INERTIAL LAW OF SYMPLECTIC FORMS ON MODULES OVER PLURAL ALGEBRA

MAREK JUKL, Olomouc*

(Received January 31, 1996)

Summary. In this paper the problem of construction of the canonical matrix belonging to symplectic forms on a module over the so called plural algebra (introduced in [5]) is solved.

Keywords: linear algebra, free module, symplectic form, symplectic basis

MSC 1991: 15A63, 51A50

I. INTRODUCTION

1. Definition. The *plural* **T**-*algebra of order* m is every linear algebra **A** on **T** having as a vector space over **T** a basis

$$\{1, \eta, \eta^2, \dots, \eta^{m-1}\} \text{ with } \eta^m = 0.$$

A plural algebra **A** is a local ring the maximal ideal of which is nilpotent. It was proved in [3] that the free finite generated **A**-module **M** (the so called **A**-space in the sense of [6]) has the following properties:

2.1. If one basis of **M** consists of n elements then each of its bases consists of the same number of n elements. (This is true in every free module over a commutative ring.)[1]

* Supported by grant No. 201/95/1631 of the Grant Agency of the Czech Republic.

[1] See [1]

2.2. From every system of generators of $\mathbf{M}$ we may select a basis of $\mathbf{M}$. (This is valid over every local ring.)[2]

Moreover, in this case:

2.3. Any linearly independent system may be completed to a basis of $\mathbf{M}$.

2.4. Every maximal linearly independent system in $\mathbf{M}$ forms a basis of $\mathbf{M}$.

3. Let $\varphi_1, \ldots, \varphi_k$ be a linearly independent system of linear forms $\mathbf{M} \to \mathbf{A}$. Then $\bigcap_{1 \leqslant i \leqslant k} \operatorname{Ker} \varphi_i$ is a free $(n-k)$-dimensional submodule of $\mathbf{M}$.

4. Let K, L be free submodules of an $\mathbf{A}$-module $\mathbf{M}$. Then $K + L$ is a free $\mathbf{A}$-submodule if and only if $K \cap L$ is a free $\mathbf{A}$-submodule and the dimensions of $\mathbf{A}$-submodules K, L, $K \cap L$, $K + L$ fulfil the relation

$$\dim(K + L) + \dim(K \cap L) = \dim K + \dim L.$$

5. Agreement. Throughout the paper we denote by $\mathbf{A}$ the plural $\mathbf{T}$-algebra introduced in this section. The capital $\mathbf{M}$ always denotes the free n-dimensional module over the algebra $\mathbf{A}$.

6. Definition. A bilinear form $\Phi\colon \mathbf{M}^2 \to \mathbf{A}$ is called a *bilinear form of order* $k\,(0 \leqslant k \leqslant m-1)$ if
(1) $\forall(\underline{X}, \underline{Y}) \in \mathbf{M}^2; \quad \Phi(\underline{X}, \underline{Y}) \in \eta^k \mathbf{A}$,
(2) $\exists(\underline{U}, \underline{V}) \in \mathbf{M}^2; \quad \Phi(\underline{U}, \underline{V}) \notin \eta^{k+1} \mathbf{A}$.

The following proposition is taken form [4].

7. Proposition. *If Φ is a bilinear form of order k then there exists at least one form Λ of order 0 such that*

$$\Phi = \eta^k \Lambda.$$

[2] See [6]

Let the dimension n of $\mathbf{M}$ be an even number.

1. Definition. Let $\Phi\colon \mathbf{M}^2 \to \mathbf{A}$ be a symplectic form[3]. If all elements of the basis $\mathcal{U} = \{\underline{U}_1, \underline{V}_1, \underline{U}_2, \underline{V}_2, \dots, \underline{U}_r, \underline{V}_r\}$ of $\mathbf{M}$ fulfil the conditions

(1) $\Phi(\underline{U}_i, \underline{U}_j) = \Phi(\underline{V}_i, \underline{V}_j) = 0$,

(2) $\Phi(\underline{U}_i, \underline{V}_i) = \{1, \eta, \eta^2, \dots, \eta^m\}$,

