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ON THE LIMITS OF SOLUTIONS OF FUNCTIONAL DIFFERENTIAL EQUATIONS

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Summary. Our aim in this paper is to obtain sufficient conditions under which for every $\xi \in R^n$ there exists a solution x of the functional differential equation

$$\dot{x}(t) = \int_c^t [\mathrm{d}_s Q(t,s)]\, f(t, x(s)), \qquad t \in [t_0, T)$$

such that $\lim\limits_{t\to T-} x(t) = \xi$.

Keywords: functional differential equation, solution, delay, completeness

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1. INTRODUCTION

Let R^n, M_n denote respectively the n-dimensional Euclidean space and the set of all $n \times n$-matrices with real entries. The norm of a vector $x \in R^n$ is denoted by $|x|$ and the norm of a matrix $A \in M_n$ is defined by $\|A\| = \sup\{\,|Ax| \mid x \in R^n,\ |x| = 1\,\}$.

Let us consider the functional differential equation

$$\dot{x}(t) = \int_c^t [\mathrm{d}_s Q(t,s)]\, f(t, x(s)), \qquad t \in [t_0, T) \tag{1}$$

where $-\infty < c \leqslant t_0 < T \leqslant \infty$ and the functions f and Q satisfy the following assumptions:

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(H_1) $f\colon [t_0,T) \times R^n \to R^n$ is a continuous function for which there exist a constant C and a continuous function $L\colon [t_0,T) \to [0,\infty)$ such that

$$(2) \qquad |f(t,x_1) - f(t,x_2)| \leqslant L(t)\,|x_1 - x_2|$$

and

$$(3) \qquad |f(t,0)| \leqslant C\,L(t)$$

hold for any $t \in [t_0,T)$ and $x_1,\, x_2 \in R^n$.

(H_2) $Q\colon [t_0,T) \times [c,T) \to M_n$ is a matrix-function of locally bounded variation in its second variable for every fixed value of its first variable. Moreover, the function

$$t \mapsto \int_c^t [\mathrm{d}_s Q(t,s)]\, y(t,s)$$

is continuous on $[t_0,T)$ whenever y is continuous on $\{\, (t,s) \mid t \in [t_0,T), s \in [c,t]\,\}$.

Throughout the paper, $q(t,s)$ denotes the total variation of $Q(t,\cdot)$ in the interval $[c,s]$ for any fixed $t \in [t_0,T)$ and $s \in (c,t]$. We put $q(t,c) = 0$, $t \in [t_0,T)$.

The integral in (1) is a Riemann-Stieltjes integral. By a *solution* of (1) we mean a function $x\colon [c,T) \to R^n$ which is continuous on $[c,t_0]$, differentiable on $[t_0,T)$ and satisfies (1).

In this paper, we seek for conditions which ensure that for every $\xi \in R^n$ there exists a solution x of (1) such that $\lim\limits_{t\to T-} x(t) = \xi$. Answering this question M. Švec [10] obtained a theorem which is essentially related to the following result known for ordinary differential equations:

Theorem 0 [6, p. 327]. *Consider the ordinary differential equation*

$$(4) \qquad \dot{x} = f(t,x), \qquad t \in [t_0,\infty)$$

where the function f satisfies condition (H_1) with $T = \infty$ and $C = 0$. If

$$(5) \qquad m_\infty = \int_{t_0}^{\infty} L(\tau)\,\mathrm{d}\tau < \infty$$

then

(i) *for every solution x of (4) there exists $\lim\limits_{t\to\infty} x(t) = \xi^* \in R^n$,*

(ii) *for every $\xi \in R^n$ there exists a solution x of (4) satisfying $\lim\limits_{t\to\infty} x(t) = \xi$.*

I. Győri has shown by an example (cf. [2, Example 3.1]) that Theorem 0 is not valid in an unaltered form for the retarded equation

$$\dot{x}(t) = f\big(t, x[t - r(t)]\big), \qquad t \geqslant t_0 \tag{6}$$

with continuous and bounded delay $r\colon [t_0, \infty) \to [0, \infty)$. We mention that in this example $m_\infty = 3$. For functional differential equations M. Švec has proved (cf. [10, Theorems 1 and 2]) the assertion of Theorem 0 under the additional assumption

$$m_\infty < \tfrac{1}{2}. \tag{7}$$

In his already mentioned work, I. Győri [2] has shown that condition (7) can be replaced by the weaker condition

$$m_\infty < 1. \tag{8}$$

It is to be noted that the above mentioned results are not applicable to differential equations with unbounded delay.

