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Mathematica Bohemica, Vol. 116 (1991), No. 4, 391–395

Persistent URL: http://dml.cz/dmlcz/126021

116 (1991) MATHEMATICA BOHEMICA No. 4, 391—395

VALUATIONS ON MODULAR LATTICES

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(Received December 8, 1989)

Summary. It is well-known that there exist infinite modular lattices possessing no non-trivial valuations. In this paper a class $\mathscr{K}$ of modular lattices is defined and it is proved that each lattice belonging to $\mathscr{K}$ has a nontrivial valuation. Next, a result of G. Birkhoff concerning valuations on modular lattices of finite length is generalized.

Keywords: modular lattice, valuation, discrete valuation.

AMS Classification: 06C05.

We denote by $\mathscr{K}$ the class of all modular lattices L which satisfy the following conditions:

(i) L has a prime interval.

(ii) If $a, b \in L$, $a < b$, then there are $a_0, a_1, a_2, \ldots, a_n$ in L such that $a = a_0 < a_1 < \ldots < a_n = b$ and for each $i \in \{1, 2, \ldots, n\}$ either a_{i-1} is covered by a_i, or the lattice $[a_{i-1}, a_i]$ has no prime interval.

It will be proved that each lattice belonging to $\mathscr{K}$ possesses a nontrivial valuation (Theorem 1). The notion of discrete valuation will be introduced. Theorem 2 concerning discrete valuations generalizes Birkhoff's theorem concerning valuations on modular lattices of finite length ([1], Chap. X, Theorem 7).

Valuations, metrics associated with valuations, and applications of this theory (including the applications in social sciences) were investigated in the expository paper [3].

In what follows we assume that L is a lattice belonging to $\mathscr{K}$.

For $a, b \in L$ with $a < b$ we denote by $S(a, b)$ the set of all finite sequences $(a_0, a_1, \ldots, a_n)$ with the properties as in the condition (ii) above. If $s = (a_0, a_1, \ldots, a_n) \in S(a, b)$, then we put $I(s) = \{i \in \{1, 2, \ldots, n\}: a_{i-1} \prec a_i\}$, where $\prec$ is the symbol denoting the covering relation.

Lemma 1. *Let* $a, b \in L$, $a < b$, $s = (a_0, a_1, \ldots, a_n) \in S(a, b)$, $s' = (b_0, b_1, \ldots, b_m) \in S(a, b)$. *Then*

(i) $\operatorname{card} I(s) = \operatorname{card} I(s')$;

(ii) *if* $\operatorname{card} I(s) \neq \emptyset$, *then there exists a one-to-one mapping* φ *of* $I(s)$ *onto* $I(s')$ *such that for each* $i \in I(s)$ *the interval* $[a_{i-1}, a_i]$ *is projective to the interval* $[b_{\varphi(i-1)}, b_{\varphi(i)}]$.

Proof. This is an immediate consequence of the Schreier-Zassenhaus Theorem; cf. also [1], Chap. III, Theorem 9, and Corollary to this theorem.

Let P be the set of all prime intervals of L. We denote by R the set of all reals. Let $f: P \to R$ be a mapping such that $f([u_1, v_1]) = f([u_2, v_2])$ whenever $[u_1, v_1]$ and $[u_2, v_2]$ are projective prime intervals of L.

For $a, b \in L$ with $a < b$ and $s = (a_0, a_1, a_2, \ldots, a_n) \in S(a, b)$ we put

$$d(a, b; f, s) = \sum f(a_{i-1}, a_i) \quad (i \in I(s)) .$$

From Lemma 1 we obtain:

Lemma 2. *Let $a, b \in L$, $a < b$. Next, let s and s' be elements of $S(a, b)$. Then $d(a, b; f, s) = d(a, b; f, s')$.*

In view of Lemma 2 we shall write $d(a, b; f)$ instead of $d(a, b; f, s)$. Next, Lemma 2 yields:

Lemma 3. *Let $a, b, c \in L$, $a < b < c$. Then $d(a, c; f) = d(a, b; f) + d(b, c; f)$.*

If $a = b$, then we set $d(a, b) = 0$.

Lemma 4. *Let $a, b, c \in L$, $a \vee b \leqq c$. Then*

$$d(a, a \vee b; f) - d(b, a \vee b; f) = d(a, c; f) - d(b, c; f) .$$

Proof. In view of Lemma 3 we have

$$d(a, c; f) = d(a, a \vee b; f) + d(a \vee b, c; f) ,$$
$$d(b, c; f) = d(b, a \vee b; f) + d(a \vee b, c; f) ,$$

which implies the assertion of the lemma.

