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SOLUTION SEMIGROUP AND INVARIANT MANIFOLDS FOR FUNCTIONAL EQUATIONS WITH INFINITE DELAY

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Summary. It is proved that parabolic equations with infinite delay generate C_0-semigroup on the space of initial conditions, such that local stable and unstable manifolds can be constructed for a fully nonlinear problems with help of usual methods of the theory of parabolic equations.

Keywords: parabolic functional equation, infinite delay, stable and unstable manifolds

AMS classification: 45K05, 35R10, 35B35, 35B40

INTRODUCTION

The aim of this paper is to prove the existence of a resolvent operator for the parabolic equation with infinite delay, such that solutions, given by this resolvent, define a C_0-semigroup with the properties, which enable to construct stable and unstable manifolds for the fully nonlinear problem

$$\dot{u}(t) = Au(t) + Lu_t + g(u(t), u_t), \tag{0.1}$$

$$u(0) = x, \quad u_0 = \varphi, \tag{0.2}$$

where u_t denotes a shift of the function u: $u_t(\tau) = u(t+\tau)$ for $\tau < 0$, A is a generator of an analytic semigroup in a Banach space X and L is a continuous linear operator from an appropriate function space Y into X. The example of the operator L we have in mind is the integral operator

$$Lu_t = \int_0^\infty k(s)(A + bI)u(t - s)\,\mathrm{d}s \tag{0.3}$$

and the nonlinearity g can take the following form:

$$g(u(t), u_t) = \int_0^\infty f(u(t-s), Au(t-s))\, \mathrm{d}\mu(s) \tag{0.4}$$

with $f(0) = 0$, $Df(0) = 0$, $\mathrm{d}\mu(s) = k_1(s)\,\mathrm{d}s + \delta_0(s)$, where k, k_1 are suitable functions and δ_0 is the Dirac function concentrated at 0. This type of equations arise e.g in investigation a heat conduction in materials with memory.

Solvability on the real line and asymptotic behaviour of the solution of the linear equation (0.1) with a more special operator L were treated in [2], [7]. (L was of the form (0.3) with the additional requirement on the Laplace transform of the kernel k to be extendible to certain sector in the complex plane). The existence of stable, unstable and center manifolds for semilinear problem (0.1), (0.2) was proved in [8]. In a fully nonlinear case, linearized stability and unstability and the existence of a local center manifold for parabolic equation was shown in [1]. In the present paper, some of these results are generalized to the equation with infinite delay. In this case, a variation of parameters formula with the analytic semigroup e^{At} is not available.

The difficulties connecting with the fully nonlinear character of the equation were in [1] overcome by replacing the space X by an interpolation space between $D(A)$ and X. These spaces and the maximal regularity properties for the linear parabolic problems were treated in [11]. Here, we make use of these spaces to prove the existence and the maximal regularity property of a resolvent operator, which satisfies the equation

$$\dot{R}(t) = AR(t) + LR_t, \quad R(0) = I, \quad R_0 = 0.$$

The operators $R(t)$ do not form a semigroup. However, we shall prove that the operators $S(t)\colon (x, \varphi) \longrightarrow (u(t), u_t)$, where u is a solution of the equation $\dot{u}(t) = Au(t) + Lu_t$, (0.2), given with help of the resolvent R, form a C_0-semigroup on a certain subspace of $X \times Y$. The equation (0.1) is then replaced by the equation

$$\dot{z}(t) = Bz(t) + h(z(t)), \quad z(t) = (u(t), u_t), \quad h(z(t)) = (g(u(t), u_t), 0)$$

and B is the generator of the semigroup S. Due to the special form of the semigroup S, estimates for projections and convolutions of S similar to those for analytic semigroups are proved. These estimates, which are not generally valid for C_0-semigroups enable us to construct stable and unstable manifolds in the usual way. The existence of a center manifold will be proved in a forthcoming paper.

The result is applied to the integrodifferential equation which can describe the heat conduction in materials with fading memory. there is a lot of papers describing the asymptotic behaviour of solutions of such equations, see e.g. [5] and references given there. The authors worked mostly in L_2-spaces with some positivity or monotonicity assumptions on the kernels, which are not necessery here, so our result does not seem to be contained in these papers.

1. Notations and preliminaries

Let A be the generator of an analytic semigroup e^{At} in a Banach space X. We introduce the interpolation spaces between $D(A)$ and X. Let $\|\cdot\|$ denote the norm in the space X, $\omega_0 = \{\sup \operatorname{Re}\lambda, \lambda \in \sigma(A)\}$. For $\omega_0 < 0, \alpha \in (0,1)$ we set

$$D_A(\alpha,\infty) = \{x \in X, |x|_\alpha = \sup_{\xi>0} \xi^{1-\alpha} \| Ae^{A\xi}x \| < \infty\},$$
$$D_A(\alpha+1,\infty) = \{x \in D(A), Ax \in D_A(\alpha,\infty)\},$$

For $\omega_0 \geqslant 0$ we set $D_A(\alpha,\infty) = D_{A-2\omega_0}(\alpha,\infty)$, $D_A(\alpha+1,\infty) = D_{A-2\omega_0}(\alpha+1,\infty)$. The closure of $D(A)$ in $D_A(\alpha,\infty)$ in the norm

$$\|x\|_\alpha = \|x\| + |x|_\alpha$$

will be denoted by X^α. It can be shown, (see [11]), that

$$X^\alpha = \{x \in X;\ \lim_{\xi\to 0+} \xi^{1-\alpha} Ae^{A\xi}x = 0\}.$$

We denote by $X^{\alpha+1}$ the closed subspace of $D_A(\alpha+1,\infty)$:

$$X^{\alpha+1} = \{x \in D(A);\ Ax \in X^\alpha\}, \quad \|x\|_{\alpha+1} = \|Ax\|_\alpha.$$

It is shown in [11] that there are constants $M > 0$, $\omega > \omega_0$, such that the following estimates hold:

$$\|R(\lambda,A)\|_{L(\bar X)} \leqslant \frac{M}{|\lambda-\omega|} \quad \text{for all } \lambda \in \mathbf{C},\ \operatorname{Re}\lambda > \omega, \tag{1.1}$$

$$\|AR(\lambda,A)\|_{L(\bar X)} \leqslant M,\ \operatorname{Re}\lambda > \omega, \tag{1.2}$$

$$\|e^{At}\|_{L(\bar X)} \leqslant Me^{\omega t}, \quad t \geqslant 0, \tag{1.3}$$

$$\|Ae^{At}\|_{L(\bar X)} \leqslant \frac{M}{t}e^{\omega t},\ t > 0, \tag{1.4}$$

$$\|Ae^{At}\|_{L(X^\alpha,X)} \leqslant \frac{M}{t^{1-\alpha}},\ t > 0, \tag{1.5}$$

$$x \in X^\alpha \ \Rightarrow\ \lim_{t\to 0+} \|e^{At}x - x\|_\alpha = 0, \tag{1.6}$$

where $\tilde{X}$ is any of the spaces $X, X^\alpha, X^{\alpha+1}$, $\alpha \in (0,1)$ and $R(\lambda, A) = (\lambda - A)^{-1}$.