(3) $\Phi(\underline{U}_i, \underline{V}_j) = 0$ for $i \neq j$,

then $\mathcal{U}$ is called *the symplectic basis of* $\mathbf{M}$ *with respect to* Φ.[4]

2. Remark. Relative to this basis the matrix of the symplectic form has the form

$$\left\| \begin{array}{cccc:cc} \boxed{\begin{matrix} 0 & \varphi_{12} \\ -\varphi_{12} & 0 \end{matrix}} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & & & \begin{matrix} 0 \\ 0 \end{matrix} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \boxed{\begin{matrix} 0 & \varphi_{34} \\ -\varphi_{34} & 0 \end{matrix}} & & & \begin{matrix} 0 \\ 0 \end{matrix} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \hdashline \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & & & \multicolumn{2}{c}{\boxed{\begin{matrix} 0 & \varphi_{n-1,n} \\ -\varphi_{n-1,n} & 0 \end{matrix}}} \end{array} \right\|$$

where $\varphi_{ij} \in \{1, \eta, \eta^2, \dots, \eta^m\}$.

3. Theorem. *Let Φ be a symplectic form on the module $\mathbf{M}$. Then there exists a symplectic basis of $\mathbf{M}$ with respect to Φ.*

Proof. By induction for $r = \frac{1}{2}n$.

1. The proposition is clear for $r = 1$.

2. Let the theorem be true for all $(n-2)$-dimensional $\mathbf{A}$-modules, $n \geqslant 4$.

(a) Let Φ be a form of order 0, i.e. $\exists(\underline{U}, \underline{V}) \in \mathbf{M}^2\colon \Phi(\underline{U}, \underline{V})$ is a unit. Let us suppose—without loss of generality—that $\Phi(\underline{U}, \underline{V}) = 1$.

This implies that $\underline{U}$, $\underline{V}$ are linearly independent. Indeed, if $\alpha\underline{U} + \beta\underline{V} = \underline{o}$ then

$$0 = \Phi(\alpha\underline{U} + \beta\underline{V}, \underline{V}) = \alpha \cdot \Phi(\underline{U}, \underline{V}) + \beta \cdot \Phi(\underline{V}, \underline{V}) = \alpha.$$

Analogously, we obtain $\beta = 0$.

Let us consider linear forms $\varphi_U(\underline{X}) \equiv \Phi(\underline{U}, \underline{X})$ and $\varphi_V(\underline{X}) \equiv \Phi(\underline{V}, \underline{X})$. Evidently, they are linearly independent. According to Proposition I.3 $\mathcal{N} = \operatorname{Ker}\varphi_U \cap \operatorname{Ker}\varphi_V$

[3] A form Φ satisfies $\Phi(\underline{X}, \underline{X}) = 0$ for all $\underline{X} \in \mathbf{M}$.

[4] For $m = 1$ (i.e. $\mathbf{A}$ is a field) we get the usual definition of a symplectic basis over fields (see [2]).

is a free $(n-2)$-dimensional submodule. Due to the induction hypothesis we may construct a symplectic basis $\{\underline{U}_1,\underline{V}_1,\underline{U}_2,\underline{V}_2,\dots,\underline{U}_{r-1},\underline{V}_{r-1}\}$ of $\mathcal{N}$ with respect to the form $\Phi|\mathcal{N}^2$.

Now, let us show $\mathbf{M}=\mathcal{N}\oplus[\underline{U},\underline{V}]$. If $\underline{X}\in[\underline{U},\underline{V}]$ then $\underline{X}=\xi\underline{U}+\zeta\underline{V}$. Consequently,

$$0=\varphi_U(\underline{X})=\Phi(\underline{U},\xi\underline{U}+\zeta\underline{V})=\xi\cdot\Phi(\underline{U},\underline{U})+\zeta\cdot\Phi(\underline{U},\underline{V})=\zeta.$$

In a similar way we get $\xi=0$. This gives $\underline{X}=\underline{O}$ and therefore $\mathcal{N}\cap[\underline{U},\underline{V}]$ is a 0-dimensional submodule. We have (by Proposition I.4) $\mathbf{M}=\mathcal{N}\oplus[\underline{Y}]$.