The following lemma is a consequence of the well-known facts concerning projectivity in modular lattices; the proof will be omitted.

Lemma 5. *Let $[a, b]$ and $[a', b']$ be projective intervals in L. Then $d(a, b; f) = d(a', b'; f)$.*

Let x_0 be a fixed element of L. For each $a \in L$ we put

$$v_f(a) = d(x_0, x_0 \vee a; f) - d(a, x_0 \vee a; f) .$$

In view of Lemma 4 we have

$$v_f(a) = d(x_0, c; f) - d(a, c; f)$$

for each $c \in L$ with $c \geqq x_0 \vee a$.

Lemma 6. *Let $a, b \in L$, $a < b$. Then*

$$v_f(b) - v_f(a) = d(a, b; f) .$$

Proof. Put $c = x_0 \vee b$. Then

$$v_f(b) = d(x_0, c; f) - d(b, c; f),$$
$$v_f(a) = d(x_0, c; f) - d(a, c; f).$$

Now it suffices to apply Lemma 3.

Lemma 7. *v_f is a valuation on the lattice L.*

Proof. Let $a, b \in L$. We have to verify that

$$v_f(a) - v_f(a \wedge b) = v_f(a \vee b) - v_f(b) \tag{1}$$

is valid. In view of Lemma 6,

$$v_f(a) - v_f(a \wedge b) = d(a \wedge b, a; f),$$
$$v_f(a \vee b) - v_f(b) = d(b, a \vee b; f).$$

Since the intervals $[a \wedge b, a]$ and $[b, a \vee b]$ are projective, in view of Lemma 5 we infer that (1) is valid.

We can choose, e.g., $v_f([a_1, b_1]) = 1$ for each prime interval of L; then, because L has at least one prime interval, the valuation v_f is nontrivial (i.e., there are $a, b \in L$ with $v_f(a) \neq v_f(b)$). Hence we obtain

Theorem 1. *Let L be a lattice belonging to the class $\mathscr{K}$. Then L possesses a nontrivial valuation.*

A valuation v on L will be said to be discrete if, whenever a, b are elements of L such that $a < b$ and the lattice $[a, b]$ has no prime interval, then $v(a) = v(b)$.

Let v be a discrete valuation on L. For each prime interval $[a_1, b_1]$ in L put

$$f([a_1, b_1]) = v(b_1) - v(a_1).$$

If $[a_1, b_1]$ and $[a_2, b_2]$ are projective prime intervals of L, then we clearly have $f([a_1, b_1]) = f([a_2, b_2])$. The mapping f will be said to be generated by the valuation v. Let x_0 be a fixed element of L; next, let v_f and d have the same meaning as above.

Lemma 8. *Let v be a discrete valuation on L and let the mapping f be generated by v. Let $a, b \in L$, $a < b$. Then $v(b) - v(a) = d(a, b; f)$.*

Proof. Choose $(a_0, a_1, \ldots, a_n) \in S(a, b)$. Then

$$v(b) - v(a) = \sum (v(a_i) - v(a_{i-1})) \quad (i = 1, 2, \ldots, n).$$

Because v is a discrete valuation, we obtain

$$v(b) - v(a) = \sum (v(a_i) - v(a_{i-1})) \quad (i \in I(S)),$$

hence $v(b) - v(a) = d(a, b; f)$.

Theorem 2. *Let L be a lattice belonging to the class $\mathscr{K}$. Assume that v is a discrete valuation on L. Let $f: P \to R$ be a mapping of the set of all prime intervals of L into R which is generated by v. Let $x_0 \in L$ and let v_f be defined as above. Then $v(a) = v(x_0) + v_f(a)$ for each $a \in L$.*

Proof. According to the definition of $v_f(a)$ and in view of Lemma 8 we have

$$v_f(a) = d(x_0, x_0 \vee a; f) - d(a, x_0 \vee a; f) =$$
$$= v(x_0 \vee a) - v(x_0) - (v(x_0 \vee a) - v(a)) = v(a) - v(x_0).$$

If L is a modular lattice such that each bounded chain in L is finite and card $L > 1$, then obviously $L \in \mathscr{K}$; moreover, each valuation on such a lattice is discrete. Hence Theorem 7 in Chap. X, [1] is a consequence of Theorem 2 above.

A valuation v on a lattice L_1 will be said to be an i-valuation if $v(x)$ is an integer for each $x \in L_1$.

By looking at the proof of Theorem 1 we see that this result can be sharpened as follows: Each lattice belonging to $\mathscr{K}$ possesses a nontrivial i-valuation.

A valuation v on a lattice L_1 will be called positive if, whenever $a, b \in L_1$ and $a < b$, then $v(a) < v(b)$.