The aim of this paper is to give some general sufficient conditions under which for every $\xi \in R^n$ there exists a solution x of the functional differential equation (1) satisfying $\lim_{t \to T-} x(t) = \xi$.

In Section 2 of this paper we present our general results. Our main theorem (cf. Theorem 1) unites and generalizes Theorem 0 and the above mentioned result of M. Švec and I. Győri. As a consequence, we obtain a theorem (cf. Theorem 2) which is a generalization of the following result due to Yu. A. Ryabov [9]:

Assume that f satisfies condition (H_1) with $T = \infty$ and $L(t) = \lambda = const.$, r: $[t_0, \infty) \to R$ $(R = R^1)$ is a continuous function such that

$$0 \leqslant r(t) \leqslant \varrho, \qquad t \geqslant t_0$$

where ϱ is a positive constant. If $\lambda \varrho e < 1$ then for every $(t^*, x^*) \in [t_0, \infty) \times R^n$ there exists a solution x of equation (6) such that $x(t^*) = x^*$.

For further discussions we refer to the work of R. D. Driver [1], J. Jarník-J. Kurzweil [7] and I. Győri [2].

In Section 3 we apply our main results to differential equations of special form and also illustrate them by examples. Among other, we show (cf. Example 1) that condition (8) cannot be improved. It should be noted that our results concern the case of time dependent, not necessarily bounded lags.

2. MAIN RESULTS

For any $t \in [t_0, T)$ define the set

$$C_t = \{ u \mid u\colon [c,t] \to R^n, \ u \text{ continuous} \}.$$

With the usual supremum norm

$$\|u\|_t = \sup_{r\in[c,t]} |u(r)|, \qquad u \in C_t$$

the sets C_t ($t \in [t_0, T)$) are Banach spaces.

Let E_{t_0} denote the set of all continuous initial functions vanishing at the point t_0 (i.e. $E_{t_0} = \{ u \in C_{t_0} \mid u(t_0) = 0\}$).

Before we formulate our main theorem, let us introduce the following notions (cf. [8], [12], [2]):

Definition 1. Let $\Phi \in E_{t_0}$. We say that equation (1) is *Φ-complete at the point T*, if for any given $\xi \in R^n$ there exist $x_0 \in R^n$ and a solution x of (1) satisfying the initial condition

$$x(t) = \Phi(t) + x_0, \qquad t \in [c, t_0] \tag{9}$$

and such that

$$\lim_{t\to T-} x(t) = \xi. \tag{10}$$

Definition 2. Let $\Phi \in E_{t_0}$. We say that equation (1) is *pointwise Φ-complete*, if for every $(t^*, x^*) \in [t_0, T) \times R^n$ there exist $x_0 \in R^n$ and a solution x of (1) satisfying the initial condition (9) and such that $x(t^*) = x^*$.

Our main result is the following

Theorem 1. *Let functions f and Q satisfy conditions (H_1) and (H_2). Assume that there exist a constant $k \in (0,1)$ and a continuous function $m : [t_0, T) \to [0, \infty)$ satisfying the following conditions:*

$$q(t,t) \leqslant m(t), \qquad t \in [t_0, T), \tag{11}$$

$$\int_{t_0}^{T} L(\tau) m(\tau)\, \mathrm{d}\tau < \infty, \tag{12}$$

$$\int_c^t \left(\int_{\max\{t_0, s\}}^{T} L(\tau) m(\tau)\, \mathrm{d}\tau \right) \mathrm{d}_s q(t,s) \leqslant k\, m(t), \qquad t \in [t_0, T). \tag{13}$$

