Miroslav Sova

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APPROXIMATIVE LIMIT AND RELATED NOTIONS

MIROSLAV SOVA, Praha

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Summary. Some notions of limit weaker than the topological one are studied.

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The purpose of this paper is to investigate certain generalized limits of functions, overlapping the topological limit which is in many cases too restrictive. The most natural and general of them is the approximative limit.

The generalized limits are useful in the study of traces of functions from various spaces of non-continuous functions.

We use the following notation:

$\mathbf{R}$ – the field of real numbers

$\mathbf{C}$ – the field of complex numbers

$\mathbf{R}^d$ – the euclidean coordinate space of dimension $d \in \{1, 2, \ldots\}$ with the norm denoted by $\|\cdot\|$,

$\mathrm{K}_h(\xi)$ – the set of all $\eta \in \mathbf{R}^d$ such that $\|\eta - \xi\| < h$ for arbitrary $\xi \in \mathbf{R}^d$ and $h \in \mathbf{R}$,

μ – the Lebesgue measure on $\mathbf{R}^d$,

$M_1 \to M_2$ – the set of all mappings of a set M_1 into the set M_2.

Let us mention that we can always understand measurable (set or function) as Borel measurable.

Lemma. *Let $\Omega \subseteq \mathbf{R}^d$ be measurable. For every $\xi \in \mathbf{R}^d$ and for every sequence $M_k \subseteq \mathbf{R}^d$, $k \in \{1, 2, \ldots\}$, such that*

(α) $\mu(\mathrm{K}_h(\xi) \cap \Omega) > 0$ *for every* $h > 0$,

(β) *the set M_k is a measurable subset of Ω for every $k \in \{1, 2, \ldots\}$,*

(γ) $\mu(\mathrm{K}_h(\xi) \cap M_k)/\mu(\mathrm{K}_h(\xi) \cap \Omega) \to 1$ $(h \to 0_+)$ *for every* $k \in \{1, 2, \ldots\}$,

there exist a set $M \subseteq \mathbf{R}^d$ and a real sequence q_k, $k \in \{1, 2, \ldots\}$, such that

(a) *the set M is a measurable subset of Ω,*

(b) $\mu(\mathrm{K}_h(\xi) \cap M)/\mu(\mathrm{K}_h(\xi) \cap \Omega) \to 1$ $(h \to 0_+)$,

(c) $q_k > 0$ *for every* $k \in \{1, 2, \ldots\}$,

(d) $\mathrm{K}_{q_k}(\xi) \cap M \subseteq M_k$ *for every* $k \in \{1, 2, \ldots\}$.

Proof. Let us fix a $\xi \in \mathbf{R}^d$ and a sequence $M_k \subseteq \mathbf{R}^d$, $k \in \{1, 2, \ldots\}$, such that the conditions $(\alpha)-(\gamma)$ hold.

For the sake of brevity, we write

(1) $\mathrm{K}_h = \mathrm{K}_h(\xi) \cap \Omega$ for $h \in \mathbf{R}$.

In view of (1), we can reformulate the conditions $(\alpha)-(\gamma)$ in the form

(2) $\mu(\mathrm{K}_h) > 0$ for every $h > 0$,

(3) the set M_k is a measurable subset of Ω for every $k \in \{1, 2, \ldots\}$,

(4) $\mu(\mathrm{K}_h \cap M_k)/\mu(\mathrm{K}_h) \to 1$ $(h \to 0_+)$ for every $k \in \{1, 2, \ldots\}$.

We see from (3) and (4) that

(5) $\mu(\mathrm{K}_h \setminus M_k)/\mu(\mathrm{K}_h) \to 0$ $(h \to 0_+)$ for every $k \in \{1, 2, \ldots\}$.

Let us now put

(6) $M_k^* = \bigcap_{j=1}^{k} M_k$ for every $k \in \{1, 2, \ldots\}$.

It is immediate from (6) that

(7) $M_k^* \subseteq M_k$ for every $k \in \{1, 2, \ldots\}$,

(8) $M_{k+1}^* \subseteq M_k^*$ for every $k \in \{1, 2, \ldots\}$.

By (2) and (6), we have

(9) the set M_k^* is measurable for every $k \in \{1, 2, \ldots\}$.

It follows from (1) and (6) that

(10) $\mathrm{K}_h \setminus M_k^* = \mathrm{K}_h \setminus \bigcap_{j=1}^{k} M_j \subseteq \bigcup_{j=1}^{k} (\mathrm{K}_h \setminus M_j)$ for every $h \in \mathbf{R}$ and $k \in \{1, 2, \ldots\}$.

Then we get from (1), (3), (9) and (10) that

(11) $\mu(\mathrm{K}_h \setminus M_k^*) \leqq \sum_{j=1}^{k} \mu(\mathrm{K}_h \setminus M_j)$ for every $h \in \mathbf{R}$ and $k \in \{1, 2, \ldots\}$.

An immediate consequence of (5) and (11) is

(12) $\mu(\mathrm{K}_h \setminus M_k^*)/\mu(\mathrm{K}_h) \to 0$ $(h \to 0_+)$ for every $k \in \{1, 2, \ldots\}$.

It is easy to see from (12) that we can fix a real sequence q_k, $k \in \{1, 2, \ldots\}$, such that

(13) $q_k > 0$ for every $k \in \{1, 2, \ldots\}$,

(14) $q_{k+1} \leqq q_k$ for every $k \in \{1, 2, \ldots\}$,

(15) $\mu(\mathrm{K}_h \setminus M_k^*)/\mu(\mathrm{K}_h) \leqq 1/k^2$

for every $h > 0$ and $k \in \{1, 2, \ldots\}$ such that $0 < h \leqq q_k$.