Since $\underline{U}_j,\underline{V}_j\in\mathcal{N}$ for every $j\in\mathbb{N}(r-1)$, hence $\Phi(\underline{U}_j,\underline{U})=\Phi(\underline{V}_j,\underline{U})=0$ and $\Phi(\underline{U}_j,\underline{V})=\Phi(\underline{V}_j,\underline{V})=0$. Thus $\{\underline{U}_1,\underline{V}_1,\underline{U}_2,\underline{V}_2,\dots,\underline{U}_{r-1},\underline{V}_{r-1},\underline{U},\underline{V}\}$ forms a symplectic basis of $\mathbf{M}$ with respect to Φ.

(b) Let Φ be a bilinear form of order $k\,(\neq 0)$. According to Proposition I.7 there exists a bilinear form Ψ of order 0 with $\Phi=\eta^k\Psi$. By (a) we can construct a symplectic basis for the form Ψ, which is also a symplectic basis for the form Φ. $\square$

4. Definition. Let Φ be a symplectic form $\mathbf{M}^2\to\mathbf{A}$ and let the basis $\mathcal{U}=\{\underline{U}_1,\underline{V}_1,\underline{U}_2,\underline{V}_2,\dots,\underline{U}_r,\underline{V}_r\}$ be symplectic with respect to Φ. Let us define a system of sets $\mathcal{J}_0,\dots,\mathcal{J}_m$ as follows:

$$\mathcal{J}_k=\{i\in\mathbb{N}(r);\Phi(\underline{U}_i,\underline{V}_i)=\eta^k\},\quad 0\leqslant k\leqslant m.$$

If we denote $\pi_k=2\operatorname{card}(\mathcal{J}_k),\quad 0\leqslant k\leqslant m$, then

$$\mathfrak{Ch}(\Phi,\mathcal{U})=(\pi_0,\dots,\pi_m)$$

is called the *characteristic of the symplectic form Φ with respect to the basis $\mathcal{U}$*.

5. Definition. For any symplectic form $\Phi\colon\mathbf{M}^2\to\mathbf{A}$ let us denote by $\mathcal{V}_k^\Phi$ the set

$$\{\underline{Y}\in\mathbf{M};\eta^k\Phi(\underline{X},\underline{Y})=0,\ \forall\underline{X}\in\mathbf{M}\},\quad 0\leqslant k\leqslant m.$$

The following lemma is evident:

6. Lemma. *If $\mathcal{U}$ is a basis of $\mathbf{M}$ and Φ is symplectic form, then*

$$\mathcal{V}_k^\Phi=\{\underline{Y}\in\mathbf{M};\eta^k\Phi(\underline{U},\underline{Y})=0,\ \forall\underline{U}\in\mathcal{U}\},\quad 0\leqslant k\leqslant m.$$

7. Proposition. *Let Φ be a symplectic form $\mathbf{M}^2\to\mathbf{A}$ and let $\mathcal{U}$ be symplectic with respect to Φ. Then a submodule $\mathcal{V}_k^\Phi$ of $\mathbf{M}$ as an $\mathbf{T}$-vector subspace has the dimension*

$$\dim_{\mathbf{T}}\mathcal{V}_k^\Phi=\sum_{j=0}^{m-k-1}(k+j)\pi_j+m\sum_{j=m-k}^{m}\pi_j,$$

where $(\pi_0, \ldots, \pi_m) = \mathfrak{Ch}(\Phi, \mathcal{U})$.