Then

(i) *for every solution x of (1) there exists*

$$\lim_{t \to T-} x(t) = \xi^* \in R^n, \tag{14}$$

(ii) *equation (1) is Φ-complete at the point T for every $\Phi \in E_{t_0}$,*

(iii) *equation (1) is pointwise Φ-complete for every $\Phi \in E_{t_0}$.*

Proof. (i) Suppose that x is a solution of (1). Then (by (2), (3), (11) and (12)) we have

$$\begin{aligned}
|x(t)| &= \left| x(t_0) + \int_{t_0}^{t} \left\{ \int_{c}^{\sigma} [\mathrm{d}_s Q(\sigma, s)]\, f\big(\sigma, x(s)\big) \right\} \mathrm{d}\sigma \right| \\
&\leqslant |x(t_0)| + \int_{t_0}^{t} \left\{ \int_{c}^{\sigma} |f\big(\sigma, x(s)\big)|\, \mathrm{d}_s q(\sigma, s) \right\} \mathrm{d}\sigma \\
&\leqslant |x(t_0)| + \int_{t_0}^{t} \left\{ \int_{c}^{\sigma} L(\sigma)[\,|x(s)| + C\,]\, \mathrm{d}_s q(\sigma, s) \right\} \mathrm{d}\sigma \\
&\leqslant |x(t_0)| + \int_{t_0}^{t} L(\sigma)[\,\|x\|_\sigma + C\,]\, q(\sigma, \sigma)\, \mathrm{d}\sigma \\
&\leqslant |x(t_0)| + C \int_{t_0}^{T} L(\sigma) m(\sigma)\, \mathrm{d}\sigma + \int_{t_0}^{t} L(\sigma) m(\sigma) \|x\|_\sigma\, \mathrm{d}\sigma
\end{aligned}$$

for all $t \in [t_0, T)$. Consequently

$$\begin{aligned}
\|x\|_t = \sup_{r \in [c,t]} |x(r)| &\leqslant \sup_{r \in [c,t_0]} |x(r)| + \sup_{r \in [t_0,t]} |x(r)| \\
&\leqslant 2\|x\|_{t_0} + C \int_{t_0}^{T} L(\sigma) m(\sigma)\, \mathrm{d}\sigma + \int_{t_0}^{t} L(\sigma) m(\sigma) \|x\|_\sigma\, \mathrm{d}\sigma
\end{aligned}$$

for $t \in [t_0, T)$. By Gronwall's lemma

$$\begin{aligned}
\|x\|_t &\leqslant \left[2\|x\|_{t_0} + C \int_{t_0}^{T} L(\sigma) m(\sigma)\, \mathrm{d}\sigma \right] \exp\left(\int_{t_0}^{t} L(\sigma) m(\sigma)\, \mathrm{d}\sigma \right) \\
&\leqslant \left[2\|x\|_{t_0} + C \int_{t_0}^{T} L(\sigma) m(\sigma)\, \mathrm{d}\sigma \right] \exp\left(\int_{t_0}^{T} L(\sigma) m(\sigma)\, \mathrm{d}\sigma \right) = K < \infty,
\end{aligned}$$

that is, x is bounded with a finite constant K.

Similarly as before, taking $t_0 \leqslant t_1 < t_2 < T$, we obtain

$$\begin{aligned}|x(t_1) - x(t_2)| &= \left| \int_{t_1}^{t_2} \left\{ \int_c^{\sigma} [\mathrm{d}_s Q(\sigma, s)]\, f(\sigma, x(s)) \right\} \mathrm{d}\sigma \right| \\ &\leqslant \int_{t_1}^{t_2} L(\sigma) m(\sigma) [C + \|x\|_{\sigma}]\, \mathrm{d}\sigma \leqslant (C + K) \int_{t_1}^{t_2} L(\sigma) m(\sigma)\, \mathrm{d}\sigma \\ &\leqslant (C + K) \int_{t_1}^{T} L(\sigma) m(\sigma)\, \mathrm{d}\sigma.\end{aligned}$$

In virtue of (12) the last estimate yields the existence of the limit (14). The proof of statement (i) is complete.

(ii) Let $\Phi \in E_{t_0}$, $h > 0$. Denote by X the Banach space of all continuous and bounded functions from $[c, T)$ into R^n with the norm

$$\|x\| = \sup_{t \in [c,T)} |x(t)| \left[\int_{\max\{t_0, t\}}^{T} L(\tau) m(\tau)\, \mathrm{d}\tau + h \right]^{-1}, \qquad x \in X.$$

If $x \in X$ then $|x(t)| \leqslant K$ for $t \in [c, T)$ with a finite constant K, and by simple calculation

$$\begin{aligned}(15)\quad &\int_t^T \left| \int_c^{\sigma} [\mathrm{d}_s Q(\sigma, s)]\, f(\sigma, x(s)) \right| \mathrm{d}\sigma \\ &\qquad\qquad \leqslant (C + K) \int_t^T L(\sigma) m(\sigma)\, \mathrm{d}\sigma < \infty, \qquad t \in [t_0, T).\end{aligned}$$