Let us now put

$$(16) \qquad \mathrm{M} = \bigcup_{k=1}^{\infty} (\mathrm{K}_{q_k} \setminus \mathrm{K}_{q_{k+1}}) \cap M_k^* .$$

It is immediate from (1), (9) and (16) that

(17) the set M is measurable.

Further, we see from (3), (7) and (16) that

$$(18) \qquad M \subseteq \bigcup_{k=1}^{\infty} M_k \subseteq \Omega .$$

Let us now recall that (14) implies

$$(19) \qquad \mathrm{K}_{q_k} \cap (\mathrm{K}_{q_j} \setminus \mathrm{K}_{q_{j+1}}) = \emptyset$$

for every $k \in \{2, 3, \ldots\}$ and $j \in \{1, 2, \ldots, k-1\}$.

It follows from (7), (8), (16) and (19) that

$$(20) \qquad \begin{aligned} \mathrm{K}_{q_k} \cap M &= \mathrm{K}_{q_k} \cap \bigcup_{j=1}^{\infty} (\mathrm{K}_{q_j} \setminus \mathrm{K}_{q_{j+1}}) \cap M_j^* = \\ &= \bigcup_{j=1}^{\infty} \mathrm{K}_{q_j} \cap (\mathrm{K}_{q_j} \setminus \mathrm{K}_{q_{j+1}}) \cap M_j^* = \\ &= \bigcup_{j=k}^{\infty} \mathrm{K}_{q_k} \cap (\mathrm{K}_{q_j} \setminus \mathrm{K}_{q_{j+1}}) \cap M_j^* \subseteq \\ &\subseteq \bigcup_{j=k}^{\infty} M_j^* = M_k^* \subseteq M_k \end{aligned}$$

for every $k \in \{1, 2, \ldots\}$.

Further, we see from (14) that

$$(21) \qquad \mathrm{K}_{q_k} = \bigcup_{j=k}^{\infty} (\mathrm{K}_{q_j} \setminus \mathrm{K}_{q_{j+1}}) \quad \text{for every} \quad k \in \{1, 2, \ldots\} .$$

Considering $h > 0$ and $k \in \{1, 2, \ldots\}$ such that $h \leqq q_k$, we get by use of (21) that

$$(22) \qquad \begin{aligned} \mathrm{K}_h \setminus M &= \mathrm{K}_h \setminus \Big(\bigcup_{j=1}^{\infty} (\mathrm{K}_{q_j} \setminus \mathrm{K}_{q_{j+1}}) \cap M_j^*\Big) \subseteq \\ &\subseteq \mathrm{K}_h \setminus \Big(\bigcup_{j=k}^{\infty} (\mathrm{K}_{q_j} \setminus \mathrm{K}_{q_{j+1}}) \cap M_j^*\Big) \subseteq \\ &\subseteq \mathrm{K}_{q_k} \setminus \Big(\bigcup_{j=k}^{\infty} (\mathrm{K}_{q_j} \setminus \mathrm{K}_{q_{j+1}}) \cap M_j^*\Big) = \\ &= \bigcup_{j=k}^{\infty} (\mathrm{K}_{q_j} \setminus \mathrm{K}_{q_{j+1}}) - \Big(\bigcup_{j=k}^{\infty} (\mathrm{K}_{q_j} \setminus \mathrm{K}_{q_{j+1}}) \cap M_j^*\Big) \subseteq \\ &\subseteq \bigcup_{j=k}^{\infty} \big((\mathrm{K}_{q_j} \setminus \mathrm{K}_{q_{j+1}}) \setminus (\mathrm{K}_{q_j} \setminus \mathrm{K}_{q_{j+1}}) \cap M_j^*\big) = \end{aligned}$$

$$= \bigcup_{j=k}^{\infty} ((\mathrm{K}_{q_j} \setminus \mathrm{K}_{q_{j+1}}) \setminus M_j^*) \subseteq \bigcup_{j=k}^{\infty} (\mathrm{K}_{q_j} \setminus M_j^*)$$

for every $h > 0$ and $k \in \{1, 2, \dots\}$ such that $h \leqq q_k$.

It is immediate from (22) that

$$(23) \qquad \begin{aligned} \mathrm{K}_h \setminus M &= \mathrm{K}_h \cap (\mathrm{K}_h \setminus M) \subseteq \\ &\subseteq \mathrm{K}_h \cap \bigcup_{j=k}^{\infty} (\mathrm{K}_{q_j} \setminus M_j^*) = \\ &= \bigcup_{j=k}^{\infty} \mathrm{K}_h \cap (\mathrm{K}_{q_j} \setminus M_j^*) \subseteq \\ &\subseteq \bigcup_{j=k}^{\infty} \mathrm{K}_h \cap \mathrm{K}_{q_j} \setminus M_j^* \end{aligned}$$

for every $h > 0$ and $k \in \{1, 2, \dots\}$ such that $h \leqq q_k$.

On the other hand, we get from (15) that

$$(24) \qquad \begin{aligned} \mu(\mathrm{K}_h \cap \mathrm{K}_{q_j} \setminus M_j^*) &= \mu(\mathrm{K}_{\min(h,q_j)} \setminus M_j^*) \leqq \\ &\leqq \frac{1}{j^2}\, \mu(\mathrm{K}_{\min(h,q_j)}) \leqq \\ &\leqq \frac{1}{j^2}\, \mu(\mathrm{K}_h) \end{aligned}$$

for every $h > 0$ and $j \in \{1, 2, \dots\}$.