Let us denote by $\mathscr{K}_1$ the class of all modular lattices L_1 such that no interval of L_1 is projective to a proper part of itself.

It is obvious that if L_2 is a lattice which does not belong to $\mathscr{K}_1$, then L_2 does not possess any positive valuation.

In [1] (Problem 8.1) the question was proposed concerning the existence of nontrivial valuations on lattices belonging to $\mathscr{K}_1$. As far as I know, this problem is still open.

On the other hand, the existence of a nontrivial valuation on a lattice does not imply that this lattice belongs to $\mathscr{K}_1$.

The following example shows that there exists $L \in \mathscr{K}$ with the property that there is an interval in L which is projective to a proper part of itself.

Example. Let C be the interval $[0, 1]$ of reals with the natural linear order. Let M be as in [2], § IV 1, Exercise 28. Next, let $A = \{0, 1\}$ be a two-element lattice and $L = M \times A$. According to Exercise 29 (ibid.), M is a modular lattice. Hence L is a modular lattice as well.

It is easy to verify that there is no prime interval in M. If $m \in M$, then $[(m, 0), (m, 1)]$ is a prime interval in L. Let $(m_1, a_1), (m_2, a_2) \in L$, $(m_1, a_1) < (m_2, a_2)$. If $a_1 = a_2$, then there is no prime interval in the lattice $[(m_1, a_1), (m_2, a_2)]$ (since this is isomorphic to the interval $[m_1, m_2]$ of M). If $a_1 < a_2$, then $[(m_1, a_1), (m_1, a_2)]$ is a prime interval and the lattice $[(m_1, a_2), (m_2, a_2)]$ does not contain any prime interval. Thus L belongs to the class $\mathscr{K}$.

Let x be a real, $0 < x < 1$. Put

$$m_1 = (0, 0, 0), \quad m_2 = (x, 0, 0), \quad m_3 = (1, 0, 0).$$

Then $m_i \in M$ $(i = 1, 2, 3)$ and clearly the interval $[m_1, m_2]$ is a proper subset of $[m_1, m_3]$. In view of Exercise 30 (ibid.) the intervals $[m_1, m_2]$ and $[m_1, m_3]$ of M are projective (the results of the Exercises quoted above are due to E. T. Schmidt [4]).

Denote $v_i = (m_i, 0)$ $(i = 1, 2, 3)$. Then the interval $[v_1, v_2]$ is a proper subset of $[v_1, v_3]$, and the two intervals are projective in L.

A prime interval $[x, y]$ of a lattice L_1 will be said to be regular if the following condition is satisfied:

(iii) Whenever $a, b \in L_1$ and $a < b$, then there are $a_0, a_1, a_2, \ldots, a_n$ in L_1 such that $a = a_0 < a_1 < a_2 < \ldots < a_n = b$ and for each $i \in \{1, 2, \ldots, n\}$ either $[a_{i-1}, a_i]$ is projective to $[x, y]$, or no subinterval of $[a_{i-1}, a_i]$ is projective to $[x, y]$.

Theorem 3. *Let L_1 be a modular lattice possessing a regular prime interval $[x, y]$. Then there exists an i-valuation v on L_1 such that $v(y) - v(x) = 1$.*

The proof requires steps analogous to those which are applied in the proof of Theorem 1 (with the distinction that the system of all prime intervals is now replaced by the system of all prime intervals which are projective to $[x, y]$). The details will be omitted.

The following questions remain open:

(1) Does there exist a lattice possessing a nontrivial valuation which has no nontrivial i-valuation?

(2) Let L_1 be a modular lattice having a prime interval; does L_1 possess a nontrivial valuation?

References

[1] *G. Birkhoff:* Lattice Theory, Providence 1967.

[2] *G. Grätzer:* General Lattice Theory, Akademie Verlag, Berlin, 1978.

[3] *B. Monjardet:* Metrics on partially ordered sets — a survey. Discrete Math. *35* (1981), 173—184.

[4] *E. T. Schmidt:* Über die Kongruenzverbände der Verbände. Publ. Math. Debrecen 9 (1962), 245—256.

Súhrn

VALUÁCIE NA MODULÁRNYCH ZVÄZOCH

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Je známe, že existujú modulárne zväzy, na ktorých nie je možné definovať netriviálnu valuáciu. V práci sa definuje trieda $\mathscr{K}$ modulárnych zväzov a dokazuje sa, že pre každý sväz tejto triedy existuje netriviálna valuácia. Ďalej sa v článku zovšeobecňuje veta G. Birkhoffa o valuáciách modulárnych zväzov konečnej dĺžky.

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