Let $\mathbf{R}^+ = [0,+\infty)$, $\mathbf{R}^- = (-\infty, 0]$. For $\eta \in \mathbf{R}$ we denote by $C_\eta(\mathbf{R}^+, \tilde{X})$ (or $C_\eta(\mathbf{R}^-, \tilde{X})$) the set of all $f\colon [0,+\infty) \to \tilde{X}$, (or $(-\infty,0], \tilde{X}$) such that $t \to e^{\eta t} f(t)$ (or $t \to e^{-\eta t} f(t)$) is continuous and bounded. These spaces are endowed with the norms:

$$\|f\|_{C_\eta(\mathbf{R}^+,\tilde{X})} = \sup_{t\geqslant 0} \|e^{\eta t} f(t)\|_{\tilde{X}}$$
$$\|f\|_{C_\eta(\mathbf{R}^-,\tilde{X})} = \sup_{t\leqslant 0} \|e^{-\eta t} f(t)\|_{\tilde{X}}$$

The following lemma is proved in [1].

Lemma 1. *Let A satisfy (1.1)–(1.3), $h \in C_\eta(\mathbf{R}^+, X^\alpha)$ for $\eta < -\omega$, $k \in C_\eta(\mathbf{R}^-, X^\alpha)$ for $\eta > \omega$. If we set*

$$u(t) = \int_0^t e^{A(t-s)} h(s)\, ds, \quad t \geqslant 0, \tag{1.7}$$

$$v(t) = \int_{-\infty}^t e^{A(t-s)} k(s)\, ds, \quad t \leqslant 0, \tag{1.8}$$

then $u \in C_\eta(\mathbf{R}^+, X^{\alpha+1})$, $v \in C_\eta(\mathbf{R}^-, X^{\alpha+1})$.

Let $\gamma > 0$. Denote by Y^α the space of all functions $\varphi\colon (-\infty, 0) \to X^\alpha$ which are strongly measurable and

$$|\varphi|_{Y^\alpha} = \sup_{\xi>0} \xi^{1-\alpha} \int_{-\infty}^0 \|e^{\gamma\tau} A e^{A\xi} \varphi(\tau)\|\, d\tau < +\infty, \tag{1.9}$$

$$\lim_{\xi\to 0+} \xi^{1-\alpha} \int_{-\infty}^0 \|e^{\gamma\tau} A e^{A\xi} \varphi(\tau)\|\, d\tau = 0 \tag{1.10}$$

with the norm

$$\|\varphi\|_{Y^\alpha} = \int_{-\infty}^0 e^{\gamma\tau} \|\varphi(\tau)\|\, d\tau + |\varphi|_{Y^\alpha}.$$

Let $Y^{\alpha+1} = \{\varphi, (\tau \to A\varphi(\tau)) \in Y^\alpha\}$ and for some $\alpha \in (0,1)$ let

(1.11) *L be a continuous linear operator from $Y^{\alpha+1}$ into X^α.*

In the sequel we shall need some informations about the operator $L(\lambda)$, which is defined by:

$$L(\lambda)\colon X^{\alpha+1} \to X^\alpha, \quad L(\lambda)x = L(\tau \to e^{\lambda\tau} x). \tag{1.12}$$

Then $L(\lambda)$ is a continuous linear operator from $X^{\alpha+1}$ into X^α for $\operatorname{Re}\lambda > -\gamma$ and

$$\|L(\lambda)x\|_\alpha \leqslant \frac{\|L\|}{\gamma + \operatorname{Re}\lambda}\|x\|_{\alpha+1}.$$

Moreover, throughout the paper we shall suppose that

$$\|L(\lambda)R(\lambda, A)\|_{L(X^\alpha)} \leqslant \frac{C}{|\gamma+\lambda|^\beta}, \quad \operatorname{Re}\lambda > -\gamma, \ \beta > 0. \tag{1.13}$$

Remark. The operator L given by

$$L\varphi = \int_0^\infty e^{-\gamma s} A\varphi(-s)\,ds$$

can serve as a simple example satisfying the assumption (1.13) with $\beta = 1$.

Now, we can define the operator

$$D(\lambda) = (\lambda - A - L(\lambda))^{-1} \tag{1.14}$$

which plays the same role in construction of a resolvent operator $R(t)$ for the equation

$$\dot{u}(t) = Au(t) + Lu_t \tag{1.15}$$

as the resolvent $R(\lambda, A)$ for the semigroup e^{At}.

For λ such that $\operatorname{Re}\lambda > -\gamma$, $|\lambda|$ large enough, we have the expression

$$D(\lambda) = R(\lambda, A) + R(\lambda, A)\sum_{n=1}^{\infty}(L(\lambda)R(\lambda, A))^n \tag{1.16}$$

so that we have estimates similar to (1.1), (1.2):

$$\|D(\lambda)\|_{L(X^\alpha)} \leqslant \frac{C}{|\lambda|}, \quad \|AD(\lambda)\|_{L(X^\alpha)} \leqslant C, \quad |\lambda| \geqslant R_0, \ \operatorname{Re}\lambda > -\gamma. \tag{1.17}$$

Due to the continuity of $D(\lambda)$, the last inequality holds for all $\lambda \in \mathbf{C}$ such that $\operatorname{dist}(\lambda, \Sigma) \geqslant \varepsilon$, where $\Sigma = \{\lambda \in \mathbf{C};\ D(\lambda) \notin L(X^\alpha, X^{\alpha+1})\}$. From now on we shall denote by C any constant.

2. Construction and estimates of the resolvent operator

We will construct the resolvent operator $R(t)$ in such a way, that the Laplace transform of R will be $D(\lambda)$. To this end we will write $D(\lambda)$ as a sum:

$$D(\lambda) = R(\lambda, A) + R(\lambda, A)L(\lambda)R(\lambda, A) + \ldots + D(\lambda)(L(\lambda)R(\lambda, A))^n. \tag{2.1}$$

Let ϱ is the domain of analyticity of the function $D(\lambda)$ which has its values in $L(X^\alpha, X^{\alpha+1})$. Then

$$R_n(t)x = \int_{\delta-i\infty}^{\delta+i\infty} e^{\lambda t} D(\lambda)(L(\lambda)R(\lambda, A))^n x \, d\lambda \tag{2.2}$$

is the inverse Laplace transform of the last term in (2.1) provided that

$$n\beta > 1, \quad \delta > \sup\{\operatorname{Re}\lambda, \ \lambda \notin \varrho\}. \tag{2.3}$$

From (2.2) we obtain the estimates:

$$\|R_n(t)x\|_{\alpha+1} \leqslant Ce^{\delta t}\|x\|_\alpha, \quad \|\dot{R}_n(t)x\|_\alpha \leqslant Ce^{\delta t}\|x\|_\alpha, \quad t \geqslant 0. \tag{2.4}$$

In the same way as in [10] we can prove that the inverse Laplace transform of $R(\lambda, A)(L(\lambda)R(\lambda, A))^k$ is a convolution ($f * g = \int_0^t f(t-s)g(s)\,ds$)

$$B_k = e^{A\cdot} * H_k, \tag{2.5}$$

where

$$H_1(t)x = L\left(\tau \to \begin{cases} 0, & \text{for } \tau \leqslant -t \\ e^{A(t+\tau)}x, & \text{for } -t < \tau < 0 \end{cases}\right), \tag{2.6}$$

$$H_{n+1}x = H_1 * H_n x, \quad x \in X^\alpha. \tag{2.7}$$

Now, we can set

$$R(t) = e^{At} + \sum_{k=1}^{n-1} B_k(t) + R_n(t), \quad t \geqslant 0. \tag{2.8}$$