Let us fix $\xi \in R^n$ and define the operator F for any $x \in X$ by

$$Fx(t) = \begin{cases} \xi - \displaystyle\int_t^T \left\{ \int_c^{\sigma} [\mathrm{d}_s Q(\sigma, s)]\, f(\sigma, x(s)) \right\} \mathrm{d}\sigma & \text{for } t \in [t_0, T) \\ \Phi(t) + \xi - \displaystyle\int_{t_0}^T \left\{ \int_c^{\sigma} [\mathrm{d}_s Q(\sigma, s)]\, f(\sigma, x(s)) \right\} \mathrm{d}\sigma & \text{for } t \in [c, t_0]. \end{cases}$$

Operator F is defined for every $x \in X$ because of (15), and $Fx \in X$. We shall show that $h > 0$ can be chosen such that F is a contraction.

For any x, $y \in X$ and $t \in [c, T)$, we have

$$\begin{aligned}|Fx(t) - Fy(t)| &= \left| \int_{\max\{t_0, t\}}^{T} \left\{ \int_c^{\sigma} [\mathrm{d}_s Q(\sigma, s)]\, [f(\sigma, x(s)) - f(\sigma, y(s))] \right\} \mathrm{d}\sigma \right| \\ &\leqslant \int_{\max\{t_0, t\}}^{T} L(\sigma) \left\{ \int_c^{\sigma} |x(s) - y(s)|\, \mathrm{d}_s q(\sigma, s) \right\} \mathrm{d}\sigma \\ &\leqslant \|x - y\| \int_{\max\{t_0, t\}}^{T} L(\sigma) \left\{ \int_c^{\sigma} \left(\int_{\max\{t_0, s\}}^{T} L(\tau) m(\tau)\, \mathrm{d}\tau + h \right) \mathrm{d}_s q(\sigma, s) \right\} \mathrm{d}\sigma \\ &\leqslant (k + h) \|x - y\| \int_{\max\{t_0, t\}}^{T} L(\sigma) m(\sigma)\, \mathrm{d}\sigma,\end{aligned}$$

the last inequality being a consequence of (11) and (13). Consequently $\|Fx - Fy\| \leqslant (k+h)\|x-y\|$. Since $k \in (0,1)$, we can choose $h > 0$ such that $k + h < 1$. Then $F\colon X \to X$ is contractive, and the proof of statement (ii) is completed by Banach's fixed point theorem.

(iii) To prove (iii) we shall use statements (i) and (ii). Let $\Phi \in E_{t_0}$ and $(t^*, x^*) \in (t_0, T) \times R^n$ be given. Consider the equation

$$\dot{x}(t) = \int_c^t [\mathrm{d}_s Q(t,s)]\, f(t, x(s)), \qquad t \in [t_0, t^*). \tag{16}$$

In virtue of (ii) equation (16) is Φ-complete at the point t^*. Consequently, there exist $x_0 \in R^n$ and a solution $\hat{x}$ of (16) satisfying the initial condition $\hat{x}(t) = \Phi(t) + x_0$, $t \in [c, t_0]$, and such that $\lim_{t \to t^*-} \hat{x}(t) = x^*$.

Let us define the function Ψ by

$$\Psi(t) = \begin{cases} \hat{x}(t) & \text{for } t \in [c, t^*) \\ x^* & \text{for } t = t^*. \end{cases}$$

Obviously, $\Psi\colon [c, t^*] \to R^n$ is continuous. It follows from known existence theorems (cf. [5, Chapter 2]) that there exists a non-extendable solution x of the equation

$$\dot{x}(t) = \int_c^t [\mathrm{d}_s Q(t,s)]\, f(t, x(s)), \qquad t \geqslant t^*$$

on some interval $[c, T^*)$ $(t^* < T^* \leqslant T)$ satisfying the initial condition $x(t) = \Psi(t)$, $t \in [c, t^*]$. According to (i) there exists $\lim_{t \to T^*-} x(t) = \xi^* \in R^n$. Since the solution x is non-extendable, $T^* = T$ (cf. [5, Chapter 2]). Obviously, x is a solution of (1) satisfying (9) and passing through (t^*, x^*).

