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L-groups versus k-groups

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L-GROUPS VERSUS k-GROUPS

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Summary. We investigate free groups over sequential spaces. In particular, we show that the free k-group and the free sequential group over a sequential space with unique limits coincide and, barred the trivial case, their sequential order is ω_1.

Keywords: sequential convergence, free sequential groups, free k-groups, sequential order

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1. INTRODUCTION

Usually, by an L-group a group equipped with a convergence of sequences compatible with the group operation is understood. The notion dates back to O. Schreier ([21]). The most important class of L-groups consists of groups in which the convergence has unique limits and satisfies the Urysohn axiom of convergence. Such groups are known as FLUSH-convergence groups (the so called Katowice notation, cf. [14]) or L^*GR_{sep} ([8]). It is known that there are nice correspondences between certain sequential convergences and certain sequential closures (cf. [7], [10]). In particular, each FUSH-convergence space can be viewed as a sequential (topological) space having unique sequential limits and each FLUSH-convergence group can be viewed as a group equipped with a sequential topology having unique sequential limits such that the group operation is sequentially continuous. Since sequential spaces are k-spaces, a natural question arises what is the relationship between L-groups and the more recent k-groups (cf. [18]). The two theories have been developed independently, though there were clues indicating that their intersection is non-void (cf. [5] and [20]). In [4] it is proved that sequential groups with unique sequential limits, hence FLUSH-convergence groups, can be identified with a subclass of weakly

Hausdorff k-groups. In the present paper we show that the identification extends to free groups. As a corollary, we show that epics in the category of FLUSH-groups are morphisms with top dense range. The identification also yields a tool for studying certain sequential properties of free k-groups and free topological groups ([19]).

2. L-GROUPS VERSUS k-GROUPS

For k-spaces and k-groups the reader is referred to [13], [1] and [18], [11], and for spaces and groups equipped with a sequential convergence to [10], [1] and [15], [9], [5], [8]. Epics in T_2 k-groups are investigated in [12].

For the reader's convenience we recall here some basic facts about sequential spaces and k-spaces.

Let X be a topological space. A subset U of X is said to be *sequentially open* if whenever a sequence $\langle x_n \rangle$ converges in X to $x \in U$, then $x_n \in U$ for all but finitely many $n \in N$; its complement is called *sequentially closed.* Open sets are sequentially open and the latter sets form a topology; the resulting space will be denoted by $s(X)$. This yields a well-known modification functor s acting on the category TOP of topological spaces and continuous maps. A topological space Y is said to be *sequential* if $s(Y) = Y$. A mapping f of a sequential space Y into X is continuous iff it is sequentially continuous. A quasi-compact space is a topological space with the property that every open cover has a finite subcover; a compact space is a T_2 quasi-compact space. A continuous map $\varphi: T \to X$ is said to be a *test* if T is compact. A subset U of X is said to be ***k**-open* (*k-closed*) if for all tests $\varphi: T \to X$, $\varphi^{-1}(U)$ is open (closed). Open sets are k-open and the latter form a topology; the resulting space will be denoted by $k(X)$. The corresponding functor acting on TOP is denoted by k. If $k(X) = X$, then X is said to be a *k-space.* Clearly, $s(X)$ is determined by all tests the domain of which is $\omega + 1$ (the space of all ordinal numbers less than or equal to ω). Indeed, a subset U of X is sequentially open (closed) iff $\varphi^{-1}(U)$ is open (closed) for all tests $\varphi: \omega + 1 \to X$. We say that X is *weakly Hausdorff*, or t_2, if for each test $\varphi: T \to X$, $\varphi(T)$ is closed in X. In a t_2 space X the set $\varphi(T)$ is compact for each test φ ([13]). The categories K of k-spaces and WHK of t_2 k-spaces (with continuous maps as morphisms) have nice properties (cf. [11]).