Proof. $\mathcal{V}_k^\Phi$ is clearly a submodule of $\mathbf{M}$. Let $\mathcal{U} = \{\underline{U}_1, \underline{V}_1, \underline{U}_2, \underline{V}_2, \ldots, \underline{U}_r, \underline{V}_r\}$ and let us consider a $\underline{X} \in \mathcal{V}_k^\Phi$, $\underline{X} = \sum_{i=1}^{r} \xi_i \underline{U}_i + \sum_{i=1}^{r} \zeta_i \underline{V}_i$. Putting $\gamma_j = \Phi(\underline{U}_j, \underline{V}_j)$, $j \in \mathbb{N}(r)$, we obtain

$$\Phi(\underline{X}, \underline{U}_j) = -\zeta_j \gamma_j \text{ and } \Phi(\underline{X}, \underline{V}_j) = \xi_j \gamma_j,$$

which yields $\underline{X} \in \mathcal{V}_k^\Phi \Leftrightarrow \forall i,\, i \in \mathbb{N}(r);\ \eta^k \Phi(\underline{X}, \underline{U}_i) = \eta^k \Phi(\underline{X}, \underline{V}_i) = 0 \Leftrightarrow \forall i,\, i \in \mathbb{N}(r);$ $\eta^k \gamma_i \zeta_i = \eta^k \gamma_i \xi_i = 0$. As every $\gamma_i = \eta^{k(i)}$ we get (according to Definition 4) that $\underline{X} \in \mathcal{V}_k^\Phi$ if and only if the following conditions are valid:

$$\begin{aligned}
&(0) \quad i \in \mathcal{J}_0 \Rightarrow \xi_i, \zeta_i \in \eta^{m-k}\mathbf{A}\\
&(1) \quad i \in \mathcal{J}_1 \Rightarrow \xi_i, \zeta_i \in \eta^{m-k-1}\mathbf{A}\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&(\mathrm{j}) \quad i \in \mathcal{J}_j \Rightarrow \xi_i, \zeta_i \in \eta^{m-k-j}\mathbf{A},\ 0 \leqslant j \leqslant m-k-1\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
(\mathrm{m}-\mathrm{k}-1) \quad &i \in \mathcal{J}_{m-k-1} \Rightarrow \xi_i, \zeta_i \in \eta\mathbf{A}\\
(\mathrm{m}-\mathrm{k}) \quad &i \in \bigcup_{s=0}^{k} \mathcal{J}_{m-s} \Rightarrow \xi_i, \zeta_i \in \mathbf{A}
\end{aligned}$$

Let us construct the following system of submodules in $\mathcal{V}_k^\Phi$:

$$\mathcal{V}_{kj}^\Phi = \{\underline{X} \in \mathbf{M};\ \underline{X} \in \mathcal{V}_k^\Phi \wedge \underline{X} = \sum_{i \in \mathcal{J}_j} \xi_i \underline{U}_i\}, \quad 0 \leqslant j \leqslant m,$$

$$\mathcal{W}_{kj}^\Phi = \{\underline{X} \in \mathbf{M};\ \underline{X} \in \mathcal{V}_k^\Phi \wedge \underline{X} = \sum_{i \in \mathcal{J}_j} \zeta_i \underline{V}_i\}, \quad 0 \leqslant j \leqslant m.$$

Clearly, $\mathcal{V}_k^\Phi = \mathcal{V}_{k0}^\Phi \oplus \mathcal{V}_{k1}^\Phi \oplus \ldots \oplus \mathcal{V}_{km}^\Phi \oplus \mathcal{W}_{k0}^\Phi \oplus \mathcal{W}_{k1}^\Phi \oplus \ldots \oplus \mathcal{W}_{km}^\Phi$.

We get [from (0)], that $\mathcal{V}_{k0}^\Phi$ or $\mathcal{W}_{k0}^\Phi$, has $\mathbf{T}$-basis

$$\bigcup_{i \in \mathcal{J}_0} \{\eta^{m-k}\underline{U}_i, \ldots, \eta^{m-1}\underline{U}_i\} \text{ or } \bigcup_{i \in \mathcal{J}_0} \{\eta^{m-k}\underline{V}_i, \ldots, \eta^{m-1}\underline{V}_i\}, \text{ respectively;}$$

therefore $\dim_{\mathbf{T}} \mathcal{V}_{k0}^\Phi = \dim_{\mathbf{T}} \mathcal{W}_{k0}^\Phi = \frac{1}{2}\pi_0 k$. Analogously, conditions (j) imply that $\dim_{\mathbf{T}} \mathcal{V}_{kj}^\Phi = \dim_{\mathbf{T}} \mathcal{W}_{kj}^\Phi = \frac{1}{2}\pi_j(k+j)$, and the condition (m − k) implies that $\dim_{\mathbf{T}} \mathcal{V}_{kj}^\Phi = \dim_{\mathbf{T}} \mathcal{W}_{kj}^\Phi = \frac{1}{2}\pi_j m$, $m-k \leqslant j \leqslant m$.