The proof of Theorem 1 is complete. □

Remark 1. The existence of the limit (14) was proved without using the assumption (13).

Remark 2. It follows from the proof that under the assumptions of Theorem 1, for every $\xi \in R^n$ and $\Phi \in E_{t_0}$ there exists a unique $x_0 \in R^n$ such that the solution x determined by the initial condition (9) satisfies (10).

Remark 3. Theorem 1 can be reformulated as follows:

Under the assumptions of Theorem 1 the functional differential equation (1) is asymptotically equivalent to the equation

$$\dot{x} = 0, \tag{17}$$

that is, for every solution y of (1) there exists a solution x of (17) such that $\lim_{t \to T-} [y(t) - x(t)] = 0$, and conversely.

Recently, asymptotic equivalence of two systems of functional differential equations has been investigated by M. Švec [11].

The following result is a consequence of Theorem 1.

Theorem 2. *Let functions f and Q satisfy (H_1) and (H_2). Assume that there exists a continuous function $\mu\colon [t_0,T) \to [0,\infty)$ such that*

$$\int_c^t \exp\left(\int_{\max\{t_0,s\}}^t L(\tau)\mu(\tau)\,\mathrm{d}\tau\right) \mathrm{d}_s q(t,s) \leqslant \mu(t), \quad t \in [t_0,T). \tag{18}$$

Then statement (iii) of Theorem 1 is valid.

In addition to (18), *assume that*

$$\int_{t_0}^T L(\tau)\mu(\tau)\,\mathrm{d}\tau < \infty. \tag{19}$$

Then statements (i) and (ii) of Theorem 1 are valid.

Proof. Assume that conditions (18) and (19) are satisfied. Put

$$k = 1 - \exp\left(-\int_{t_0}^T L(\tau)\mu(\tau)\,\mathrm{d}\tau\right),$$

$$m(t) = \mu(t)\exp\left(\int_t^T L(\tau)\mu(\tau)\,\mathrm{d}\tau\right), \qquad t \in [t_0,T).$$

It can be easily seen that (11) and (12) are satisfied. We shall show that (13) also holds. For $t \in [t_0,T)$ we have

$$\begin{aligned}
&\int_c^t \left(\int_{\max\{t_0,s\}}^T L(\tau)m(\tau)\,\mathrm{d}\tau\right) \mathrm{d}_s q(t,s) \\
&= \int_c^t \left\{\exp\left(\int_{\max\{t_0,s\}}^T L(\tau)\mu(\tau)\,\mathrm{d}\tau\right) - 1\right\} \mathrm{d}_s q(t,s) \\
&\leqslant k \int_c^t \exp\left(\int_{\max\{t_0,s\}}^T L(\tau)\mu(\tau)\,\mathrm{d}\tau\right) \mathrm{d}_s q(t,s) \\
&= k\exp\left(\int_t^T L(\tau)\mu(\tau)\,\mathrm{d}\tau\right) \int_c^t \exp\left(\int_{\max\{t_0,s\}}^t L(\tau)\mu(\tau)\,\mathrm{d}\tau\right) \mathrm{d}_s q(t,s) \\
&\leqslant k\mu(t)\exp\left(\int_t^T L(\tau)\mu(\tau)\,\mathrm{d}\tau\right) = k\,m(t)
\end{aligned}$$

where the last inequality is a consequence of (18). By the application of Theorem 1 we conclude that statements (i) and (ii) are valid.

Statement (iii) can be proved by using (i), (ii) and employing a similar argument as in the proof of Theorem 1; we omit the details. The proof of Theorem 2 is complete. □

Corollary 1. *Consider the delay differential equation*

$$\dot{x}(t) = f\big(t, x[g(t)]\big), \qquad t \in [t_0, T) \tag{20}$$

where f satisfies condition (H_1), $g\colon [t_0, T) \to R$ is a continuous function such that $g(t) \leqslant t$ for $t \in [t_0, T)$ and $\inf_{t \in [t_0, T)} g(t) > -\infty$. If

$$\int_{t_0}^{T} L(\tau)\,\mathrm{d}\tau < 1 \tag{21}$$

then

(i) *for every solution x of* (20) *the limit (14) exists,*

(ii) *equation* (20) *is Φ-complete at the point T for every $\Phi \in E_{t_0}$,*