Let us now fix a function $\mathrm{k}(\varepsilon)$, $\varepsilon > 0$, $\mathrm{k}(\varepsilon) \in \{1, 2, \dots\}$, such that

$$(25) \qquad \sum_{j=\mathrm{k}(\varepsilon)}^{\infty} \frac{1}{j^2} \leqq \varepsilon \quad \text{for every} \quad \varepsilon > 0\,.$$

We get from (23), (24) and (25) that

$$(26) \qquad \begin{aligned} \frac{\mu(\mathrm{K}_h \setminus M)}{\mu(\mathrm{K}_h)} &\leqq \frac{1}{\mu(\mathrm{K}_h)}\, \mu\Big(\bigcup_{j=\mathrm{k}(\varepsilon)}^{\infty} \mathrm{K}_h \cap \mathrm{K}_{q_j} \setminus M_j^*\Big) \leqq \\ &\leqq \frac{1}{\mu(\mathrm{K}_h)} \sum_{j=\mathrm{k}(\varepsilon)}^{\infty} \mu(\mathrm{K}_h \cap \mathrm{K}_{q_j} \setminus M_j^*) \leqq \sum_{j=\mathrm{k}(\varepsilon)}^{\infty} \frac{1}{j^2} \leqq \varepsilon \end{aligned}$$

for every $\varepsilon > 0$ and $0 < h \leqq q_{\mathrm{k}(\varepsilon)}$.

But (13) and (26) immediately give

$$(27) \qquad \mu(\mathrm{K}_h \setminus M)/\mu(\mathrm{K}_h) \to 0 \;\; (h \to 0_+)\,.$$

By (2), (18) and (27) we have

$$(28) \qquad \mu(\mathrm{K}_h \cap M)/\mu(\mathrm{K}_h) \to 1 \;\; (h \to 0_+)\,.$$

The conclusion of Lemma is now immediate from (1), (13), (17), (18), (20) and (28).

Remark 1. The reasoning in the proof of Lemma is due to Denjoy [1], p. 167. Cf. also [3], p. 288.

Remark 2. The above lemma is of some independent interest and has many applications. Moreover, this formal result brings significant clarification into many proofs. For these reasons, we presented this lemma as an independent item.

Proposition 1. *Let $\Omega \subseteq \mathbf{R}^d$ be measurable and $f \in \Omega \to \mathbf{C}$. If the function f is measurable, then for every $\xi \in \mathbf{R}^d$ such that $\mu(\mathrm{K}_h(\xi) \cap \Omega) > 0$ for $h > 0$ and for every $z \in \mathbf{C}$ the following statements (A) and (B) are equivalent:*

(A) *there exists a measurable set $M \subseteq \Omega$ such that $f(\eta) \to z$ $(\eta \to \xi, \eta \in M)$,*

$$\mu(\mathrm{K}_h(\xi) \cap M)/\mu(\mathrm{K}_h(\xi) \cap \Omega) \to 1 \ (h \to 0_+)\,,$$

(B) *there exists a sequence of sets $S_k \subseteq \mathbf{R}^d$, $k \in \{1, 2, \ldots\}$, and a real sequence r_k, $k \in \{1, 2, \ldots\}$, such that*

  (i) *$r_k > 0$ for every $k \in \{1, 2, \ldots\}$,*
  (ii) *$r_k \to 0$ $(k \to \infty)$,*
  (iii) *$S_k \subseteq \{\eta : \eta \in \Omega, (|f(\eta) - z| \leqq r_k\}$ for every $k \in \{1, 2, \ldots\}$,*
  (iv) *for every $k \in \{1, 2, \ldots\}$ and every measurable set $\tilde{S}_k \subseteq \Omega$ such that $\tilde{S}_k \supseteq S_k$, we have*

$$\mu(\mathrm{K}_h(\xi) \cap \tilde{S}_k)/\mu(\mathrm{K}_h(\xi) \cap \Omega) \to 1 \ (h \to 0_+)\,.$$

Proof. (A) $\Rightarrow$ (B): It suffices to put, for $k \in \{1, 2, \ldots\}$:

$$S_k = \{\eta : \eta \in M, \ |f(\eta) - z| \leqq 1/k\}\,,$$
$$r_k = 1/k\,.$$

Then the properties (B) (i)–(iv) are immediate from (A) by virtue of the presumed measurability of f.

(B) $\Rightarrow$ (A): Let us first fix sequences S_k, r_k, $k \in \{1, 2, \ldots\}$, with properties (B) (i)–(iv).

Further, we put

(1) $$M_k = \{\eta : \eta \in \Omega, \ |f(\eta) - z| \leqq r_k\} \quad \text{for} \quad k \in \{1, 2, \ldots\}\,.$$

It is immediate from (1) by virtue of the presumed measurability of f that

(2) the set M_k is measurable for every $k \in \{1, 2, \ldots\}$.

Further, we see from (1) that

(3) $$M_k \subseteq \Omega \quad \text{for every} \quad k \in \{1, 2, \ldots\}\,.$$

Finally, we get from (iii), (iv), (1) and (2) that

(4) $$\mu(\mathrm{K}_h(\xi) \cap M_k)/\mu(\mathrm{K}_h(\xi) \cap \Omega) \to 1 \ (h \to 0_+) \quad \text{for every} \quad k \in \{1, 2, \ldots\}\,.$$

Since $\mu(\mathrm{K}_h(\xi) \cap \Omega) > 0$ for every $h > 0$, we see from (2), (3) and (4) that Lemma can be applied. Hence we fix a set $M \subseteq \mathbf{R}^d$ and a real sequence q_k, $k \in \{1, 2, \ldots\}$, such that

(5) the conditions (a)–(d) from Lemma are fulfilled.