For X, $Y \in$ TOP their topological product will be denoted by $X \times Y$ and, for X, $Y \in$ K, $k(X \times Y)$ is their product in K; it will be denoted by $X \times_k Y$ and called the *k-product* of X and Y (in [11] the topological product of X and Y is denoted by $X \times_c Y$ and the k-product is denoted by $X \times Y$). Similarly, if X and Y are sequential, then $s(X \times Y)$ is their *sequential product* and will be denoted by $X \times_s Y$.

A *k-group* is a group G with a k-topology such that inversion is continuous and multiplication is continuous on the k-product.

A group G equipped with a sequential topology such that inversion is continuous and such that multiplication is continuous on the sequential product is said to be a *sequential group* ([4]).

It is known that there is a one-to-one relationship between FUSH-convergences and sequential topologies with unique limits. Clearly, this induces a one-to-one relationship between FLUSH-convergence groups and sequential groups with unique sequential limits. (Recall that (L) stands for the compatibility of the convergence: if $\langle x_n \rangle$ converges to x and $\langle y_n \rangle$ converges to y, then $\langle x_n^{-1} \rangle$ converges to x^{-1} and $\langle x_n y_n \rangle$ converges to xy.) Indeed, if we start with a FLUSH-convergence in a group, sequentially open sets form a sequential topology having the same convergent sequences as the original convergence. Axiom (L) guarantees that the group equipped with the induced sequential topology is a sequential group with unique sequential limits.

Note that in the FLUS-convergence group theory (the uniqueness of limits is not assumed) the invariants fail to be "topological" (cf. Remark 2.1 in [5]). In fact, a group can be equipped with two non-isomorphic FLUS-convergences inducing the same sequential topology. The first example of such a group is due to A. Kaminski.

In the sequel, the following results from [4] will be needed.

Theorem 0. (i) *A sequential space is t_2 iff it has unique sequential limits.*

(ii) *Let X and Y be sequential spaces with unique sequential limits. Then their sequential product $X \times_s Y$ has unique sequential limits and coincides with their k-product $X \times_k Y$.*

(iii) *Let G be a k-group. Then $s(G)$ is a sequential group. If G is a t_2 space, then $s(G)$ has unique sequential limits.*

(iv) *Let G be a sequential group with unique sequential limits. Then G is a t_2 k-group.*

Remark 0. The assertion (ii) in Theorem 0 follows by Theorem 3.6 in [22] (s-products and k-products of finitely many spaces coincide) and the fact that unique sequential limits are preserved by products. Observe that while Theorem 3.6 in [22] is proved by categorical arguments, in [4] a simple topological proof of (ii) in Theorem 0 is given.

Remark 1. Denote by WHSG the category of all sequential groups having unique sequential limits, with sequentially continuous homomorphisms as morphisms. It follows from (iv) in Theorem 0 that WHSG is a full subcategory of WHKG (consisting of all t_2 k-groups, see [11]). This in turn induces an isomorphism between FLUSH-convergence groups and the corresponding full subcategory of WHKG.

Remark 2. Since the k-product of two k-groups is a k-group, it follows from (ii) in Theorem 0 that the k-product of two t_2 sequential groups is a t_2 sequential group. The assertion can be easily extended to finite products. The k-product of uncountably many t_2 sequential groups need not be a sequential group. E.g., let X be an uncountable power of the one-dimensional torus. The k-product topology is the usual product topology for X; it is a k-group topology but fails to be sequential. The question arises whether every k-product of countably many t_2 sequential groups is a sequential group. The answer is "yes" provided the topological product of countably many compact sequential spaces with unique sequential limits is a sequential space.

The relevant information on free FLUSH-convergence groups (commutative, noncommutative, pointed commutative and pointed noncommutative) can be found in [5] and [6]. For free k-groups and their relationship to free topological groups, the reader is referred to [18] and [11]. Interesting facts about free continuous algebras are contained in [20].

