ON 2-EXTENDABILITY OF GENERALIZED PETERSEN GRAPHS

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Summary. Let $GP(n,k)$ be a generalized Petersen graph with $(n,k) = 1$, $n > k \geqslant 4$. Then every pair of parallel edges of $GP(n,k)$ is contained in a 1-factor of $GP(n,k)$. This partially answers a question posed by Larry Cammack and Gerald Schrag [Problem 101, Discrete Math. 73(3), 1989, 311–312].

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A simple loopless graph G with an even number of vertices is said to be 2-*extendable* if G has a pair of parallel edges and if every such pair is contained in a 1-factor of G. Let $k < n$ be natural numbers. The generalized Petersen graph $GP(n,k)$ is the graph with a vertex set $V = \{u_i, v_i \colon 1 \leqslant i \leqslant n\}$ and the edge set $O \cup I \cup C$, where

$$O = \{u_i u_{i+1} \colon 1 \leqslant i \leqslant n\},\ I = \{v_i v_{i+k} \colon 1 \leqslant i \leqslant n\} \text{ and } C = \{u_i v_i \colon 1 \leqslant i \leqslant n\}.$$

Here $i+1$ and $i+k$ are taken modulo n. The edges in O, I and C are referred to as the outer edges, the inner edges and the spokes, respectively.

In 1989, G. Schrag and L. Cammack [1] proved that

(i) $GP(n,1)$ is 2-extendable if and only if n is even,

(ii) $GP(n,2)$ is 2-extendable if and only if $n \neq 5, 6, 8$,

(iii) $GP(2k,k)$ is not 2-extendable for all $k \geqslant 2$,

(iv) $GP(3k,k)$ is not 2-extendable for all $k \geqslant 3$,

(v) if $3 \leqslant k \leqslant 7$, then $GP(n,k)$ is 2-extendable if and only if $n \neq 3k$.

(vi) if $k \geqslant 4$, then any pair of parallel edges containing a spoke can be extended to a 1-factor, and

(vii) $GP(n,k)$ is 2-extendable for all $k \geqslant 2$, $n \geqslant 3k+5$.

They conjectured that $GP(n,k)$ is 2-extendable for all $k \geqslant 3$ and $n \neq 2k, 3k$. In this note we prove that $GP(n,k)$ is 2-extendable for all $n, k \geqslant 4$ such that $(n,k) = 1$. While many cases considered here are covered by [1], we give a uniform treatment which covers several additional cases including the important cases $n = 2k+1$, $3k-1$, $3k+1$.

Theorem. *If $n, k \geqslant 4$ are natural numbers such that $(n,k) = 1$, then $GP(n,k)$ is 2-extendable.*

Proof. Let e and f be two given parallel edges of $GP(n,k)$, where $(n,k) = 1$. We divide the problem into six possibilities:

$$\begin{array}{lll} P(1)\colon e,f \in O, & P(2)\colon e \in O, f \in I, & P(3)\colon e \in O, f \in C, \\ P(4)\colon e,f \in I, & P(5)\colon e \in I, f \in C, & P(6)\colon e,f \in C. \end{array}$$

In $P(6)$, the set C consisting of all spokes is the required 1-factor. Moreover, since $(n,k) = 1$, O and I play the same role in $GP(n,k)$. Hence we have only to consider $P(1)$, $P(2)$, $P(3)$. If $k > n/2$, then $GP(n,k)$ is isomorphic to $GP(n, n-k)$. Thus we can assume that $k < n/2$. Without loss of generality, we can take $e = u_1u_2$. We shall denote the desired 1-factor containing e and f by F.

Case 1: n is even

In this case, I as well as O can be written as a union of two disjoint 1-factors. Moreover, if we remove two adjacent points from either O or I, then the resulting path, which is of odd length, has a unique 1-factor.

$P(1)$: Let $f = u_ru_{r+1}$, $3 \leqslant r \leqslant n-1$.

If r is odd, then F is obtained by taking the 1-factor of O containing u_1u_2 and u_ru_{r+1} together with any one of the two 1-factors of I.

If r is even with $r+k-1 \leqslant n$, then let $F = F_1 \cup F_2 \cup F_3 \cup F_4$, where F_1 is the unique 1-factor of $I - v_{r-1} - v_{r-1+k}$,

$$F_4 = \{u_{r+k}u_{r+k+1}, u_{r+k+2}u_{r+k+3}, \dots, u_1u_2, \dots, u_{r-3}u_{r-2}\},$$
$$F_2 = \{u_{r-1}v_{r-1}, u_{r-1+k}v_{r-1+k}\}, \quad F_3 = \{u_ru_{r+1}, u_{r+2}u_{r+3}, \dots, u_{r+k-3}u_{r+k-2}\}.$$

If r is even with $r+k-1 > n$, then clearly $r+2-k \geqslant 3$. Again F is obtained by taking $F_1 \cup F_2 \cup F_3 \cup F_4$, where F_1 is the unique 1-factor of $I - v_{r+2} - v_{r+2-k}$, $F_2 = \{u_{r+2}v_{r+2}, u_{r+2-k}v_{r+2-k}\}$, $F_3 = \{u_{r+3-k}u_{r+4-k}, \dots, u_ru_{r+1}\}$, and $F_4 = \{u_{r+3}u_{r+4}, u_{r+5}u_{r+6}, \dots, u_1u_2, \dots, u_{r-k}u_{r-k+1}\}$.