By (3) and (5) sub (a), we have

(6) the set M is a measurable subset of Ω.

Further, it follows from (1) and (5) sub (d) that

(7) $|f(\eta) - z| \leqq r_k$ for every $k \in \{1, 2, \ldots\}$ and $\eta \in \mathrm{K}_{q_k} \cap M$.

Now we conclude from (i), (ii), (5) sub(c) and (7) that

(8) $f(\eta) \to z \ (\eta \to \xi,\ \eta \in M)$.

The statement (A) follows from (5) sub(b), (6) and (8).

Remark 3. The property (A) from Proposition 1 describes the notion usually called the *approximative limit.*

More precisely, let $\Omega \subseteq \mathbf{R}^d$ be measurable, $f \in \Omega \to \mathbf{C}$, $\xi \in \mathbf{R}^d$ and $z \in \mathbf{C}$. We say that the function f has the approximative limit z at the point ξ if $\mu(\mathrm{K}_h(\xi) \cap \Omega) > 0$ for $h > 0$ and the condition (A) holds.

Cf. also [2] and [6].

Remark 4. Proposition 1 is a slight modification of a classical result of Denjoy [1], p. 167.

Theorem. *Let $\Omega \subseteq \mathbf{R}^d$ be measurable and $f \in \Omega \to \mathbf{C}$. If the function f is boundedly integrable over Ω (integrable over bounded measurable subsets of Ω), then for every $\xi \in \mathbf{R}^d$ such that $\mu(\mathrm{K}_h(\xi) \cap \Omega) > 0$ for $h > 0$ and for every $z \in \mathbf{C}$, the following statements* (A) *and* (B) *are equivalent:*

(A) (I) *there exists a measurable set $M \subseteq \Omega$ such that $f(\eta) \to z$ $(\eta \to \xi,\ \eta \in M)$,*

$$\mu(\mathrm{K}_h(\xi) \cap M)/\mu(\mathrm{K}_h(\xi) \cap \Omega) \to 1 \ (h \to 0_+),$$

(II) *for every measurable set $Q \subseteq \Omega$ such that*

$$\mu(\mathrm{K}_h(\xi) \cap Q)/\mu(\mathrm{K}_h(\xi) \cap \Omega) \to 0 \ (h \to 0_+),$$

we have

$$\frac{1}{\mu(\mathrm{K}_h(\xi) \cap \Omega)} \int_{\mathrm{K}_h(\xi) \cap Q} |f(\eta)| \, \mathrm{d}\eta \to 0 \ (h \to 0_+).$$

(B) $$\frac{1}{\mu(\mathrm{K}_h(\xi) \cap \Omega)} \int_{\mathrm{K}_h(\xi) \cap \Omega} |f(\eta) - z| \, \mathrm{d}\eta \to 0 \ (h \to 0_+).$$

Proof. (A) $\Rightarrow$ (B): Let us first fix a measurable set $M \subseteq \Omega$ satisfying (A) (I). We then get from (A) (I) that

(1) $$\frac{1}{\mu(\mathrm{K}_h(\xi) \cap \Omega)} \int_{\mathrm{K}_h(\xi) \cap M} |f(\eta) - z| \, \mathrm{d}\eta \to 0 \ (h \to 0_+).$$

Let us now denote

(2) $Q = \Omega \setminus M$.

By (A) (I), it is clear from (2) that

$$(3) \qquad \mu(\mathrm{K}_h(\xi) \cap Q)/\mu(\mathrm{K}_h(\xi) \cap \Omega) \to 0 \ (h \to 0_+) .$$

In view of (3), we obtain from (A) (II) that

$$(4) \qquad \frac{1}{\mu(\mathrm{K}_h(\xi) \cap \Omega)} \int_{\mathrm{K}_h(\xi) \cap Q} |f(\eta)| \,\mathrm{d}\eta \to 0 \ (h \to 0_+) .$$

It is immediate that

$$(5) \qquad \frac{1}{\mu(\mathrm{K}_h(\xi) \cap \Omega)} \int_{\mathrm{K}_h(\xi) \cap Q} |f(\eta) - z| \,\mathrm{d}\eta \leqq$$

$$\frac{1}{\mu(\mathrm{K}_h(\xi) \cap \Omega)} \int_{\mathrm{K}_h(\xi) \cap Q} |f(\eta)| \,\mathrm{d}\eta + |z| \frac{\mu(\mathrm{K}_h(\xi) \cap Q)}{\mu(\mathrm{K}_h(\xi) \cap \Omega)} \quad \text{for every} \quad h > 0 .$$

It follows from (3), (4) and (5) that

$$(6) \qquad \frac{1}{\mu(\mathrm{K}_h(\xi) \cap \Omega)} \int_{\mathrm{K}_h(\xi) \cap Q} |f(\eta) - z| \,\mathrm{d}\eta \to 0 \ (h \to 0_+) .$$

Summing up (1) and (6), we immediately obtain (B) by virtue of (3).

(B) $\Rightarrow$ (A): We begin with

(B) $\Rightarrow$ (A) (I): Let us first define

$$(7) \qquad M_k = \{\eta \colon \eta \in \Omega,\ |f(\eta) - z| \leqq 1/k\} \quad \text{for} \quad k \in \{1, 2, \ldots\} .$$

It is clear from (7) that

(8) the set M_k is a measurable subset of Ω for every $k \in \{1, 2, \ldots\}$.

Now we need to prove that

$$(9) \qquad \mu(\mathrm{K}_h(\xi) \cap M_k)/\mu(\mathrm{K}_h(\xi) \cap \Omega) \to 1 \ (h \to 0_+) \quad \text{for every} \quad k \in \{1, 2, \ldots\} .$$

To this aim, let us proceed indirectly and suppose that (9) does not hold.