(iii) *equation* (20) *is pointwise Φ-complete for every $\Phi \in E_{t_0}$.*

P r o o f. Let $-\infty < c < \inf_{t \in [t_0, T)} g(t)$ and define an $n \times n$ matrix-function U by

$$U(r) = \begin{cases} 0 & \text{for } r < 0 \\ I & \text{for } r \geqslant 0 \end{cases}$$

where I denotes the $n \times n$ identity matrix. Put

$$Q(t, s) = U\big(s - g(t)\big), \qquad t \in [t_0, T),\ s \in [c, T).$$

Then $q(t, s) = u\big(s - g(t)\big)$ where $u(r) = 0$ for $r < 0$ and $u(r) = 1$ for $r \geqslant 0$. Observe that conditions (H_1) and (H_2) are satisfied, equation (1) reduces to (20), and conditions (11)–(13) become

$$1 \leqslant m(t), \qquad t \in [t_0, T), \tag{11$'$}$$

$$\int_{t_0}^{T} L(\tau) m(\tau)\,\mathrm{d}\tau < \infty, \tag{12$'$}$$

$$\int_{\max\{t_0, g(t)\}}^{T} L(\tau) m(\tau)\,\mathrm{d}\tau \leqslant k\, m(t), \qquad t \in [t_0, T). \tag{13$'$}$$

Now taking $m(t) = 1$, $t \in [t_0, T)$, condition (21) implies that conditions (11)$'$–(13)$'$ are satisfied ($k = \int_{t_0}^{T} L(\tau)\,\mathrm{d}\tau$) and the result follows from Theorem 1. □

The importance and the strictness of condition (21) in Corollary 1 is shown by the following

E x a m p l e 1. Consider the scalar equation

$$\dot{x}(t) = -p(t)\, x(t-1), \qquad t \geqslant 0 \tag{22}$$

where

$$p(t) = \begin{cases} 2\sin^2 \pi t & \text{for } t \in [2,3] \\ 0 & \text{for } t \in [0,2) \cup (3,\infty). \end{cases}$$

In this case $L(t) = p(t)$, $t_0 = 0$, $T = \infty$. One can easily verify that

$$\int_{t_0}^{T} L(\tau)\, \mathrm{d}\tau = \int_0^{\infty} p(\tau)\, \mathrm{d}\tau = 1.$$

We shall show that for every solution x of (22), $x(t) = 0$ for $t \geqslant 3$.

Let x be an arbitrary solution of (22). Then $x(t) = x(2)$ for $t \in [0,2]$. Hence

$$x(3) = x(2) - \int_2^3 p(\tau)x(\tau-1)\, \mathrm{d}\tau = x(2)\left[1 - 2\int_2^3 \sin^2\pi\tau\, \mathrm{d}\tau\right] = 0.$$

Consequently, in virtue of the definition of p, $x(t) = 0$ for $t \geqslant 3$.

Corollary 2. *Let functions f and g be such as in Corollary* 1. *If*

$$\int_{\max\{t_0, g(t)\}}^{t} L(\tau)\, \mathrm{d}\tau \leqslant \frac{1}{e}, \qquad t \in [t_0, T) \tag{23}$$

then statement (iii) *of Corollary 1 is valid.*

In addition to (23), *assume that*

$$\int_{t_0}^{T} L(\tau)\, \mathrm{d}\tau < \infty. \tag{24}$$

Then statements (i) *and* (ii) *of Corollary 1 are valid.*

P r o o f. Let Q be defined as in the proof of Corollary 1. Then conditions (18) and (19) of Theorem 2 can be rewritten as follows:

$$\exp\left(\int_{\max\{t_0, g(t)\}}^{t} L(\tau)\mu(\tau)\, \mathrm{d}\tau\right) \leqslant \mu(t), \qquad t \in [t_0, T), \tag{18$'$}$$

$$\int_{t_0}^{T} L(\tau)\mu(\tau)\, \mathrm{d}\tau < \infty. \tag{19$'$}$$

Conditions (23) and (24) imply that (18)$'$ and (19)$'$ are fulfilled with $\mu(t) = e$, $t \in [t_0, T)$. The result follows from Theorem 2. □

R e m a r k 4. Corollary 2 unites and generalizes Theorem 0 and the result of Yu. A. Ryabov mentioned in Section 1.

The following example shows the strictness of condition (23) in Corollary 2.