Definition. Let X be a sequential space having unique sequential limits with a distinguished point e and let FX be a sequential group having unique sequential limits which contains X as a subspace and has e as its identity element. Then FX is said to be the *pointed (Graev) free sequential group over* X if each continuous map of X into a sequential group G with unique sequential limits, sending e to the identity element of G, can be uniquely extended to a continuous homomorphism of FX into G. The (non-pointed) *free sequential group*, the *abelian free sequential group* and the *pointed abelian free sequential group* are defined in the obvious way.

Remark 3. Let X be a sequential space with unique sequential limits. The existence and properties of all four types of free sequential groups FX follow directly from the corresponding results for FLUSH-convergence free groups ([5]).

Theorem 1. *Let X be a sequential space with unique sequential limits. Let FX be the free k-group generated by X. Then FX is a sequential group with unique sequential limits.*

Proof. Consider the sequential modification $s(FX)$ of FX. Since X is a t_2 space, FX is a t_2 space as well (cf. Corollary 2.13 in [11]). By (iii) in Theorem 0, $s(FX)$ is a sequential group with unique sequential limits and, by (iv) in Theorem 0, it is a t_2 k-group. Since the identity mapping of $s(FX)$ into FX is continuous, necessarily $s(FX) = FX$. □

Corollary 1. *Let X be a sequential space with unique sequential limits. Let FX be the free k-group (abelian, pointed, poinetd abelian) over X. Then FX is the free sequential (abelian, pointed, pointed abelian) group over X.*

Proof. The free sequential group over X (cf. Remark 3) is a t_2 k-group ((iv) in Theorem 0), hence it has to coincide with FX. □

R e m a r k 4. Corollary 1 yields an alternative construction of the free FLUSH-convergence group via the free k-group. Indeed, if X is a FUSH-convergence space, then the free FLUSH-convergence group (pointed, abelian, pointed abelian) generated by X can be constructed via applying successively the topological modification functor, then the free k-group functor and then the sequential convergence functor assigning to each topological space (in general to each filter convergence space) the associated sequential (FUS-) convergence.

Theorem 2. *Let $f: G \to H$ be an epic in the category of sequential groups with unique sequential limits. Then $f(G)$ is dense in H.*

P r o o f. Clearly, f is an epic in the category WHKG. By Corollary 2.48 in [11], $f(G)$ is dense in H. □

Corollary 2. *Epics in* WHSG *are exactly morphisms with dense range.*

Corollary 3. *Epics in the category of* FLUSH-*convergence groups are exactly morphisms with top-dense range.*

3. Sequential conditions in free groups

Let X be a k-space with a distinguished point (basepoint) e. The pointed (Graev) free k-group over X will be denoted by $F_K X$ (recall that if X is t_2, then $F_K X$ is t_2 as well). Similarly, for a Tychonoff space X with a distinguished point e the pointed (Graev) free topological group over X will be denoted by $F_G X$. As a rule, the non-pointed (Markov) free groups are covered as a special case with e isolated. Since the choice of e plays no role in our considerations, we usually do not mention it.

Theorem 3. *Let X be a t_2 k-space. Then X is sequential iff $F_K X$ is sequential.*

P r o o f. If $F_K X$ is sequential, then so it its closed subspace X. The converse follows from Corollary 1. □

Recall ([16]) that a topological space is called a k_ω-space if it is a direct limit of an expanding sequence of compact (i.e. Hausdorff) subspaces. Since $F_G X = F_K X$ whenever X is a k_ω-space, Theorem 3 generalizes Theorem 3.1 in [19] stating that a k_ω-space X is sequential iff $F_G X$ is sequential. Obviously, the most natural generalization of Theorem 3.1 in [19] leads to the class $\{X; F_G X = F_K X\}$ of all Tychonoff k-spaces X for which $F_G X$ and $F_K X$ coincide. It would be interesting to establish the basic properties of this class (cf. [2], [16]).