Then we can fix a $k \in \{1, 2, \ldots\}$, an $\varepsilon > 0$ and a real sequence h_r, $r \in \{1, 2, \ldots\}$, such that

$$(10) \qquad h_r \to 0_+ \ (r \to \infty) ,$$

$$(11) \qquad \mu(\mathrm{K}_{h_r}(\xi) \cap M_k)/\mu(\mathrm{K}_{h_r}(\xi) \cap \Omega) \leqq 1 - \varepsilon \quad \text{for every} \quad r \in \{1, 2, \ldots\} .$$

We get from (7), (8) and (11) that

$$(12) \qquad \frac{1}{\mu(\mathrm{K}_{h_r}(\xi) \cap \Omega)} \int_{A \cap \Omega} |f(\eta) - z| \,\mathrm{d}\eta \geqq$$

$$\geqq \frac{1}{\mu(\mathrm{K}_{h_r}(\xi) \cap \Omega)} \int_{A \cap \Omega \setminus M_k} |f(\eta) - z| \,\mathrm{d}\eta \geqq$$

$$\geqq \frac{1}{\mu(\mathrm{K}_{h_r}(\xi) \cap \Omega)} \frac{1}{k} \mu(\mathrm{K}_{h_r}(\xi) \cap \Omega \setminus M_k) =$$

$$= \frac{1}{k} \frac{\mu(\mathrm{K}_{h_r}(\xi) \cap \Omega \setminus M_k)}{\mu(\mathrm{K}_{h_r}(\xi) \cap \Omega)} =$$

$$= \frac{1}{k}\left(1 - \frac{\mu(\mathrm{K}_{h_r}(\xi) \cap M_k)}{\mu(\mathrm{K}_{h_r}(\xi) \cap \Omega)}\right) = \frac{\varepsilon}{k}$$

for every $r \in \{1, 2, \ldots\}$ (where $A = \mathrm{K}_{h_r}(\xi)$).

Because $k \in \{1, 2, \ldots\}$ and $\varepsilon < 0$, we see that (10) and (12) contradict (B) and this proves (9).

Taking the assumption $\mu(\mathrm{K}_h(\xi) \cap \Omega) > 0$ for $h > 0$ into account, we see from (8) and (9) that we can apply Lemma and consequently, we fix a set $M \subseteq \mathbf{R}^d$ and a real sequence q_k, $k \in \{1, 2, \ldots\}$, such that

(13) $\quad M$ is a measurable subset of Ω,

(14) $$\mu(\mathrm{K}_h(\xi) \cap M)/\mu(\mathrm{K}_h(\xi) \cap \Omega) \to 1 \ (h \to 0_+)\,,$$

(15) $$q_k > 0 \quad \text{for every} \quad k \in \{1, 2, \ldots\}\,,$$

(16) $$\mathrm{K}_{q_k}(\xi) \cap M \subseteq M_k \quad \text{for every} \quad k \in \{1, 2, \ldots\}\,.$$

Now we need to prove that

(17) $$f(\eta) \to z(\eta \to \xi,\ \eta \in M)\,.$$

Indeed, let $\varepsilon > 0$. We fix a $k \in \{1, 2, \ldots\}$ such that $1/k \leqq \varepsilon$. Then by (7), $|f(\eta) - z| \leqq \varepsilon$ for every $\eta \in M_k$. By (16), we get that $|f(\eta) - z| \leqq \varepsilon$ for every $\eta \in M$ such that $\|\eta - \xi\| < q_k$. Since $q_k > 0$ by (15), the statement (17) is proved.

The desired statement (A) (I) is contained in (13), (14) and (17).

We continue by

(B) $\Rightarrow$ (A) (II): To this aim, let us fix a measurable set $Q \subseteq \Omega$ such that

(18) $$\mu(\mathrm{K}_h(\xi) \cap Q)/\mu(\mathrm{K}_h(\xi) \cap \Omega) \to 0 \ (h \to 0_+)\,.$$

On the other hand, we can write

(19) $$\frac{1}{\mu(\mathrm{K}_h(\xi) \cap \Omega)} \int_{\mathrm{K}_h(\xi) \cap Q} |f(\eta)|\, \mathrm{d}\eta =$$

$$= \frac{1}{\mu(\mathrm{K}_h(\xi) \cap \Omega)} \int_{\mathrm{K}_h(\xi) \cap Q} |f(\eta) - z + z|\, \mathrm{d}\eta \leqq$$

$$\leqq \frac{1}{\mu(\mathrm{K}_h(\xi) \cap \Omega)} \int_{\mathrm{K}_h(\xi) \cap Q} |f(\eta) - z|\, \mathrm{d}\eta + |z| \frac{\mu(\mathrm{K}_h(\xi) \cap Q)}{\mu(\mathrm{K}_h(\xi) \cap \Omega)} \leqq$$

$$\leqq \frac{1}{\mu(\mathrm{K}_h(\xi) \cap \Omega)} \int_{\mathrm{K}_h(\xi) \cap \Omega} |f(\eta) - z|\, \mathrm{d}\eta + |z| \frac{\mu(\mathrm{K}_h(\xi) \cap Q)}{\mu(\mathrm{K}_h(\xi) \cap \Omega)}$$

for every $h > 0$.

The statement (A) (II) is now an immediate consequence of (18) and (19) if we use (B).

Remark 5. The property (A) (II) from Theorem seems not to be considered earlier. It describes a certain moderation of the function f at the point ξ.