$P(2)$: Let $f = v_rv_{r+k}$, $1 \leqslant r \leqslant n$.

In this case F is obtained by taking the union of the 1-factor of O containing u_1u_2 and the 1-factor of I containing v_rv_{r+k}.

$P(3)$: Let $f = u_r v_r$, $3 \leqslant r \leqslant n$.

Let $2t$ be the greatest even integer less than r and $s = \min\{2t, 2k\}$. We can now take F to be $F_1 \cup F_2 \cup F_3$, where

$$F_1 = \{u_i v_i : i \neq s, s-1, \ldots, s-2k+1\},$$
$$F_2 = \{u_s u_{s-1}, u_{s-2} u_{s-3}, \ldots, u_{s-2k+2} u_{s-2k+1}\},$$
$$F_3 = \{v_s v_{s-k}, v_{s-1} v_{s-k-1}, \ldots, v_{s-k+1} v_{s-2k+1}\}.$$

Clearly f is in F_1 and e is in F_2.

Case 2: n is odd

In this case, $O - u_i$ as well as $I - v_i$ have a unique 1-factor for each i.

$P(1)$: Let $f = u_r u_{r+1}$, $3 \leqslant r \leqslant n-1$.

If r is odd, then take $i = n$. If r is even, then take $i = 3$. Let $F = F_1 \cup F_2 \cup F_3$, where F_1 is the unique 1-factor of $O - u_i$, $F_2 = \{u_i v_i\}$ and F_3 is the unique 1-factor of $I - v_i$.

$P(2)$: Let $f = v_r v_{r+k}$, $1 \leqslant r \leqslant n$.

In this case, $n \geqslant 2k+1$. Here we have to handle the cases $n = 2k+1$, $2k+3$, $3k-1$, and $n = 3k+1$ carefully. In what follows, F_1 will always be the set of all spokes not on points of F_2 and F_3. For $n \neq 3k-1, 3k+1$, we take $F = F_1 \cup F_2 \cup F_3$ with $F_2 = \{u_i u_{i+1}, u_j u_{j+1}, u_{i+k} u_{i+1+k}, u_{j+k} u_{j+1+k}\}$, $F_3 = \{v_i v_{i+k}, v_{i+1} v_{i+1+k}, v_j v_{j+k}, v_{j+1} v_{j+1+k}\}$, where i and j are given in the following table:

		i	j
$n = 2k+1$	$r \in \{k, k+1, k+2, k+3\}$	$k+2$	k
	$r \notin \{1, 2, k, k+1, k+2, k+3, n, n-1\}$	1	r
$2k+3 < n$	$r \in \{k+1, k+2, n-k+1, n-k+2\}$	$n+1-k$	$k+1$
$n \neq 3k+1, 3k+1$	$r \notin \{1, 2, k, k+1, k+2, n-k$	1	r
	$n-k+1, n+2-k, n\}$		
$n = 2k+3$	$r = 2k+3$	$k+4$	$k+2$

See figure 1 for the case $k = 5$, $n = 2k+3 = 13$, $r = n-k = 8$ and Figure 2 for the case $k = 5$, $n = 2k+1 = 11$, $r = n-1 = 10$.

Here the cases $f = v_1 v_{1+k}$, $v_2 v_{2+k}$ are not considered since these edges appear along with the edge $u_1 u_2$ in most of the 1-factors given by the table. Also, the cases $f = v_k v_{2k}$ (when $n \neq 2k+1$), $v_{n-k} v_n$ (when $n \neq 2k+3$), $v_n v_k$ are not considered, since these edges appear along with the edge $u_1 u_2$ in the 1-factor for the values

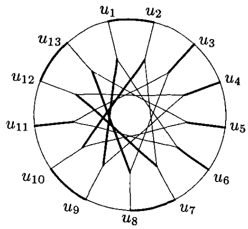

Fig. 1

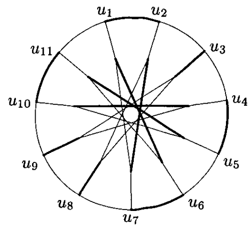

Fig. 2

$r = k-1$, $n-k-1$, $n-1$ respectively, given by the table. For several values of r, this table in fact gives two distinct 1-factors containing u_1u_2 and v_rv_{r+k}.

Let $n = 3k-1$.

If $r \in \{2, k+1, n-k+1 = 2k\}$, we take $F_2 = \{u_1u_2, u_{k+1}u_{k+2}, u_{2k}u_{2k+1}\}$, $F_3 = \{v_{2k}v_1, v_2v_{k+2}, v_{k+1}v_{2k+1}\}$. If $r \in \{1, k+2, n-k+2 = 2k+1\}$, we take $F_2 = \{u_1u_2, u_{k+1}u_{k+2}, u_{2k+1}u_{2k+2}\}$, $F_3 = \{v_1v_{1+k}, v_{2k+1}v_2, v_{k+2}v_{2k+2}\}$.