Our final topic is the sequential order in free groups. For the reader's convenience we start with some general remarks.

Let X be a nonempty set equipped with an FS-convergence of sequences. Then to each subset A of X we can assign the set $\mathrm{cl}\, A$ of all limits of sequences ranging in A. The corresponding convergence closure operator cl need not be idempotent. For each ordinal number α not exceeding ω_1 we define inductively α-$\mathrm{cl}\, A$, $A \subset X$ as follows:

$0\text{-cl}\, A = A$,

$\alpha\text{-cl}\, A = \bigcup\{\mathrm{cl}(\beta\text{-cl}\, A)\,;\ \beta < \alpha\}$.

Then $\omega_1\text{-cl}\, A = \mathrm{cl}(\omega_1\text{-cl}\, A)$ for all $A \subset X$ and ω_1-cl is a closure operator satisfying all four axioms of Kuratowski. The sequential order of the convergence (of the underlying space) is the least ordinal number α such that $\mathrm{cl}(\alpha\text{-cl}\, A) = \alpha\text{-cl}\, A$ for all $A \subset X$. For a limit ordinal number α, α-cl is sometimes defined by $\alpha\text{-cl}\, A = \mathrm{cl} \bigcup\{\beta\text{-cl}\, A\,;\ \beta < \alpha\}$. Obviously, the two corresponding notions of the sequential order are slightly different. However, the fact that the sequential order of a convergence is ω_1 does not depend on the way how α-cl is defined for limit ordinal numbers.

Iterations of cl in various types of continuous groups have been investigated in, e.g., [15], [17], [19], [3]. P. Nyikos asked in [17] whether in a sequential topological group the sequential order may be anything between 1 and ω_1. Since the sequential order of all known continuous groups is either 0, 1 or ω_1, it is natural to ask the same question in this more general setting.

Theorem 4. *Let X be a t_2 k-space and suppose that there is a one-to-one sequence $\langle x_n \rangle$ converging in X to a point x. Let T be the subspace of X the underlying set of which is $\{x_n\,;\ n = 1, 2, \dots\} \cup \{x\} \cup \{e\}$. Then*

(i) *T is a closed compact subspace of X;*

(ii) *The subgroup of $F_K X$ generated by T is $F_K T$ and it is closed in $F_K X$;*

(iii) *The sequential order of $F_K X$ is ω_1.*

Proof. (i) Observe that T is a continuous image of a compact space. Since X is t_2, T is a closed subspace. Consequently (cf. 2.1 in [13]), T is compact.

(ii) The assertion follows directly from Proposition 5.3 in [18].

(iii) Since T is compact, we have $F_G T = F_K T$. By Corollary 3.8 in [18], $F_G T$ contains a closed subspace homeomorphic to the well-known sequential space S_ω the sequential order of which is ω_1. Thus the sequential order of $F_K X$ is ω_1 as well. □

Corollary 4. (i) *Let X be a nondiscrete* FUSH*-convergence space and let FX be the free* FLUSH*-convergence group over X. Then the sequential order of FX is ω_1.*

(ii) *Let X be a nondiscrete sequential space with unique sequential limits. Let FX be the free sequential group over X. Then the sequential order of FX is ω_1.*

In [19], the proof that the sequential order of a topological group G is ω_1 is carried out by embedding S_ω into G as a closed subspace. In [3] a general inductive construction is used to show that the sequential order of various FUSH-convergence spaces and FLUSH-convergence groups is ω_1. As an illustration of the inductive construction we prove that the sequential order of the free commutative FLUSH-convergence group over a nondiscrete FUSH-convergence space is ω_1.

Let X be a nondiscrete FUSH-convergence space. Let $\langle x_n \rangle$ be a one-to-one sequence converging in X to a point x, $x_n \neq x$ for all $n \in N$. Let FCX be the free commutative FLUSH-convergence group over X. Then for each $k \in N$, the sequence $\langle k(x_n - x) \rangle = \langle S_k(n) \rangle = S_k$ converges in FCX to 0, but no diagonal subsequence converges in FCX to 0. Hence (cf. [15]) FCX fails to be a Fréchet space.