More precisely, let $\Omega \subseteq \mathbf{R}^d$ be measurable, let $f \in \Omega \to \mathbf{C}$ be boundedly integrable and $\xi \in \mathbf{R}^d$. We will say that the function f is moderated at the point ξ if $\mu(\mathrm{K}_h(\xi) \cap \cap \Omega) > 0$ for $h > 0$ and the condition (A) (II) holds.

Remark 6. The property (B) from Theorem seems to have no name even if a special case appears often in literature with Ω open, $\xi \in \Omega$ and $z = f(\xi)$ when one speaks of the Lebesgue point ξ of the function f. One possible term for the property (B) is the *limit in the absolute mean.*

More precisely, let $\Omega \subseteq \mathbf{R}^d$ be measurable, let $f \in \Omega \to \mathbf{C}$ be boundedly integrable, $\xi \in \mathbf{R}^d$ and $z \in \mathbf{C}$. We will say that the function f has the limit in the absolute mean z at the point ξ if $\mu(\mathrm{K}_h(\xi) \cap \Omega) > 0$ for $h > 0$ and the condition (B) holds.

Remark 7. Let $\Omega \subseteq \mathbf{R}^d$ be open. It is immediate that every measurable essentially bounded function is moderated at every point of $\bar{\Omega}$ which proves one essentially known result that every measurable essentially bounded function has the limit in the absolute mean at every point of $\bar{\Omega}$ at which it has the approximative limit (and naturally both these limits coincide).

Remark 8. Let $\Omega \subseteq \mathbf{R}^d$ be open and let $f \in \Omega \to \mathbf{C}$ be boundedly integrable. It is easy to show that the upper and lower boundedness of a real function f at a point $\xi \in \Omega$ (in the sense of [4], p. 263) implies that the function f is moderated at the point ξ, which proves another essentially known result (see [4], p. 264): if the function f has the approximative limit z at the point ξ and if it is upper and lower bounded at this point, then it has the same limit in the absolute mean at this point.

Proposition 2. *The conditions* (A) (I) *and* (A) (II) *of Theorem are independent in the framework of functions boundedly integrable over* Ω.

Consequently, the implication (A) (I) $\Rightarrow$ (B) *is not generally true for functions boundedly integrable over* Ω.

Proof by example. Let $\Omega = (0, 1)$, $f(\xi) = 2^r$ for $\xi \in (1/2^r - 1/2^{2r}, 1/2^r)$ and $r \in \in \{1, 2, \ldots\}$, and $f(\xi) = 0$ otherwise.

It is easy to see that the function f is integrable over Ω.

We easily verify that the function f satisfies the condition (A) (I) with $\xi = 0$ and $z = 0$ if we choose

$$M = (0, 1) \setminus \bigcup_{r=1}^{\infty} (1/2^r - 1/2^{2r}, 1/2^r)\,.$$

It remains to prove that the function f does not satisfy the condition (A) (II) for $\xi = 0$. To this aim, it suffices to take $Q = \bigcup_{r=1}^{\infty} (1/2^r - 1/2^{2r}, 1/2^r)$ and use the evident fact that $2^r \int_0^{1/2^r} \to 2 (r \to \infty)$.

Remark 9. The example in the proof of Proposition 2 was adapted from [3], p. 291, footnote 30.

Proposition 3. *Let $\Omega \subseteq \mathbf{R}^d$ be measurable and $f \in \Omega \to \mathbf{C}$. If the function f is boundedly integrable over Ω, then for every $\xi \in \mathbf{R}^d$ such that $\mu(\mathrm{K}_h(\xi) \cap \Omega) > 0$ for $h > 0$ and for every $z \in \mathbf{C}$, the following statements* (A) *and* (B) *are equivalent:*

(A) $$\frac{1}{\mu(\mathrm{K}_h(\xi) \cap \Omega)} \int_{\mathrm{K}_h(\xi) \cap \Omega} |f(\eta) - z| \, \mathrm{d}\eta \to 0 \ (h \to 0_+),$$

(B) *for every sequence of measurable sets $Z_k \subseteq \mathbf{R}^d$, $k \in \{1, 2, \ldots\}$, for which there exists a real sequence h_k, $k \in \{1, 2, \ldots\}$, such that*

(i) $h_k > 0$ *for every* $k \in \{1, 2, \ldots\}$,
(ii) $h_k \to 0$ $(k \to \infty)$,
(iii) $Z_k \subseteq \mathrm{K}_{h_k}(\xi) \cap \Omega$ *for every* $k \in \{1, 2, \ldots\}$,
(iv) $$\inf_{k \in \{1,2,\ldots\}} \frac{\mu(Z_k)}{\mu(\mathrm{K}_{h_k}(\xi) \cap \Omega)} > 0,$$

we have

$$\frac{1}{\mu(Z_k)} \int_{Z_k} f(\eta) \, \mathrm{d}\eta \to z \ (k \to \infty).$$

Proof. (A) $\Rightarrow$ (B): Let $Z_k \subseteq \mathbf{R}^d$, $k \in \{1, 2, \ldots\}$, be a sequence of measurable sets for which we can fix a real sequence h_k, $k \in \{1, 2, \ldots\}$, such that the conditions (B) (i)–(iv) are satisfied.