Proposition 1. *Let (1.1)–(1.6), (1.11), (1.13), (1.17) be fulfilled. Let $x \in X^\alpha$. Then $R(t)x \in X^{\alpha+1}$ for $t > 0$, R is differentiable on $(0, +\infty)$ and there is $K > 0$ such that*

$$\|AR(t)x\|_\alpha + \|\dot{R}(t)x\|_\alpha \leqslant \frac{K}{t} e^{dt}\|x\|_\alpha, \quad t > 0, \tag{2.9}$$

where $d > \max(\delta, -\frac{1}{2}\gamma)$, *with* δ *given in* (2.3). *Moreover, if* $x \in X^{\alpha+1}$, *then* $R(\cdot)x \in C(\mathbf{R}^+, X^{\alpha+1}) \cap C^1(\mathbf{R}^+, X^\alpha)$ *and*

$$\|R(t)x\|_{\alpha+1} + \|\dot{R}(t)x\|_\alpha \leqslant K e^{dt} \|x\|_{\alpha+1}, \quad t \geqslant 0. \tag{2.10}$$

Proof. First, let us suppose that (1.1)–(1.4) hold with $\omega = -\gamma$. Then it is sufficient to prove the estimate (2.9) for B_k only. The first and the last terms in (2.8) have been estimated in (1.4) and (2.4) respectively.

Let us estimate $H_1(t)x$ for $x \in X^\alpha$. According to (2.6), it means to estimate the $Y^{\alpha+1}$-norm of the function ψ_t, where

$$\psi_t(\tau) = \begin{cases} 0, & \text{for } \tau \leqslant -t \\ e^{A(t+\tau)}x, & \text{for } -t < \tau < 0 \end{cases}$$

Making use of (1.3), (1,5) we get

$$\int_{-\infty}^0 e^{\gamma\tau} \|A\psi_t(\tau)\| \, d\tau = \int_0^t e^{-\gamma s} \|e^{A\frac{t-s}{2}} A e^{A\frac{t-s}{2}} x\| \, ds$$
$$\leqslant C \int_0^t e^{-\gamma s} e^{\gamma \frac{t-s}{2}} \left(\frac{t-s}{2}\right)^{\alpha-1} \|x\|_\alpha ds \leqslant C e^{-\gamma \frac{t}{2}} \|x\|_\alpha,$$

$$\begin{aligned}
|\psi_t|_{Y^{\alpha+1}} &= \sup_{\xi>0} \xi^{1-\alpha} \int_{-\infty}^0 \|e^{\gamma\tau} A^2 e^{A\xi} \psi(\tau)\| \, d\tau \\
&= \sup_{\xi>0} \xi^{1-\alpha} \int_0^t \|e^{-\gamma s} A^2 e^{A\xi} e^{A(t-s)} x\| \, ds \\
&= \sup_{\xi>0} \xi^{1-\alpha} \int_0^t \|e^{-\gamma s} A e^{A(\xi+t-s)/2} A e^{A(\xi+t-s)/2} x\| \, ds \\
&\leqslant \sup_{\xi>0} \xi^{1-\alpha} C \int_0^t e^{-\gamma s} . 2 e^{-\gamma(\xi+t-s)/2} . (\xi + t - s)^{-1} . \left(\frac{\xi+t-s}{2}\right)^{\alpha-1} \|x\|_\alpha \, ds \\
&\leqslant \sup_{\xi>0} \xi^{1-\alpha} C e^{-\gamma t/2} . \int_0^t (\xi + t - s)^{-2+\alpha} ds \|x\|_\alpha \leqslant C e^{-\gamma t/2} \|x\|_\alpha .
\end{aligned}$$

Next, we have to prove that

$$\lim_{\xi \to 0+} \xi^{1-\alpha} \int_{-\infty}^0 \|e^{\gamma\tau} A^2 e^{A\xi} \psi_t(\tau)\| \, d\tau = 0.$$

Let us choose $\varepsilon > 0$. Then there is $\delta_1 > 0$ such that $\|Ae^{As}x\| < s^{\alpha-1}\cdot\frac{\varepsilon(1-\alpha)}{8M}$ for $s < \delta_1$ and M given in (1.4). Then for $\xi < \delta_1$ we get:

$$
\begin{aligned}
&\xi^{1-\alpha}\int_{t-\delta_1}^{t}\|e^{-\gamma s}A^2e^{A(t-s+\xi)}x\|\,ds \leqslant \\
(2.11)\qquad &\leqslant \xi^{1-\alpha}\int_{t-\delta_1}^{t}\left(\frac{t-s+\xi}{2}\right)^{-2+\alpha} ds\cdot\frac{\varepsilon(1-\alpha)}{8} < \frac{\varepsilon}{2}.
\end{aligned}
$$

Now we choose $\delta \leqslant \delta_1$ such that $\xi^{1-\alpha}\|Ae^{A\xi}x\| < \frac{\varepsilon\gamma\delta_1}{2M}$ whenever $\xi < \delta$. Then

$$
\xi^{1-\alpha}\int_0^{t-\delta_1}\|e^{-\gamma s}A^2e^{A(t-s+\xi)}x\|\,ds \leqslant \int_0^{t-\delta_1} e^{-\gamma s}(t-s)^{-1}\cdot\frac{\varepsilon\gamma\delta_1}{2}\,ds < \frac{\varepsilon}{2}.
$$

As $H_1(t) \doteq L\psi_t$ and $H_k = H_{k-1} * H_1$, we get the estimate:

$$
(2.12)\qquad \|H_k(t)\|_{L(X^\alpha)} \leqslant Ct^{k-1}e^{-\gamma t/2}.
$$

To prove the continuity of H_1, let us consider the difference $\psi_{t+h} - \psi_t$.

$$
\begin{aligned}
&\|\psi_{t+h} - \psi_t\|_{Y^{\alpha+1}} \\
&\leqslant \int_{-t}^{0}\|e^{\gamma\tau}Ae^{A(t+\tau)}(e^{Ah}-I)x\|\,d\tau + \int_{-t-h}^{-t}\|e^{\gamma\tau}Ae^{A(t+h+\tau)}x\|\,d\tau \\
&+ \sup_{\xi>0}\xi^{1-\alpha}\left[\int_{-t}^{0}\|e^{\gamma\tau}A^2e^{A(t+\tau+\xi)}(e^{Ah}-I)x\|\,d\tau + \int_{-t-h}^{-t}\|e^{\gamma\tau}A^2e^{A(t+h+\tau+\xi)}x\|\,d\tau\right].
\end{aligned}
$$

In the first and the third terms we make use of (1.6), the second term we estimate with help of (1.5). The last term we estimate for small $\xi < \delta$ in the same way as in (2.11) and then we realize that $\sup_{\xi\geqslant\delta}\xi^{1-\alpha}\int_0^h(s+\xi)^{-2+\alpha}ds\|x\|_\alpha \to 0$ for $h \to 0$.

