CENTRALLY DETERMINED STATES ON VON NEUMANN ALGEBRAS

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Summary. It is shown that every von Neumann algebra whose centre determines the state space is already abelian.

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The following question was posed in [4]: Is every von Neumann algebra with centrally determined state space abelian? The aim of this note is to establish a positive answer to this question.

Let $\mathcal{A}$ be an arbitrary von Neumann algebra and let Z be its centre. Let $\mathcal{P}(\mathcal{A})$ and $\mathcal{P}(Z)$ stand for the orthomodular lattices of all projections in $\mathcal{A}$ and Z, respectively (see [6]). Let us call a mapping $\eta\colon \mathcal{A} \to Z$ a *centre state* if it is positive, $\eta(C) = C$ and $\eta(CA) = C\eta(A)$ for every $C \in Z$ and $A \in \mathcal{A}$ (see [1]). Further, let us call a mapping $s\colon \mathcal{P}(\mathcal{A}) \to \langle 0,1\rangle$ a *state* if $s(I) = 1$ (I is an identity in A) and $s\Big(\sum\limits_{n\in N} P_n\Big) = \sum\limits_{n\in N} s(P_n)$, whenever (P_n) is sequence of mutually orthogonal elements of $\mathcal{P}(\mathcal{A})$. Finally, let us say that $\mathcal{A}$ has a *centrally determined state space* (see [2, 3]) if states s_1 and s_2 on $\mathcal{P}(\mathcal{A})$ coincide whenever they agree on $\mathcal{P}(Z)$.

Theorem. *A von Neumann algebra $\mathcal{A}$ has a centrally determined state space if and only if it is abelian.*

P r o o f. The sufficiency is obvious. Let us take up the necessity. Suppose that $\mathcal{A}$ is not abelian. Looking for a contradiction let us assume that $\mathcal{A}$ has centrally determined state space. Let us choose $A \in \mathcal{A} \setminus Z$. According to [1, Lemma 8.2.3, p. 512] $\mathcal{A}$ admits an ultraweakly continuous centre state $\eta\colon \mathcal{A} \to Z$, that is, $\mathcal{A}$ admits such a central state η which is additive with respect to any system of mutually orthogonal projections. Then, obviously $\eta(A) \neq A$. Let ω be a normal state of $\mathcal{A}$ such that $\omega(\eta(A)) \neq \omega(A)$. Put $\tilde{\omega} = \omega \circ \eta$. Then $\tilde{\omega}$ is a normal state of $\mathcal{A}$ again and we have $\omega|Z = \tilde{\omega}|Z$, $\omega(A) \neq \tilde{\omega}(A)$. However, using the spectral theorem, we see that ω and $\tilde{\omega}$ do not coincide on $\mathcal{P}(\mathcal{A})$. We have obtained two distinct states $\omega|\mathcal{P}(\mathcal{A})$ and $\tilde{\omega}|\mathcal{P}(\mathcal{A})$ which coincide on the centre $\mathcal{P}(Z)$. This is a contradiction and the proof is complete. □

It is easy to observe that the latter theorem holds even in more general situation, i.e., for instance it holds for any C^*-algebra $\mathcal{A}$ whose projections generate a dense subspace in $\mathcal{A}$. It should be also noted that our result may be relevant to the noncommutative measure theory on von Neumann algebras. Namely, our theorem complemented with results in [2, 3] implies that the classical version of the Radon-Nikodym theorem holds exactly in the classical (i.e. commutative) case (compare also with [5]).

References

[1] *R. V. Kadison, J. R. Ringrose*: Fundamentals of the Theory of Operator Algebras, Academic Press, 1986.
[2] *M. Navara*: Stavový Prostor Kvantových Logik, Ph.d. thesis, Prague, 1987.
[3] *M. Navara, P. Pták*: On the Radon-Nikodým property in quantum logics, Proceedings of the Conference Topology and Measure VI, Wardeminde, 1991, to appear.
[4] Proceedings of the 2nd Winter School on Measure Theory, Liptovský Ján, January 1990.
[5] *S. Sakai*: A Radon-Nikodym theorem in W^*-algebras, Bull. Amer. Math. Soc. *71* (1965), 149–151.
[6] *V. Varadarajan*: Geometry of Quantum Theory I, Van Nostrand, Princeton, New Jersey, 1968.

Souhrn

CENTRÁLNĚ DETERMINOVANÉ STAVY NA VON NEUMANNOVÝCH ALGEBRÁCH

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Je ukázáno, že každá von Neumannova algebra, jejíž centrum určuje stavový prostor, je abelovská.

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