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α-COMPACT FUZZY TOPOLOGICAL SPACES

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Summary. The purpose of this paper is to introduce and discuss the concept of α-compactness for fuzzy topological spaces.

Keywords: fuzzy topological spaces, compactness, α-compactness, α-open, fuzzy α-continuity

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1. INTRODUCTION

The concept of fuzzy sets and fuzzy set operations was first introduced by Zadeh in his classical paper [7]. Subsequently several authors have applied various basic concepts from general topology to fuzzy sets and developed a theory of fuzzy topological spaces. The concept of α-compactness for topological spaces has been discussed in [5]. The purpose of this paper is to introduce and study α-compactness for fuzzy topological spaces, thus filling a gap in the existing literature on fuzzy topological spaces.

2. DEFINITIONS

Throughout this paper (X,T) will denote a fuzzy topological space. If A is a fuzzy set in a fuzzy topological space then the closure and interior of A will be as usual denoted by $\mathrm{Cl}\, A$ and $\mathrm{Int}\, A$, respectively. We now introduce the following definitions.

Definition 2.1. Let A be a fuzzy subset of a fuzzy topological space X. A is said to be fuzzy α-open if $A \subset \mathrm{Int}\,\mathrm{Cl}\,\mathrm{Int}\, A$. The set of all fuzzy α-open subsets of X will be denoted by $F\alpha(X)$.

Definition 2.2. In a fuzzy topological space (X, T), a family v of fuzzy subsets of X is called an α-covering of X iff v covers X and $v \subset F\alpha(X)$.

Definition 2.3. A fuzzy topological space (X, T) is said to be α-compact if every α-open cover of X has a finite subcover.

Definition 2.4. Let (X, T) and (Y, S) be fuzzy topological spaces. A mapping $f \colon X \to Y$ is called fuzzy α-continuous if the inverse image of each fuzzy open set in Y is fuzzy α-open in X.

Definition 2.5. A mapping $f \colon X \to Y$ is said to be fuzzy α-irresolute if the inverse image of every fuzzy α-open set in Y is fuzzy α-open in X.

Definition 2.6. Let (X, T) and (Y, S) be fuzzy topological spaces and let T_ξ be a fuzzy topology on X which has $F\alpha(X)$ as a subbase. A mapping $f \colon X \to Y$ is called fuzzy ξ-continuous if $f \colon (X, T_\xi) \to (Y, S)$ is fuzzy continuous; f is said to be fuzzy ξ'-continuous if $f \colon (X, T_\xi) \to (Y, S_\xi)$ is fuzzy continuous.

3. Results

Theorem 3.1. *Let (X, T) and (Y, S) be fuzzy topological spaces and let T_ξ be a fuzzy topology on X which has $F\alpha(X)$ as a subbase. If $f \colon (X, T) \to (Y, S)$ is fuzzy α-continuous, then f is fuzzy ξ-continuous.*

Proof. Let f be fuzzy α-continuous and let $V \in S$. Then $f^{-1}(V) \in F\alpha(X)$ and so $f^{-1}(V) \in T_\xi$. Thus f is fuzzy ξ-continuous and this completes the proof. □

Theorem 3.2. *Let (X, T) and (Y, S) be fuzzy topological spaces. Let T_ξ and S_ξ be respectively the fuzzy topologies on X and Y which have $F\alpha(X)$ and $F\alpha(Y)$ as subbases. If $f \colon (X, T) \to (Y, S)$ is fuzzy α-irresolute then f is fuzzy ξ'-continuous.*

Proof. Let f be fuzzy α-irresolute and let $V \in S_\xi$. Then

$$V = \bigcup_i \left(\bigcap_{i=1}^{n} S_{i_{n_i}} \right) \quad \text{where } S_{i_{n_i}} \in F\alpha(Y, S),$$

and

$$f^{-1}(V) = f^{-1}\left(\bigcup_i \left(\bigcap_{i=1}^{n} S_{i_{n_i}} \right) \right) = \bigcup_i \left(\bigcap_{i=1}^{n} f^{-1}(S_{i_{n_i}}) \right).$$

Since f is fuzzy α-irresolute, $f^{-1}(S_{i_{n_i}}) \in F\alpha(X, T)$. This implies that $f^{-1}(V) \in T_\xi$ and thus f is fuzzy ξ'-continuous. □

Theorem 3.3. *A fuzzy topological space X is α-compact if and only if every family of fuzzy α-closed subsets of X with finite intersection property has non-empty intersection.*