Using the same procedure as above and making use of Lemma 1, we obtain the estimate for $B_k = e^{A\cdot} * H_k$:

$$
(2.13)\qquad \|B_k(t)x\|_{\alpha+1} \leqslant Ct^{k-1}e^{-\gamma t/2}\|x\|_\alpha.
$$

If A has its spectrum also on the right of the line $\operatorname{Re}\lambda = -\gamma$, we take $\tilde{A} = A - aI$, $\tilde{L}(\lambda) = L(\lambda) + aI$ so that $\tilde{A}$ fulfills the estimates (1.1)–(1.4) with $\omega = -\gamma$. Then

$$
\begin{aligned}
D(\lambda) = (\lambda - \tilde{A} - \tilde{L}(\lambda))^{-1} &= R(\lambda,\tilde{A}) + aR(\lambda,\tilde{A})^2 + R(\lambda,\tilde{A})L(\lambda)R(\lambda,\tilde{A}) \\
&+ R(\lambda,\tilde{A})((L(\lambda)+aI)R(\lambda,\tilde{A}))^2 + \ldots + D(\lambda)(\tilde{L}(\lambda)R(\lambda,\tilde{A}))^n
\end{aligned}
$$

and

$$
R(t) = e^{\tilde{A}t} + ate^{\tilde{A}t} + \sum_{k=1}^{n-1}\tilde{B}_k(t) + B_n(t) + \tilde{R}_n(t).
$$

Here $\tilde{B}_k$, $\tilde{R}_n$ correspond to B_k, R_n respectively (see (2.5), (2.2)) with A replaced by $\tilde{A}$ and B_n is the inverse Laplace transform of the remaining terms. These terms are analytic in $\operatorname{Re}\lambda > -\gamma$ and have sufficient decay so that the inverse Laplace transform exists and the estimate

$$\|B_n(t)x\|_{\alpha+1} \leqslant C\mathrm{e}^{-\gamma t/2}\|x\|_\alpha \tag{2.14}$$

holds. Further,

$$\|\tilde{L}(\lambda)R(\lambda,\tilde{A})x\|_\alpha \leqslant \|L(\lambda)R(\lambda,\tilde{A})x\|_\alpha + a\|R(\lambda,\tilde{A})x\|_\alpha \leqslant \frac{C}{|\lambda+\gamma|^\beta}\|x\|_\alpha,$$

for $\operatorname{Re}\lambda > \gamma$, which implies that the estimate (2.4) remains valid with $\tilde{R}_n$ instead of R_n. Now, (1.4) and (2.11) with $\tilde{A}$, $\tilde{B}_k$ instead of A, B_k, (2.4) with $\tilde{R}_n$ instead of R_n and (2.12) give the estimate of $\|AR(t)x\|_\alpha$. The same estimate for the derivative $\dot{R}(t)$ we get analogously if we realize that $H_k \in C(R^+, X^\alpha)$ and then

$$\dot{B}_k(t)x = H_k(t)x + \int_0^t A\mathrm{e}^{A(t-s)}H_k(s)x\,\mathrm{d}s.$$

The estimate (2.10) follows immediately from (1.3), (2.4), (2.13). □

The following lemma is similar to the Lemma 1.

Lemma 2. *Let the assumptions of the Proposition 1 hold. Let $h \in C_\eta(\mathbf{R}^+, X^\alpha)$ with $\eta < \min\left(\frac{1}{2}\gamma, -\delta\right)$, $k \in C_\mu(\varrho, X^\alpha)$ with $\mu > \max\left(-\frac{1}{2}\gamma, \delta\right)$. Set*

$$u(t) = \int_0^t R(t-s)h(s)\,\mathrm{d}s, \quad t \geqslant 0, \tag{2.15}$$

$$v(t) = \int_{-\infty}^t R(t-s)k(s)\,\mathrm{d}s, \quad t \leqslant 0. \tag{2.16}$$

Then $u \in C_\eta(\mathbf{R}^+, X^{\alpha+1}) \cap C^1_\eta(\mathbf{R}^+, X^\alpha)$, $v \in C_\eta(\mathbf{R}^-, X^{\alpha+1}) \cap C^1_\eta(\mathbf{R}^-, X^\alpha)$ and

$$\sup_{t\geqslant 0}\|\mathrm{e}^{\eta t}\dot{u}(t)\|_\alpha + \sup_{t\geqslant 0}\|\mathrm{e}^{\eta t}u(t)\|_{\alpha+1} \leqslant C_1(\eta)\sup_{t\geqslant 0}\|\mathrm{e}^{\eta t}h(t)\|_\alpha, \tag{2.17}$$

$$\sup_{t\leqslant 0}\|\mathrm{e}^{-\mu t}\dot{v}(t)\|_\alpha + \sup_{t\leqslant 0}\|\mathrm{e}^{-\mu t}v(t)\|_{\alpha+1} \leqslant C_2(\mu)\sup_{t\leqslant 0}\|\mathrm{e}^{-\mu t}k(t)\|_\alpha. \tag{2.18}$$

Proof. The proof is similar to that of the Proposition 1. Again, Lemma 1 and the estimates (2.4), (2.14) prove the assertion for two parts of R. Now, using (2.12) and arguing as before we prove that

$$\begin{aligned}\Big\|\mathrm{e}^{\eta t}\int_0^t B_k(t-s)h(s)\,\mathrm{d}s\Big\|_{\alpha+1} &= \Big\|\int_0^t\int_0^{t-s} \mathrm{e}^{\eta(t-s)}\mathrm{e}^{A(t-s-\sigma)}H_k(\sigma)\mathrm{e}^{\eta s}h(s)\,\mathrm{d}\sigma\,\mathrm{d}s\Big\|_{\alpha+1}\\ &\leqslant \sup_{\xi>0}\xi^{1-\alpha}\int_0^t\int_0^{t-s}\|A^2\mathrm{e}^{A(t-s-\sigma+\xi)}\mathrm{e}^{\eta(t-s)}H_k(\sigma)\mathrm{e}^{\eta s}h(s)\|\,\mathrm{d}\sigma\,\mathrm{d}s\\ &\leqslant C\sup_{\xi>0}\xi^{1-\alpha}\int_0^t\int_0^{t-s}\Big(\frac{t-s-\sigma+\xi}{2}\Big)^{-2+\alpha}\mathrm{e}^{-\gamma\frac{t-s-\sigma+\xi}{2}}\sigma^{k-1}\mathrm{e}^{-\frac{\gamma\sigma}{2}+\eta(t-s)}\mathrm{d}\sigma\mathrm{d}s\\ &\qquad\times\sup_{s\geqslant 0}\|\mathrm{e}^{\eta s}h(s)\|_\alpha \leqslant C\sup_{t\geqslant 0}\|\mathrm{e}^{\eta t}h(t)\|_\alpha.\end{aligned}$$

The derivative $\dot{u}(t)$ exists in X and $\dot{u}(t) = h(t)+\int_0^t \dot{R}(t-s)h(s)\,\mathrm{d}s$. Again, decomposing $\dot{R}$ in three terms we get the rest of the estimate (2.17). The proof of (2.18) is analogous.

