu=0}^{n-1}\Big[(x_{\nu+1} + x_\nu)\tan\frac{\pi}{n} - (y_{\nu+1} - y_\nu)\Big]^2$$

$$+ \frac{1}{2}\sum_{\nu=0}^{n-1}\Big[(y_{\nu+1} + y_\nu)\tan\frac{\pi}{n} + (x_{\nu+1} - x_\nu)\Big]^2$$

$$+ \frac{1}{2}\sum_{\nu=0}^{n-1}\Big[(x_{\nu+1} - x_\nu)^2 - (x_{\nu+1} + x_\nu)^2\tan^2\frac{\pi}{n}\Big]$$

$$+ \frac{1}{2}\sum_{\nu=0}^{n-1}\Big[(y_{\nu+1} - y_\nu)^2 - (y_{\nu+1} + y_\nu)^2\tan^2\frac{\pi}{n}\Big] \geqslant \frac{n}{2}\tan^2\frac{\pi}{n}|A_p + A_{p+m}|^2.$$

It is easy to prove that the sign of equality occurs in (6) only for the regular n-gon. □

Corollary. *Let $\mathscr{A} = A_0, A_1, \ldots, A_{n-1}$ denote a plane equilateral closed n-gon of area F and perimeter L. Let $n = 2m$. Let us denote by d_i the distance of the center of A_iA_{i+m} and the centroid of $\mathscr{A}$. Then*

$$L^2 - 4n\tan\frac{\pi}{n}F \geqslant 2n^2\tan^2\frac{\pi}{n}d_i^2 \tag{7}$$

with the equality holding only for the regular n-gon.

Remark. The continuous case of inequality (7) was investigated by L. Boček [1].

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