Proof. Evident. □

Theorem 3.4. *Let (X,T) be a fuzzy topological space and T_ξ a fuzzy topology on X which has $F\alpha(X)$ as a subbase. Then (X,T) is α-compact if and only if (X,T_ξ) is compact.*

Proof. Let (X,T_ξ) be compact. Then, since $F\alpha(X) \subset T_\xi$, it follows that (X,T) is α-compact. □

Theorem 3.5. *Let (X,T) be a fuzzy topological space which is α-compact. Then each T_ξ-closed fuzzy set in X is α-compact.*

Proof. Let U be any T_ξ-closed fuzzy set in X. Let $\{V_{\beta_i} : \beta_i \in I\}$ be a T_ξ-open cover of U. Since $X - U$ is T_ξ-open, $\{V_{\beta_i} : \beta_i \in I\} \cup (X - U)$ is a T_ξ-open cover of X. Since X is T_ξ-compact, by Theorem 3.4 there exists a finite subset $I_0 \subset I$ such that

$$X = \bigcup\{V_{\beta_i} : \beta_i \in I_0\} \cup (X - U).$$

This implies that

$$U \subset \bigcup\{V_{\beta_i} : \beta_i \in I_0\}.$$

Hence U is α-compact relative to X and this completes the proof. □

Theorem 3.6. *Let a fuzzy topological space (X,T) be α-compact. Then every family of T_ξ-closed fuzzy subsets of X with finite intersection property has non-empty intersection.*

Proof. Let X be α-compact. Let $U = \{B_{\beta_i} : \beta_i \in I\}$ be any family of T_ξ-closed fuzzy subsets of X with finite intersection property. Suppose $\bigcap\{B_{\beta_i} : \beta_i \in I\} = \emptyset$.

Then $\{X - B_{\beta_i} : \beta_i \in I\}$ is a T_ξ-open cover of X. Hence it must contain a finite subcover $\{X - B_{\beta_{i_j}} : j = 1,2,\dots,n\}$ for X. This implies that $\bigcap\{B_{\beta_{i_j}} : j = 1,2,\dots,n\} = \emptyset$ and contradicts the assumption that U has finite intersection property. □

Theorem 3.7. *Let X and Y be fuzzy topological spaces and let $f\colon X \to Y$ be fuzzy ξ'-continuous. If a fuzzy subset G of X is α-compact relative to X, then $f(G)$ is α-compact relative to Y.*

Proof. Let $\{V_{\beta_i} : \beta_i \in I\}$ be a cover of $f(G)$ by S_ξ-open fuzzy sets in Y. Then $\{f^{-1}(V_{\beta_i}) : \beta_i \in I\}$ is a cover of G by T_ξ-open fuzzy sets in X. G is α-compact relative to X. Hence by Theorem 3.4, G is T_ξ-compact. So there exists a finite subset $I_0 \subset I$ such that

$$G \subset \bigcup\{f^{-1}(V_{\beta_i}) \colon \beta_i \in I_0\}$$

and so

$$f(G) \subset \{V_{\beta_i} \colon \beta_i \in I_0\}.$$

Hence $f(G)$ is T_ξ-compact relative to Y. Thus $f(G)$ is α-compact relative to Y and this completes the proof. □

Corollary 3.8. *If $f \colon (X,T) \to (Y,S)$ is a fuzzy ξ'-continuous surjective function and X is α-compact, then Y is α-compact.*

Corollary 3.9. *If $f \colon (X,T) \to (Y,S)$ is a fuzzy α-irresolute surjective function and X is α-compact then Y is α-compact.*

Theorem 3.10. *Let A and B be fuzzy subsets of a fuzzy topological space X such that A is α-compact relative to X and B is T_ξ-closed in X. Then $A \cap B$ is α-compact relative to X.*

Proof. Let $\{V_{\beta_i} \colon \beta_i \in I\}$ be a cover of $A \cap B$ by T_ξ-open fuzzy subsets of X. Since $X - B$ is a T_ξ-open fuzzy set,

$$\{V_{\beta_i} \colon \beta_i \in I\} \cup (X - B)$$

is a cover of A. A is α-compact and thus T_ξ-compact relative to X. Hence there exists a finite subset $I_0 \subset I$ such that

$$A \subset \bigcup\{V_{\beta_i} \colon \beta_i \in I_0\} \cup (X - B).$$

Therefore

$$A \cap B \subset \bigcup\{V_{\beta_i} \colon \beta_i \in I_0\}.$$

Hence $A \cap B$ is T_ξ-compact. Therefore $A \cap B$ is α-compact and this completes the proof. □

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