In the following, we shall define $R(t)$ by (2.8) for $t \geqslant 0$, $R(t) = 0$ for $t < 0$. Then we can define $R_t\colon (-\infty, 0) \to L(X^\alpha)$, $R_t(\tau)x = R(t+\tau)x$. Similarly as above we can prove that $R_t x \in Y^{\alpha+1}$ provided that $x \in X^\alpha$ and

$$\|R_t x\|_{Y^{\alpha+1}} \leqslant C\mathrm{e}^{\delta t}\|x\|_\alpha,\quad \|R_t x\|_{Y^{\alpha+1}} \to 0 \text{ for } t\to 0. \tag{2.19}$$

The decomposition (2.1) with $n = 1$ yields that R satisfies the equation

$$R(t)x = \mathrm{e}^{At}x + \int_0^t \mathrm{e}^{A(t-s)}LR_s x\,\mathrm{d}s,\quad x\in X^\alpha,\ t\geqslant 0$$

and Proposition 1 now implies that

$$\dot{R}(t)x = AR(t)x + LR_t x,\quad x\in X^\alpha,\ t>0,\quad (x\in X^{\alpha+1},\ t\geqslant 0). \tag{2.20}$$

□

Now, a solution of a nonhomogeneous linear initial-value problem can be given with help of R.

Lemma 3. *Let $x \in X^{\alpha+1}$, $\varphi \in Y^{\alpha+1}$, $h \in C_\eta(\mathbf{R}^+, X^\alpha)$ with $\eta < \min\left(\frac{1}{2}\gamma, -\delta\right)$. Let us define $\varphi(t) = 0$ for $t \geqslant 0$. Then the problem*

$$\dot{u}(t) = Au(t) + Lu_t + h(t)\quad t>0, \tag{2.21}$$

$$u(0) = x,\quad u(\tau) = \varphi(\tau)\ \ \text{for } \tau < 0 \tag{2.22}$$

has a unique solution $u \in C_\eta(\mathbf{R}^+, X^{\alpha+1}) \cap C^1_\eta(\mathbf{R}^+, X^\alpha)$ given by

$$u(t) = R(t)x + \int_0^t R(t-s)(L\varphi_s + h(s))\,\mathrm{d}s. \tag{2.23}$$

Proof. The only thing to be proved is that the function $l(s) = L\varphi_s$ belongs to $C_\eta(R^+, X^\alpha)$ for $\varphi \in Y^{\alpha+1}$.

$$\begin{aligned}\|\varphi_s\|_{Y^{\alpha+1}} &= \int_{-\infty}^{-s} e^{\gamma\tau}\|\varphi(s+\tau)\|\,d\tau + \sup_{\xi>0}\xi^{1-\alpha}\int_{-\infty}^{-s}\|e^{\gamma\tau}A^2e^{A\xi}\varphi(s+\tau)\|\,d\tau \\ &= \int_{-\infty}^{0} e^{\gamma(\tau-s)}\|\varphi(\tau)\|\,d\tau + \sup_{\xi>0}\xi^{1-\alpha}\int_{-\infty}^{0}\|e^{\gamma(\tau-s)}A^2e^{A\xi}\varphi(\tau)\|\,d\tau \\ &= e^{-\gamma s}\|\varphi\|_{Y^{\alpha+1}}.\end{aligned} \tag{2.24}$$

The continuity of l can be proved in a similar way as the continuity of H_1.

$$\begin{aligned}|\varphi_{t+h}-\varphi_t|_{Y^{\alpha+1}} = \sup_{\xi>0}\xi^{1-\alpha}\Big[\int_{-\infty}^{-t-h} & e^{\gamma\tau}\|A^2e^{A\xi}\varphi(t+h+\tau)-\varphi(t+\tau)\|\,d\tau \\ &+\int_{-t-h}^{-t}e^{\gamma\tau}\|A^2e^{A\xi}\varphi(t+\tau)\|\,d\tau\Big] = \max\big(\sup_{0<\xi<\delta}(\ldots),\sup_{\xi\geqslant\delta}(\ldots)\big).\end{aligned}$$

First, we choose δ so that the first supremum is sufficiently small and then we find h to make the second one small enough. The assertion now follows easily. □

3. Solution semigroup

The solution of the problem (2.21), (2.22) is given with help of the resolvent operator $R(t)$, which has most of the properties of the analytic semigroup e^{At}, but the operators $R(t)$, $t \geqslant 0$ do not form a semigroup. However, if we define the operator $S(t)\colon (x,\varphi) \to (u(t), u_t)$, where u is a solution of the problem with $h = 0$, we get a semigroup on the space $Z^\alpha = X^\alpha \times Y^{\alpha+1}$:

$$S(t)\begin{pmatrix}x\\ \varphi\end{pmatrix} = \begin{pmatrix}R(t)x + \int_0^t R(t-s)L\varphi_s\,ds\\ \varphi_t + R_t x + \int_0^t R_{t-s}L\varphi_s\,ds\end{pmatrix}. \tag{3.1}$$

Proposition 2. *Let $S(t)$ be defined by (3.1) for $t \geqslant 0$. Then $\{S(t)\}$ is a C_0-semigroup of linear operators in the space $Z^\alpha = X^\alpha \times Y^{\alpha+1}$. Its generator B is given by:*

$$D(B) = \{(x,\varphi) \in Z^\alpha,\ x \in X^{\alpha+1},\ \dot\varphi \in Y^{\alpha+1},\ \lim_{\tau\to 0}\varphi(\tau) = x\}, \tag{3.2}$$

$$B\begin{pmatrix}x\\ \varphi\end{pmatrix} = \begin{pmatrix}Ax + L\varphi\\ \dot\varphi\end{pmatrix}. \tag{3.3}$$

$\lambda \in \mathbf{C}$ is in $\varrho(B)$, the resolvent set of B iff $\operatorname{Re}\lambda > -\gamma$ *and $D(\lambda) \in L(X^\alpha, X^{\alpha+1})$.*

Proof. The semigroup property and the continuity of S follow from its definition, (2.8), (2.19) and the continuity of $t \to \varphi_t$. Let Δ be defined by the right hand side of (3.2) and let $(x, \varphi) \in \Delta$. Then according to Lemma 3

$$u(t) = R(t)x + \int_0^t R(t-s)L\varphi_s \, \mathrm{d}s$$

is a strict solution of the equation (2.19) with $h = 0$. It follows that

$$\begin{gathered} \dot u(0) = Ax + L\varphi, \quad \frac{\mathrm{d}}{\mathrm{d}t} u_t|_{t=0}(\tau) = \dot\varphi(\tau), \quad \tau < 0 \\ \Rightarrow \ \Delta \subseteq D(B), \ B\begin{pmatrix} x \\ \varphi \end{pmatrix} = \begin{pmatrix} Ax + L\varphi \\ \dot\varphi \end{pmatrix}. \end{gathered}$$