Let M be a nonempty subset of N. Denote by $W(M)$ the set of all elements of FCX of the form $\sum_{i=1}^{m} w_i$, where $m \in N$ and for each i, $i = 1, \ldots, m$, there are $k(i) \in M$ and $n(i) \in N$ such that $w_i = S_{k(i)}(n(i)) = k(i)(x_{n(i)} - x)$.

Lemma. *Let $\langle M_n \rangle$ be a sequence of disjoint nonempty subsets of N. Let $\langle v_n \rangle$ be a sequence such that $v_n \in W(M_n)$, $n = 1, 2, \ldots$. Then no subsequence of $\langle v_n \rangle$ converges in FCX.*

Proof. The assertion follows from the fact that the complexity of words v_n (the number of occurrences of a generator in v_n) tends to infinity. □

Theorem 5. *The sequential order of FCX is ω_1.*

Proof. It suffices to prove that for each ordinal number α, $\alpha < \omega_1$, the following proposition holds true:

$P(\alpha)$ For each infinite set $M \subset N$ there exists a set $A \subset W(M)$ such that $\alpha\text{-cl}\, A \subset W(M) \setminus \{0\}$ and $(\alpha + 1)\text{-cl}\, A = \{0\} \cup \alpha\text{-cl}\, A$.

We shall proceed by transfinite induction. Let M be an infinite subset of N.

Let $\alpha = 0$. Clearly, it suffices to choose $k \in M$ and put $A = \{S_k(n);\, n \in N\}$. Now, let $\alpha > 0$ and assume that $P(\beta)$ holds true for all ordinal numbers β, $\beta < \alpha$.

Let $\alpha = \beta + 1$. Let $\langle M_n \rangle$ be a sequence of infinite disjoint subsets of M. By the inductive assumption, for each M_n, $n = 1, 2, \ldots$, there is a set $A_n \subset W(M_n)$ such that $\beta\text{-cl}\, A_n \subset W(M_n) \setminus \{0\}$ and $(\beta + 1)\text{-cl}\, A_n = \{0\} \cup \beta\text{-cl}\, A_n$. By Lemma, no subsequence of any diagonal sequence $\langle v_n \rangle$, $v_n \in \beta\text{-cl}\, A_n$, $n = 1, 2, \ldots$, converges in FCX. Fix $m \in M$ and let A be the union of the sets $S_m(n) + A_n$, $n \in N$. It is easy to verify that $(\beta + 1)\text{-cl}\, A \subset W(M) \setminus \{0\}$ and $(\beta + 2)\text{-cl}\, A = \{0\} \cup (\beta + 1)\text{-cl}\, A$.

Let α be a limit ordinal number and let $\langle \alpha_n \rangle$ be a sequence of ordinal numbers converging to α, $\alpha_n < \alpha$ for all $n \in N$. The construction of a suitable set $A \subset W(M)$

is similar to that for isolated α, $\alpha > 0$, and it is omitted. This completes the proof. □

E. T. Ordman and B. V. Smith-Thomas raised a question whether, if $F_G X$ contains a nontrivial convergent sequence, then also X contains a nontrivial convergent sequence (Question 3.11 in [19]). The following two related questions seem to be natural. What happens if $F_G X$ is replaced by $F_K X$? What is the relationship between compact sets in X and compact sets in $F_G X$ (or $F_K X$)?

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Súhrn

L-GRUPY VERSUS k-GRUPY

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V tomto článku sú vyšetrované voľné grupy nad sekvenčnými priestorami. Je dokázané, že voľná k-grupa a L-grupa nad sekvenčným priestorom s jednoznačnými limitami splývajú a že v netriviálnom prípade ich sekvenčný rád je ω_1.

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