E x a m p l e 2 [3, Example 3.1]. Consider the equation

$$\dot{x}(t) = -x\big(g_n(t)\big), \qquad t \geqslant -n \tag{25}$$

where $n \geqslant 1$ is a given integer and $g_n(t) = t - \tau_n(t)$ with

$$\tau_n(t) = \begin{cases} t + \big(n(-t)^{n-1}\big)^{\frac{1}{n}} & \text{for } t \in [-n, 0) \\ 0 & \text{for } t \geqslant 0. \end{cases}$$

We have $t_0 = -n$, $T = \infty$ and $L(t) = 1$ for $t \geqslant -n$. A simple calculation shows that the general solution of equation (25) can be given by the formula

$$x(t) = \begin{cases} \alpha t^n & \text{for } t \in [-n, 0) \\ 0 & \text{for } t \geqslant 0 \end{cases}$$

where α is an arbitrary real constant. Consequently, equation (25) is not pointwise Φ-complete for any $\Phi \in E_{-n}$. By simple calculation it can be seen that

$$\begin{aligned} s_n = \sup_{t \in [t_0, T)} \int_{\max\{t_0, g_n(t)\}}^{t} L(\tau)\,\mathrm{d}\tau &= \sup_{t \geqslant -n} \tau_n(t) \\ &= \tau_n\left(\frac{-(n-1)^n}{n^{n-1}}\right) = \left(\frac{n-1}{n}\right)^{n-1} > \frac{1}{\mathrm{e}}, \end{aligned}$$

and

$$\lim_{n \to \infty} s_n = \frac{1}{\mathrm{e}}.$$

Corollary 3. *Consider an equation with several delays*

$$\dot{x}(t) = \sum_{i=1}^{\ell} P_i(t) x\big(g_i(t)\big), \qquad t \in [t_0, T) \tag{26}$$

where $P_i\colon [t_0, T) \to M_n$, $i = 1, 2, \ldots, \ell$ are continuous functions, $g_i\colon [t_0, T) \to R$ are continuous functions such that $g_i(t) \leqslant t$ for $t \in [t_0, T)$ and $\inf_{t \in [t_0, T)} g_i(t) > -\infty$, $i = 1, 2, \ldots, \ell$. If

$$\int_{t_0}^{T} \sum_{i=1}^{\ell} \|P_i(\tau)\|\,\mathrm{d}\tau < 1 \tag{27}$$

then

(i) *for every solution x of* (26) *the limit* (14) *exists,*

(ii) *equation* (26) *is Φ-complete at the point T for any $\Phi \in E_{t_0}$,*

(iii) *equation* (26) *is pointwise Φ-complete for every $\Phi \in E_{t_0}$.*

Proof. Put $g(t) = \min_{1 \leqslant i \leqslant \ell} g_i(t)$ for $t \in [t_0, T)$ and choose a constant c such that $-\infty < c < \inf_{t \in [t_0, T)} g(t)$. Let U and u have the same meaning as in the proof of Corollary 1. Put $f(t,x) = x$ and

$$Q(t,s) = \sum_{i=1}^{\ell} P_i(t) U\big(s - g_i(t)\big), \qquad t \in [t_0, T),\ s \in [c, T).$$

Then conditions (H_1) and (H_2) are satisfied ($L(t) = 1$ for $t \in [t_0, T)$, $C = 0$), and equation (1) reduces to (26). For $t \in [t_0, T)$ let $I(t)$ denote the set of all indeces $i \in \{1, 2, \dots, \ell\}$ satisfying the following property: if $g_j(t) = g_i(t)$ for some $j \in \{1, 2, \dots, \ell\}$ then $j \leqslant i$. Then

$$q(t,s) = \sum_{i \in I(t)} \Big\| \sum_{j \mid g_j(t) = g_i(t)} P_j(t) \Big\| u\big(s - g_i(t)\big).$$

Conditions (11)–(13) become

$$\text{(11)}'' \qquad \sum_{i \in I(t)} \Big\| \sum_{j \mid g_j(t) = g_i(t)} P_j(t) \Big\| \leqslant m(t), \qquad t \in [t_0, T),$$

$$\text{(12)}'' \qquad \int_{t_0}^{T} m(\tau)\,\mathrm{d}\tau < \infty,$$

$$\text{(13)}'' \qquad \sum_{i \in I(t)} \Big\| \sum_{j \mid g_j(t) = g_i(t)} P_j(t) \Big\| \int_{\max\{t_0, g_i(t)\}}^{T} m(\tau)\,\mathrm{d}\tau \leqslant k\, m(t), \quad t \in [t_0, T).$$