On the other hand, let $(x, \varphi) \in D(B)$. Then there are $\lambda \in \varrho(B)$, $(y, \psi) \in X^\alpha \times Y^{\alpha+1}$, such that $(x, \varphi) = (\lambda - B)^{-1}(y, \psi)$. A direct computation yields the expression for $(\lambda - B)^{-1}$, $\lambda \in \varrho(B)$:

$$(3.4)\ (\lambda - B)^{-1}\begin{pmatrix} y \\ \psi \end{pmatrix} = \begin{pmatrix} D(\lambda)(y + L(\theta \to \int_\theta^0 \mathrm{e}^{\lambda(\theta-\sigma)}\psi(\sigma)\,\mathrm{d}\sigma)) \\ \tau \to \mathrm{e}^{\lambda\tau} D(\lambda)(y + L(\theta \to \int_\theta^0 \mathrm{e}^{\lambda(\theta-\sigma)}\psi(\sigma)\,\mathrm{d}\sigma)) + \int_\tau^0 \mathrm{e}^{\lambda(\tau-\sigma)}\psi(\sigma)\,\mathrm{d}\sigma \end{pmatrix}$$

As $D(\lambda)\colon X^\alpha \to X^{\alpha+1}$, we get $x \in X^{\alpha+1}$. Further $\dot\varphi(\tau) = \lambda\varphi(\tau) - \psi(\tau)$, $\varphi(0) = x \Rightarrow (x, \varphi) \in \Delta$. □

The assertion about the spectrum of the operator B follows easily from the expression (3.4).

Now, if we denote $z(t) = (u(t), u_t)$, then the problem (2.21), (2.22) can be rewritten in the following form:

$$(3.5) \qquad \dot z(t) = Bz(t) + \begin{pmatrix} h(t) \\ 0 \end{pmatrix}, \quad z(0) = \begin{pmatrix} x \\ \varphi \end{pmatrix}.$$

In the sequel we shall suppose that

$$(3.6) \qquad \sigma(B) \cap \mathrm{i}\mathbf{R} = \emptyset, \quad \sup \operatorname{Re} \sigma^-(B) < \lambda_1 < 0 < \lambda_2 < \inf \operatorname{Re} \sigma^+(B),$$

where $\sigma^-(B)\,(\sigma^+(B))$ denote the corresponding parts of $\sigma(B)$ with negative (positive) real parts.

We shall denote by P^+ the projection operator

$$(3.7) \qquad P^+ = \frac{1}{2\pi\mathrm{i}} \int_\Gamma R(\lambda, B)\,\mathrm{d}\lambda,$$

where Γ is a suitable path around the bounded set $\sigma^+(B)$ with $\operatorname{Re} z > \lambda_2$ for $z \in \Gamma$. Further, let $P^- = I - P^+$, $Z^- = P^-(Z^\alpha)$, $Z^+ = P^+(Z^\alpha)$,

$$(3.8) \qquad S^+(t) = \frac{1}{2\pi i}\int_\Gamma e^{\lambda t} R(\lambda, B)\,d\lambda, \quad t \in \mathbf{R}, \quad S^-(t) = S(t) - S^+(t), \quad t \geqslant 0.$$

From this expression we get the following estimate of the operator $S^+(t)$:

$$(3.9) \qquad \|S^+(t)\|_{L(Z^\alpha, D(B))} \leqslant C e^{\lambda_2 t}, \; t \leqslant 0.$$

To get the estimate for the operator $S^-(t)$ we need the decomposition of $D(\lambda)$.

As we have seen in the proof of Proposition 1, we can assume, without loss of generality, that $\operatorname{Re}\sigma(A) < -\gamma$. Then in the decomposition (2.1) of $D(\lambda)$ all terms but the last one are analytic in the halfplane $\operatorname{Re}\lambda > -\gamma$. Let us define

$$(3.10) \qquad R_n(\lambda) = D(\lambda)(L(\lambda)R(\lambda, A))^n,$$

$$(3.11) \qquad R^+(t)x = \int_\Gamma e^{\lambda t} D(\lambda)x\,d\lambda = \int_\Gamma e^{\lambda t} R_n(\lambda)x\,d\lambda,$$

$$(3.12) \qquad R^-(t) = R(t) - R^+(t) \quad \text{for } t \in \mathbf{R}.$$

It means that $R^-(t) = -R^+(t)$ for $t < 0$ and

$$R^-(t) = e^{At} + \sum_{k=1}^{n-1} B_k(t) + \int_{\lambda_1 - i\infty}^{\lambda_1 + i\infty} e^{\lambda t} R_n(\lambda)\,d\lambda \quad \text{for } t \geqslant 0.$$

Then (1.3), (1.4) with $\omega = -\gamma$, (2.4), (2.17) with $\delta = \lambda_1$, (2.9), (3.10)–(3.12) yield the estimates

$$(3.13) \qquad \|R^-(t)x\|_{\alpha+1} \leqslant \frac{C}{t} e^{-at}\|x\|_\alpha \quad \text{for } t > 0, \; a < \min\left(\frac{1}{2}\gamma, -\lambda_1\right),$$

$$(3.14) \qquad \|R^-(t)x\|_{\alpha+1} \leqslant C e^{-at}\|x\|_{\alpha+1}, \quad t \geqslant 0,$$

$$(3.15) \qquad \|R^-(\tau)x\|_{\alpha+1} \leqslant C e^{\lambda_2 \tau}\|x\|_\alpha, \quad \tau < 0,$$

$$(3.16) \qquad \|R_t^- x\|_{Y^{\alpha+1}} \leqslant C e^{-at}\|x\|_\alpha \quad t \geqslant 0.$$

For $\lambda > -\gamma$ we denote by Φ_λ the function

$$\Phi_\lambda(\tau) = \int_\tau^0 e^{\lambda(\tau - \sigma)}\varphi(\sigma)\,d\sigma = \int_0^\infty e^{\lambda t}\varphi_\tau(t)\,dt.$$

Then $\|\Phi\|_{Y^{\alpha+1}} \leqslant \frac{1}{\gamma + \operatorname{Re}\lambda}\|\varphi\|_{Y^{\alpha+1}}$ and we get the following formula for the operator $S^-(t) = S(t) - S^+(t)$, $t \geqslant 0$:

$$S^-(t)\begin{pmatrix} x \\ \varphi \end{pmatrix} = \begin{pmatrix} y(t) \\ \psi(t) \end{pmatrix},$$

where

$$y(t) = R^-(t)x + \int_0^t [e^{A(t-s)} + \sum_{k=1}^{n-1} B_k(t-s)]L\varphi_s ds + \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} e^{\lambda t} R_n(\lambda) L\Phi_\lambda d\lambda,$$

$$\psi(t)(\tau) = \begin{cases} y(t+\tau) & \text{for } -t \leqslant \tau < 0 \\ R^-(t+\tau)x + \varphi(t+\tau) - \frac{1}{2\pi i}\int_\Gamma e^{\lambda(t+\tau)} R_n(\lambda) L\Phi_\lambda d\lambda & \text{for } \tau < -t \end{cases}$$

With help of this expression, (2.24), (3.13)–(3.16) we get the estimate of $S^-(t)$:

$$\|S^-(t)\|_{L(\tilde{Z})} \leqslant Ce^{-at}, \quad t \geqslant 0, \tag{3.17}$$

where $\tilde{Z}$ is any of the spaces Z^α, Z,

$$\begin{aligned} Z &= \{z = (x,\varphi) \in Z^\alpha;\ x \in X^{\alpha+1},\ \lim_{\tau\to 0^-} \varphi(\tau) = x\}, \\ \|z\|_Z &= \|x\|_{\alpha+1} + \|\varphi\|_{Y^{\alpha+1}}. \end{aligned} \tag{3.18}$$