If $r \notin \{1, 2, k, k+1, k+2, 2k = n-k+1, 2k+1, n-k = 2k-1, n\}$, we can take $F_2 = \{u_1u_2, u_ru_{r+1}, u_{1+k}u_{2+k}, u_{r+k}u_{r+1+k}\}$, $F_3 = \{v_1v_{1+k}, v_2v_{2+k}, v_rv_{r+k}, v_{r+1}v_{r+1+k}\}$. Here it may appear that the cases $f = v_kv_{2k}$, $v_{n-k}v_n$, v_nv_{n+k} are not considered. But we note that these edges appear along with u_1u_2 in the 1-factors for $r = k-1$, $n-k-1$, $n-1$, respectively, except when $k = 4$, $r = n-k = 7$. But in this case the edges u_1u_2 and v_7v_{11} appear in the 1-factor

$$\{u_1u_2, u_4u_5, u_6u_7, u_8u_9, u_{10}u_{11}, u_3v_3, v_7v_{11}, v_4v_8, v_1v_5, v_9v_2, v_6v_{10}\}$$

Finally, let $n = 3k+1$.

If $r \in \{2, k+1, n+1-k = 2k+2,\}$, we take $F_2 = \{u_1u_2, u_{k+1}u_{k+2}, u_{2k+1}u_{2k+2}\}$, $F_3 = \{v_{2+2k}v_1, v_{2+k}v_2, v_{k+1}v_{2k+1}\}$.

If $r \in \{1, k+2, n-k+2 = 2k+3\}$, we take $F_2 = \{u_1u_2, u_{k+1}u_{k+2}, u_{2k+2}u_{2k+3}\}$, $F_3 = \{v_1v_{k+1}, v_{k+2}v_{2k+2}, v_{n+2-k}v_2\}$.

If $r \notin \{1, 2, k, k+1, k+2, n-k = 2k+1, 2k+2, 2k+3, n\}$, we take $F_2 = \{u_1u_2, u_ru_{r+1}, u_{1+k}u_{2+k}, u_{r+k}u_{r+1+k}\}$, $F_3 = \{v_1v_{1+k}, v_2v_{2+k}, v_rv_{r+k}, v_{r+1}v_{r+1+k}\}$. Again the edges $f = v_nv_{2k}, v_{2k+1}v_n, v_kv_n$ appear along with the edge u_1u_2 in the 1-factors for $r = k-1, 2k, n-1$, respectively.

$P(3)$: Let $f = u_rv_r$, $3 \leqslant r \leqslant n$.

If r is odd, then take $i = r$ and F as in $P(1)$.

If r is even, consider four points u_{k+1}, u_{k+2}, u_{n+1-k}, u_{n+2-k}. Since n is odd, $k+1 \neq n+1-k$ and $k+2 \neq n+2-k$. Moreover, $n > 2k-1$ implies that

$k+1 \neq n+2-k$. If u_r is different from u_{k+1}, u_{k+2}, take $i = k+1$. If u_r is one of u_{k+1}, u_{k+2} but $r \neq n-k+1, n-k+2$, we take $i = n+1-k$. We then let $F = \{u_1u_2, u_iu_{i+1}, v_1v_i, v_2v_{i+1}\} \cup \{u_jv_j : j \neq 1, 2, i, i+1\}$.

Finally, suppose $r = k+2 = n+1-k$. Here $k = r-2$ and hence it is even. Let $F_1 = \{u_1u_2, u_rv_r, u_3v_3, u_nv_n, v_1v_{k+1} = v_1v_{r-1}, v_2v_{r+2-k} = v_2v_{r+1}\}$. To construct F_2, consider the path from u_4 to u_{r-1} in the cycle O. This path contains $k-2$ points, where $k-2$ is even. Hence this path of odd length has a unique 1-factor. Similarly, the path from u_{r+1} to u_{n-1} on the cycle O also has unique 1-factor. Take F_2 to be the union of these two 1-factors. Let $F_3 = \{v_4v_{4+k}, v_5v_{5+k}, \ldots, v_{r-2}v_{r-2+k} = v_kv_{2k}\}$. Now let $F = F_1 \cup F_2 \cup F_3$. See Figure 3 for the case $k = 6$, $n = 2k+1 = 13$, $r = 8$.

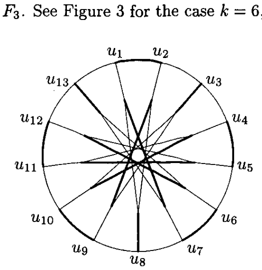

Fig. 3

This completes the proof of the theorem. □

Remark. We have assumed $k \geqslant 4$ but the construction given here can be suitably modified for the cases (i) $k = 2$, $n \geqslant 7$, n odd and (ii) $k = 3$. Note that the Petersen graph $GP(5, 2)$ is not 2-extendable.

References

[1] *G. Schrag and L. Cammack*: On the 2-extendability of the generalized Petersen graphs. Discrete Math. *78* (1989), 169–177.

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