Then we get from (A) using the properties (B) (i)–(iv) that

$$\left| \frac{1}{\mu(Z_k)} \int_{Z_k} f(\eta) \, \mathrm{d}\eta - z \right| \leqq$$

$$\leqq \frac{1}{\mu(Z_k)} \int_{Z_k} |f(\eta) - z| \, \mathrm{d}\eta \leqq$$

$$\leqq \frac{\mu(\mathrm{K}_{h_k}(\xi) \cap \Omega)}{\mu(Z_k)} \frac{1}{\mu(\mathrm{K}_{h_k}(\xi) \cap \Omega)} \int_{Z_k} |f(\eta) - z| \, \mathrm{d}\eta \leqq$$

$$\leqq \frac{\mu(\mathrm{K}_{h_k}(\xi) \cap \Omega)}{\mu(Z_k)} \frac{1}{\mu(\mathrm{K}_{h_k}(\xi) \cap \Omega)} \int_{A \cap \Omega} |f(\eta) - z| \, \mathrm{d}\eta \to 0 \ (k \to \infty)$$

(where $A = \mathrm{K}_{h_k}(\xi)$),

which proves (B).

(B) $\Rightarrow$ (A): It is easy to see that without loss of generality, we can suppose f and z real.

Let us first consider an arbitrary sequence h_k, $k \in \{1, 2, \ldots\}$, such that

(1) $h_k > 0$ for every $k \in \{1, 2, \ldots\}$,

(2) $h_k \to 0$ $(k \to \infty)$.

We first have to prove that

(3) $$\frac{1}{\mu(\mathrm{K}_{h_k}(\xi) \cap \Omega)} \int_{A \cap \Omega} |f(\eta) - z| \, \mathrm{d}\eta \to 0 \quad (k \to \infty).$$

To this aim, let us define

(4) $$Z_k = \{\eta\colon \eta \in \mathrm{K}_{h_k}(\xi) \cap \Omega,\ f(\eta) \geqq z\} \quad \text{for} \quad k \in \{1, 2, \ldots\}.$$

It is clear from (4) that

(5) Z_k is a measurable subset of Ω for every $k \in \{1, 2, \ldots\}$.

(6) $Z_k \subseteq \mathrm{K}_{h_k}(\xi) \cap \Omega$ for every $k \in \{1, 2, \ldots\}$.

It follows from (6) that

(7) for every sequence $k(i)$, $i \in \{1, 2, \ldots\}$, $k(i) \in \{1, 2, \ldots\}$, such that $k(i) \to \infty$ $(i \to \infty)$, we can find a subsequence $\bar{k}(i)$ such that the sequence

$$\frac{\mu(Z_{\bar{k}(i)})}{\mu(\mathrm{K}_{h_{\bar{k}(i)}}(\xi) \cap \Omega)}, \quad i \in \{1, 2, \ldots\}, \quad \text{is convergent}.$$

Let us first consider one possible case of (7), i.e.

(7′) $$\lim_{i \to \infty} \frac{\mu(Z_{\bar{k}(i)})}{\mu(\mathrm{K}_{h_{\bar{k}(i)}}(\xi) \cap \Omega)} = 0.$$

From (6) and (7′) we immediately get

(8) $$\lim_{i \to \infty} \frac{\mu(\mathrm{K}_{h_{\bar{k}(i)}}(\xi) \cap \Omega \setminus Z_{\bar{k}(i)})}{\mu(\mathrm{K}_{h_{\bar{k}(i)}}(\xi) \cap \Omega)} = 1.$$

In view of (1), (2), (5), (6) and (8) we conclude from (B) that

(9) $$\frac{1}{\mu(\mathrm{K}_{h_{\bar{k}(i)}}(\xi) \cap \Omega \setminus Z_{\bar{k}(i)})} \int_{B \cap \Omega \setminus Z_{\bar{k}(i)}} (f(\eta) - z) \, \mathrm{d}\eta \to 0 \quad (i \to \infty),$$

(where $B = \mathrm{K}_{h_{\bar{k}(i)}}(\xi)$).

Now (8) and (9) imply

(10) $$\frac{1}{\mu(\mathrm{K}_{h_{\bar{k}(i)}}(\xi) \cap \Omega)} \int_{B \cap \Omega \setminus Z_{\bar{k}(i)}} (f(\eta) - z) \, \mathrm{d}\eta \to 0 \ (i \to \infty).$$

On the other hand, we easily see that in view of (1), (2) and (7), (B) implies

(11) $$\frac{}{\mu(\mathrm{K}_{h_{\bar{k}(i)}}(\xi) \cap \Omega)} \int_{B \cap \Omega} (f(\eta) - z) \, \mathrm{d}\eta \to 0 \ (i \to \infty).$$

Combining (10) and (11), we get

(12) $$\frac{}{\mu(\mathrm{K}_{h_{\bar{k}(i)}}(\xi) \cap \Omega)} \int_{Z_{\bar{k}(i)}} (f(\eta) - z) \, \mathrm{d}\eta \to 0 \ (i \to \infty).$$

By (4), (6) and (12), we get

$$(13)\qquad \frac{1}{\mu(\mathrm{K}_{h_{\bar{k}(i)}}(\xi)\cap\Omega)}\int_{B\cap\Omega}\max\left(f(\eta)-z,0\right)\mathrm{d}\eta\to 0\ (i\to\infty)$$

if the assumption (7′) holds.

Let us now consider the second possible case of (7), i.e.

$$(7'')\qquad \lim_{i\to\infty}\frac{\mu(Z_{\bar{k}(i)})}{\mu(\mathrm{K}_{h_{\bar{k}(i)}}(\xi)\cap\Omega)}>0\,.$$

In view of (1), (2), (5), (6) and (7″) we conclude from (B) that

$$(14)\qquad \frac{1}{\mu(Z_{\bar{k}(i)})}\int_{Z_{\bar{k}(i)}}(f(\eta)-z)\,\mathrm{d}\eta\to 0\ (i\to\infty)\,.$$

Now it is immediate from (7″) and (14) that

$$(15)\qquad \frac{1}{\mu(\mathrm{K}_{h_{\bar{k}(i)}}(\xi)\cap\Omega)}\int_{Z_{\bar{k}(i)}}(f(\eta)-z)\,\mathrm{d}\eta\to 0\ (i\to\infty)\,.$$

By (4), (6) and (15) we can write

$$(16)\qquad \frac{1}{\mu(\mathrm{K}_{h_{\bar{k}(i)}}(\xi)\cap\Omega)}\int_{B\cap\Omega}\max\left(f(\eta)-z,0\right)\mathrm{d}\eta\to 0\ (i\to\infty)$$

if the assumption (7″) holds.