It is easily seen from (3.4), (3.8) that

$$S^+(t)\begin{pmatrix} x \\ 0 \end{pmatrix} = \begin{pmatrix} R^+(t)x \\ R_t^+ x \end{pmatrix} \Rightarrow S^-(t)\begin{pmatrix} x \\ 0 \end{pmatrix} = \begin{pmatrix} R^-(t)x \\ R_t^- x \end{pmatrix}. \tag{3.19}$$

Next, in the same way as in Lemma 2 we prove that

$$\sup_{t\geqslant 0} \left\| e^{\eta t} \int_0^t S^-(t-s) \begin{pmatrix} h(s) \\ 0 \end{pmatrix} ds \right\|_Z \leqslant C(\eta) \sup_{t\geqslant 0} \|e^{\eta t} h(t)\|_\alpha, \tag{3.20}$$

$$\sup_{t\leqslant 0} \left\| e^{-\mu t} \int_{-\infty}^t S^-(t-s) \begin{pmatrix} k(s) \\ 0 \end{pmatrix} ds \right\|_Z \leqslant C(\mu) \sup_{t\leqslant 0} \|e^{-\mu t} k(t)\|_\alpha, \tag{3.21}$$

provided that $\eta < a$, $\mu > -a$, $h \in C_\eta(\mathbf{R}^+, X^\alpha)$, $k \in C_\mu(\mathbf{R}^-, X^\alpha)$.

Now we can prove the existence of a stable and unstable manifolds for the equation

$$\dot{z}(t) = Bz(t) + \begin{pmatrix} g(z(t)) \\ 0 \end{pmatrix}, \quad z(0) = z_0, \tag{3.22}$$

$$g \in C^1(Z, X^\alpha), \quad g(0) = 0, \quad Dg(0) = 0. \tag{3.23}$$

Theorem. *Assume that A is a generator of an analytic semigroup in X, L satisfies* (1.11), (1.13) *and* (3.6), (3.23) *hold. Then there exists $r > 0$ and two differentiable functions*

$$h\colon B(0,r) \subset Z^- \cap Z \to Z,$$

$$k\colon B(0,r) \subset Z^+ \to Z,$$

such that, setting

$$\mathscr{S} = \{h(\zeta);\ \zeta \in B(0,r) \subset Z^- \cap Z\},$$
$$\mathscr{U} = \{k(\tilde{\zeta});\ \tilde{\zeta} \in B(0,r) \subset Z^+\},$$

we obtain the following conclusions:

(i) $\mathscr{S}(\mathscr{U})$ *is tangent to* $Z^-(Z^+)$ *at the origin.*

(ii) *For any* $z_0 \in \mathscr{S}(z_0 \in \mathscr{U})$ *there is a mild solution* z *of the equation* (3.22) *which satisfies the initial condition* $z(0) = z_0$. *This solution is defined on* $\mathbf{R}^+(\mathbf{R}^-)$ *and* $\lim_{t\to+\infty} z(t) = 0$ ($\lim_{t\to-\infty} z(t) = 0$)

(iii) *The trajectory of a solution* z *belongs to* $\mathscr{S}(\mathscr{U})$ *provided that* $z(0) \in \mathscr{S}$ $(z(0) \in U)$, $\|z(0)\|_Z$ *is small enough.*

Proof. Consider the operator Π given by

(3.24)
$$\Pi(\zeta,z)(t) = z(t) - S^-(t)\zeta - \int_0^t S^-(t-s)\begin{pmatrix} g(z(s)) \\ 0 \end{pmatrix} \mathrm{d}s + \int_t^\infty S^+(t-s)\begin{pmatrix} g(z(s)) \\ 0 \end{pmatrix} \mathrm{d}s$$

It follows from the definition, the estimates of S^+, S^- and (3.23) that this operator maps a neighbourhood of zero in the space $(Z\cap Z^-)\times C_\eta(\mathbf{R}^+,Z)$ into $C_\eta(\mathbf{R}^+,Z)$, it is continuously differentiable in both variables, $\Pi(0,0) = 0$, $D_2\Pi(0,0) = \mathrm{id}$. Using the Implicite Function Theorem we get the existence of $r > 0$, $r_1 > 0$ such that for each $\zeta \in B(r, Z\cap Z^-)$ $(B(r,X) = \{x \in X;\ \|x\|_X < r\})$ there is a unique $z(\zeta) \in B(r_1, C_\eta(\mathbf{R}^+,Z))$ with $\Pi(\zeta, z(\zeta)) = 0$ and z is continuously differentiable with respect to ζ.

Now we can define the function h: $h(\zeta) = z(\zeta)(0)$. The set $\mathscr{S}$ is a graph of a map Φ: $\Phi(\zeta) = -\int_0^\infty S^+(-s)\begin{pmatrix} g(z(s)) \\ 0 \end{pmatrix} ds$ for $\zeta \in B(r, Z\cap Z^-)$. As $\dot{\Phi}(0) = 0$, we get the assertion (i).

For $z_0 \in \mathscr{S}$ we have a function $z \in C_\eta(\mathbf{R}^+,Z)$, such that $\Pi(P^-z_0, z) = 0$. According to (3.24) $P^+z(0) = -\int_0^\infty S^+(-s)\begin{pmatrix} g(z(s)) \\ 0 \end{pmatrix} ds$ and

$$\begin{aligned} z(t) &= S^-(t)P^-z_0 + \int_0^t S^-(t-s)\begin{pmatrix} g(z(s)) \\ 0 \end{pmatrix} - \int_t^\infty S^+(t-s)\begin{pmatrix} g(z(s)) \\ 0 \end{pmatrix} \\ &= S(t)\left(P^-z_0 - \int_0^\infty S^+(-s)\begin{pmatrix} g(z(s)) \\ 0 \end{pmatrix}\right) + \int_0^t S(t-s)\begin{pmatrix} g(z(s)) \\ 0 \end{pmatrix} \\ &= S(t)z_0 + \int_0^t S(t-s)\begin{pmatrix} g(z(s)) \\ 0 \end{pmatrix} \end{aligned} \tag{3.25}$$

which proves (ii).

Let $z_0 \in \mathcal{S}$, $t_0 > 0$, $z(P^- z_0)$ be a solution of (3.22) given by (3.24). As the equation is autonomous, the function $u(t) = z(P^- z_0)(t + t_0)$ is also a solution of (3.22). Then

$$u(t) = [S^-(t) + S^+(t)]u(0) + \int_0^t [S^-(t-s) + S^+(t-s)] \begin{pmatrix} g(u(s)) \\ 0 \end{pmatrix} \mathrm{d}s.$$

Multiplying by $S^+(-t)$ and limiting for $t \to \infty$ we get

$$P^+ u(0) = -\int_0^\infty S^+(s) \begin{pmatrix} g(u(s)) \\ 0 \end{pmatrix} \mathrm{d}s$$

and in the same way as in (3.25) we obtain $\Pi(P^- u(0), u) = 0$. It follows that $u(0) = z(t_0) \in \mathcal{S}$ provided that $\|z_0\|_Z$ is so small that $\|P^- u(0)\|_Z < r$.