The relation for the $\mathbf{T}$-dimension of $\mathcal{V}_k^\Phi$ is now evident. □

8. Theorem (inertial law). *Let a symplectic form* $\Phi\colon \mathbf{M}^2 \to \mathbf{A}$ *be given. If* $\mathcal{U}$, $\mathcal{V}$ *are arbitrary symplectic bases of* $\mathbf{M}$ *with respect to this form, then*

$$\mathfrak{Ch}(\Phi,\mathcal{U}) = \mathfrak{Ch}(\Phi,\mathcal{V}).$$

Proof. Let $\mathfrak{Ch}(\Phi,\mathcal{U}) = (\pi_0,\dots,\pi_m)$. Then Proposition II.7 implies

$$\dim_{\mathbf{T}} \mathcal{V}_k^{\Phi} = \sum_{j=0}^{m-k} \pi_j(k+j) + \sum_{j=m-k+1}^{m} \pi_j \cdot m,$$

$$\dim_{\mathbf{T}} \mathcal{V}_{k-1}^{\Phi} = \sum_{j=0}^{m-k} \bigl(\pi_j(k+j) - \pi_j\bigr) + \sum_{j=m-k+1}^{m} \pi_j \cdot m.$$

Consequently, we have $\dim_{\mathbf{T}} \mathcal{V}_k^{\Phi} - \dim_{\mathbf{T}} \mathcal{V}_{k-1}^{\Phi} = \sum\limits_{j=0}^{m-k} \pi_j$. Let $\mathfrak{Ch}(\Phi,\mathcal{V}) = (\nu_0,\dots,\nu_m)$. Then we obtain $\dim_{\mathbf{T}} \mathcal{V}_k^{\Phi} - \dim_{\mathbf{T}} \mathcal{V}_{k-1}^{\Phi} = \sum\limits_{h=0}^{m-k} \nu_h$, i.e. $\sum\limits_{j=0}^{m-k} \pi_j = \sum\limits_{h=0}^{m-k} \nu_h$. Putting $k = m$, $m-1$, …, 0, we get

$$\pi_0 = \nu_0,\ \ \pi_0 + \pi_1 = \nu_0 + \nu_1,\ \ \dots,\ \ \sum_{j=0}^{m-k} \pi_j + \pi_m = \sum_{h=0}^{m-k} \nu_h + \nu_m,$$

which successively yields $\pi_0 = \nu_0$, $\pi_1 = \nu_1$, …, $\pi_m = \nu_m$. □

References

[1] *F. W. Anderson, F. K. Fuller*: Rings and Categories of Modules. Springer-Verlag, New York, 1973.

[2] *E. Artin*: Geometric Algebra. Nauka, Moskva, 1969. (In Russian.)

[3] *M. F. Atiyah, I. G. MacDonald*: Introduction to Commutative Algebra. Mir, Moskva, 1972. (In Russian.)

[4] *M. Jukl*: Grassmann formula for certain type of modules. Acta Univ. Palack. Olomuc. Fac. Rerum Natur. Math. *34* (1995), 69–74.

[5] *M. Jukl*: Inertial laws of quadratic forms on modules over plural algebra. Math. Bohem. *120* (1995), 255–263.

[6] *M. Jukl*: Linear forms on free modules over certain local ring. Acta Univ. Palack. Olomuc. Fac. Rerum Natur. Math. *32* (1993), 49–62.

[7] *B. R. McDonald*: Geometric Algebra over Local Rings. Pure and applied mathematics, New York, 1976.

Author's address: *Marek Jukl*, Department of Algebra and Geometry, Faculty of Science, Palacký University, Tomkova 40, 779 00 Olomouc, Czech Republic, e-mail: `jukl@matnw.upol.cz`.