We conclude from (7), (7′), (7″), (13) and (16) that

(17) for every sequence $k(i)$, $i\in\{1,2,\ldots\}$, $k(i)\in\{1,2,\ldots\}$, such that $k(i)\to\infty$ $(i\to\infty)$, we can find a subsequence $\bar{k}(i)$ such that

$$\frac{1}{\mu(\mathrm{K}_{h_{\bar{k}(i)}}(\xi)\cap\Omega)}\int_{B\cap\Omega}\max\left(f(\eta)-z,0\right)\mathrm{d}\eta\to 0\ (i\to\infty)\,.$$

An immediate consequence of (17) is

$$(18)\qquad \frac{1}{\mu(\mathrm{K}_{h_k}(\xi)\cap\Omega)}\int_{A\cap\Omega}\max\left(f(\eta)-z,0\right)\mathrm{d}\eta\to 0\ (k\to\infty)\,.$$

Now, since the real sequence h_k, $k\in\{1,2,\ldots\}$, satisfying (1) and (2) was arbitrary, we get from (18) that

$$(19)\qquad \frac{1}{\mu(\mathrm{K}_h(\xi)\cap\Omega)}\int_{\mathrm{K}_h(\xi)\cap\Omega}\max\left(f(\eta)-z,0\right)\mathrm{d}\eta\to 0\ (h\to 0_+)\,.$$

In a completely similar way, we get

$$(20)\qquad \frac{1}{\mu(\mathrm{K}_h(\xi)\cap\Omega)}\int_{\mathrm{K}_h(\xi)\cap\Omega}\max\left(z-f(\eta),0\right)\mathrm{d}\eta\to 0\ (h\to 0_+)\,.$$

Since $\max\left(f(\eta)-z,0\right)+\max\left(z-f(\eta),0\right)=\left|f(\eta)-z\right|$ for every $\eta\in\Omega$, we see that (19) and (20) give (A).

Remark 10. As to the property (B) in Proposition 3, compare also the notion of regular derivative in [3], 17.4, which essentially originates from Lebesgue.

Remark 11. Let $\Omega \subseteq \mathbf{R}^d$ be measurable, $f \in \Omega \to \mathbf{C}$ boundedly integrable, $\xi \in \mathbf{R}^d$ and $z \in \mathbf{C}$. We say that the function f has the *limit in the mean* z at the point ξ if $\mu(\mathrm{K}_h(\xi) \cap \Omega) > 0$ for $h > 0$ and

$$\frac{1}{\mu(\mathrm{K}_h(\xi) \cap \Omega)} \int_{\mathrm{K}_h(\xi) \cap \Omega} f(\eta) \, \mathrm{d}\eta \to z \ (h \to 0_+) \, .$$

Remark 12. The limit in the mean is generally incomparable with the approximative limit in the framework of boundedly integrable functions, i.e. the existence of the limit in the mean does not imply the existence of approximative limit and conversely. (Use the following example: $\Omega = (-1, 1)$, $f(\eta) = -1$ for $\eta \in (-1, 0)$, $f(0) = 0$, $f(\eta) = 1$ for $\eta \in (0, 1)$ for the first statement and the example from Proposition 2 for the second statement.)

Remark 13. The above introduced notions of limits (approximative, in the absolute mean, in the mean) of functions are particularly important in the definition of traces of functions on the boundary, especially for functions from Sobolev and related spaces where the topological limit is too restrictive.

In this sense, the approximative limit was used in [8], p. 284, the limit in the absolute mean in [7], p. 192 and the limit in the mean in [5], p. 281.

We know from previous results that the limit in the absolute mean can always be replaced by the approximative limit.

As to the limit in the mean used by Burago and Mazja in [5], we will show in a subsequent paper that this limit can also be replaced by the approximative limit in the framework of the assumptions frequently introduced in Burago-Mazja's theory.

Remark 14. The results of this paper remain valid also for the more general situation when the set $\bar{\Omega}$ and the Lebesgue measure μ on $\bar{\Omega}$ are replaced by a general measure space (X, Σ, λ), and the system $\mathrm{K}_h(\xi) \cap \Omega$ (or better $\mathrm{K}_h(\xi) \cap \bar{\Omega}$), $\xi \in \bar{\Omega}$, $h > 0$, is replaced by appropriate filters associated to points $\xi \in \bar{\Omega}$ (filter differentiation basis). See e.g. [4], p. 162, 203 and elsewhere.

The above proofs require only some natural modifications.

Comment. The implication (B) $\Rightarrow$ (A) (I) in Theorem is known, usually in the form of continuity, i.e. $z = f(\xi)$. See e.g. [3], § 18. Our proof seems considerably simpler and more transparent thanks to the use of Lemma.

The condition (A) (II) in Theorem seems new and solves the problem of an additional condition necessary and sufficient for the approximative limit to be the limit in the absolute mean for boundedly integrable function.

Proposition 2 seems also new, at least we were not able to find it in the literature.

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Author's address: 289 14 Poříčany 371, Czechoslovakia.