In the similar way, by solving the equation

$$(3.26)\quad z(t) = S^+(t)\tilde{\zeta} + \int_0^t S^+(t-s) \begin{pmatrix} g(z(s)) \\ 0 \end{pmatrix} \mathrm{d}s + \int_{-\infty}^t S^-(t-s) \begin{pmatrix} g(z(s)) \\ 0 \end{pmatrix} \mathrm{d}s$$

in a neighbourhood of zero in $Z^+ \times C_\mu(\mathbf{R}^-, Z)$, we obtain a backward solution which tends exponentially to zero when $t \to -\infty$. □

Remark. For the original problem (0.1), (0.2) we get the following assertions:

(i) For any $(x, \varphi) \in \mathcal{S}$ the solution of (0.1), (0.2) exists in the large. It belongs to $C_\eta(\mathbf{R}^+, X^{\alpha+1}) \cap C^1_\eta(\mathbf{R}^+, X^\alpha)$ with $\|e^{\eta t} u(t)\|_{\alpha+1} \leqslant r_1$. Conversely, if (x, φ) is such that $\|P^-(x, \varphi)\|_Z \leqslant r$, $u(., (x, \varphi)) \in C_\eta(\mathbf{R}^+, X^{\alpha+1})$ and $\|e^{\eta t} u(t)\|_{\alpha+1} \leqslant r_1$ for $t \geqslant 0$, then $(x, \varphi) \in \mathcal{S}$.

(ii) Any $(x, \varphi) \in \mathcal{U}$ satisfies the equation (0.1) for $t < 0$.

4. Example

Consider the problem

$$(4.1)\quad \begin{aligned} \dot{u}(t,x) &= \Delta u(t,x) + bu(t,x) + \int_0^\infty k_1(s)(\Delta u(t-s,x) + cu(t-s,x))\,\mathrm{d}s \\ &\quad + f(u(t,x), \Delta u(t,x)) + \int_0^\infty k_2(s) h(u(t-s,x), \Delta u(t-s,x))\,\mathrm{d}s, \end{aligned}$$

$$u(t,x) = 0 \text{ for } x \in \partial\Omega,\ t \in \mathbf{R},$$
$$u(0,x) = u_0(x) \text{ for } x \in \Omega,$$
$$u(\tau,x) = \varphi(\tau,x) \text{ for } \tau < 0,\ x \in \Omega.$$

We suppose that Ω is a bounded open set in $\mathbf{R}^n$ with a smooth boundary, f, h are smooth functions vanishing at zero together with their first derivatives,

$$(4.2) \qquad |h(p,q)| \leqslant C(|p|+|q|) \text{ for } p,q \in \mathbf{R},$$

$$(4.3) \qquad |k_i(s)| \leqslant C_i e^{-\gamma s} \quad \text{for } i=1,2 \ \ |\hat{k}_1(\lambda)| \leqslant \frac{C}{\lambda^\beta}, \ \beta > 0.$$

Now, we can rewrite the equation (4.1) in the form (0.1), setting

$$A = \Delta + bI,$$

$$L\psi(x) = \int_0^\infty k_1(s)(\Delta\psi(-s,x) + c\psi(-s,x))\,ds$$

$$g(v,\psi)(x) = f(v(x), \Delta v(x)) + \int_0^\infty k_2(s) h(\psi(-s,x), \Delta\psi(-s,x))\,ds.$$

It was shown in [5] that, taking $X = C(\bar{\Omega})$, $D(A) = \{u \in C^2(\bar{\Omega}),\ u\big|_{\partial\Omega} = 0\}$, we get $X^\alpha = h_0^{2\alpha}(\bar{\Omega})$, $X^{\alpha+1} = h_0^{2\alpha+2}(\bar{\Omega})$, where $h_0^\theta(\bar{\Omega})$ is the space of all functions $v\colon \bar{\Omega} \to \mathbf{R}$, such that $v/_{\partial\Omega} = 0$ and

$$\lim_{\delta\to 0} \sup_{|x-y|\leqslant\delta} \frac{|u(x)-u(y)|}{|x-y|^\theta} = 0, \quad h_0^{2+\theta} = \{u \in C^2(\bar{\Omega}), \Delta u \in h_0^\theta\}.$$

Then, owing to the assumptions on the functions f, h, (4.2), (4.3) it is easy to verify that g maps the space Z into X^α, L is a continuous linear operator from $Y^{\alpha+1}$ into X^α satisfying (1.13) and A is a generator of an analytic semigroup in X.

The relation between the eigenvalues of the Laplace operator and the Laplace transform of the kernel k_1 yields the values of the spectrum of the equation. In fact, for $v \in X^{\alpha+1}$ we have

$$L(\lambda)v = \int_0^\infty k_1(s) e^{-\gamma s}(\Delta + c)v\,ds = \hat{k}_1(\lambda)(\Delta + c)v.$$

Let $0 > \mu_1 > \mu_2 > \mu_3 > \dots$ be eigenvalues of the operator Δ. Then $\lambda \in \mathbf{C}$, such that $\operatorname{Re}\lambda > -\gamma$ is in the spectrum of the operator B (see (3.3)) iff

$$(\hat{k}_1(\lambda) + 1)\mu_n = \lambda - c\hat{k}_1(\lambda) - b \ \text{ for some } n \in \mathbf{N}.$$

It follows that for $b \leqslant 0$, $c \leqslant 0$, k_1 nonnegative, nonincreasing, the spectrum of B lies in the halfplane with negative real parts and 0 is asymptotically stable solution of (4.1).

If we take $k_1(s) = e^{-\gamma s}$, we have $\hat{k}_1(\lambda) = \frac{1}{\gamma+\lambda}$ and we get an instability of the zero solution whenever $b > \gamma - \mu_1$ or if $c + \mu_1 + \gamma(\mu_1 + b) > 0$. If, moreover, $c + \mu_n + \gamma(\mu_n + b) \neq 0$ for $n = 2, 3, \dots$, then we can apply Theorem 1 to get the saddle point property of the zero solution.

Remark. It is also possible to deal with integral operators with singular kernels of the type $t^{-\beta}e^{-\gamma t}$ for $\beta < 1$. The weight function $e^{\gamma\tau}$ in the definition (1.9) of the space Y^{α} should then be replaced by the function $(-\tau)^{-\beta}e^{\gamma\tau}$. All results remains valid with this change, only the proofs are a bit more complicated. The operator L given by

$$L\varphi = \int_0^{\infty} s^{-\beta} e^{-\gamma s} A\varphi(s)\,\mathrm{d}s$$

then satisfies the assumptions (1.11), (1.13).

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Souhrn

ŘEŠÍCÍ SEMIGRUPA A INVARIANTNÍ VARIETY PRO FUNKCIONÁLNÍ ROVNICE S NEKONEČNÝM ZPOŽDĚNÍM

HANA PETZELTOVÁ

V práci je ukázáno, že počáteční úloha pro funkcionální diferenciální rovnice parabolického typu definuje C_0-semigrupu na prostoru počátečních podmínek, jejíž vlastnosti dovolují zkonstruovat stabilní a nestabilní variety pro plně nelineární rovnice obvyklými metodami.

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