Evidently

$$\sum_{i \in I(t)} \Big\| \sum_{j \mid g_j(t) = g_i(t)} P_j(t) \Big\| \leqslant \sum_{i \in I(t)} \sum_{j \mid g_j(t) = g_i(t)} \|P_j(t)\| = \sum_{i=1}^{\ell} \|P_i(t)\|.$$

This together with (27) implies that setting

$$m(t) = \sum_{i=1}^{\ell} \|P_i(t)\|, \qquad t \in [t_0, T)$$

conditions (11)″–(13)″ are satisfied ($k = \int_{t_0}^{T} \sum_{i=1}^{\ell} \|P_i(\tau)\|\,\mathrm{d}\tau$). The result follows from Theorem 1. □

Corollary 4. *Let functions P_i, g_i, $i = 1, 2, \dots, \ell$ be such as in Corollary 3. If the condition*

$$(28) \qquad \int_{\max\{t_0, g(t)\}}^{t} \sum_{i=1}^{\ell} \|P_i(\tau)\| \, \mathrm{d}\tau \leqslant \frac{1}{\mathrm{e}}, \qquad t \in [t_0, T)$$

is satisfied, where $g(t) = \min\limits_{1 \leqslant i \leqslant \ell} g_i(t)$, $t \in [t_0, T)$, then statement (iii) of Corollary 3 is valid.

In addition to (28), assume that

$$(29) \qquad \int_{t_0}^{T} \sum_{i=1}^{\ell} \|P_i(\tau)\| \, \mathrm{d}\tau < \infty.$$

Then statements (i) and (ii) of Corollary 3 are valid.

Proof. Let c, Q, f and $I(t)$ ($t \in [t_0, T)$) be defined as in the proof of Corollary 3. Then conditions (18) and (19) can be written in the form

$$(18)'' \quad \sum_{i \in I(t)} \Big\| \sum_{j \mid g_j(t) = g_i(t)} P_j(t) \Big\| \exp\left(\int_{\max\{t_0, g_i(t)\}}^{t} \mu(\tau) \, \mathrm{d}\tau \right) \leqslant \mu(t), \quad t \in [t_0, T),$$

$$(19)'' \qquad \int_{t_0}^{T} \mu(\tau) \, \mathrm{d}\tau < \infty.$$

From (28) and (29) it follows that (18)″ and (19)″ are fulfilled with

$$\mu(t) = \mathrm{e} \sum_{i=1}^{\ell} \|P_i(t)\|, \qquad t \in [t_0, T).$$

The result follows from Theorem 2. □

Corollary 5. *Let functions P_i, g_i, $i = 1, 2, \dots, \ell$ be such as in Corollary 3. Assume that there exist positive constants K_i, C_i, $i = 1, 2, \dots, \ell$ and a continuous function $\varkappa\colon [t_0, T) \to [0, \infty)$ satisfying the following conditions:*

$$(30) \qquad \|P_i(t)\| \leqslant K_i \, \varkappa(t), \qquad t \in [t_0, T), \ i = 1, 2, \dots, \ell,$$

$$(31) \qquad \int_{\max\{t_0, g_i(t)\}}^{t} \varkappa(\tau) \, \mathrm{d}\tau \leqslant C_i, \qquad t \in [t_0, T), \ i = 1, 2, \dots, \ell.$$

Furthermore, assume that

$$(32) \qquad \lambda = \sum_{i=1}^{\ell} K_i \exp(\lambda C_i) \quad \textit{has a positive root.}$$

Then statement (iii) *of Corollary 3 is valid.*

In addition to (30)–(32), *assume that*

$$\int_{t_0}^{T} \varkappa(\tau)\,\mathrm{d}\tau < \infty. \tag{33}$$

Then statements (i) *and* (ii) *of Corollary 3 are valid.*

Proof. Observe that if (30)–(33) hold, then setting

$$\mu(t) = \lambda_0\,\varkappa(t), \qquad t \in [t_0, T),$$

where λ_0 is a positive root of equation (32), conditions (18)″ and (19)″ of Corollary 4 are satisfied and the proof is complete. □

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