Mathematica Bohemica

Corneliu A. Marinov; Gheorghe Moroşanu

Consistent models for electrical networks with distributed parameters

Mathematica Bohemica, Vol. 117 (1992), No. 2, 113–122

Persistent URL: http://dml.cz/dmlcz/125904

117 (1992) MATHEMATICA BOHEMICA No. 2, 113–122

CONSISTENT MODELS FOR ELECTRICAL NETWORKS WITH DISTRIBUTED PARAMETERS

CORNELIU A. MARINOV, Bucharest, GHEORGHE MOROŞANU, Iaşi

(Received August 29, 1988)

Summary. A system of one-dimensional linear parabolic equations coupled by boundary conditions which include additional state variables, is considered. This system describes an electric circuit with distributed parameter lines and lumped capacitors all connected through a resistive multiport. By using the monotony in a space of the form $L^2(0,T; H^1)$, one proves the existence and uniqueness of a variational solution, if reasonable engineering hypotheses are fulfilled.

Keywords: parabolic equations, initial-boundary value problem, monotone operators, variational solution

AMS classification: 35K45, 35K50, 47B44

1. INTRODUCTION

The structures with distributed electrical parameters are described by the well known hyperbolic type telegraph equations ([4]), treated in various works ([3], [12], [13]).

Our paper refers to the models of digital integrated semiconductor circuits in which the distributed resistance, capacitance and even conductance naturally arise. Since the inductance is practically absent, the equations are of parabolic type. In this case, most of engineering oriented works (for instance [17], [18], [5]) approximate the distributed element by a lumped parameter circuit, which means to study a system of ordinary differential equations. However, there are papers ([6–11], [15], [16]) in which the modelling is "exact" in the sence that the distributed parameter structure (i.e. the parabolic partial differential equations) is retained. This is necessary, for example, when one studies the problem of the delay time in MOS digital circuits ([17], [18]) where the correct modelling of device interconnections is essential.

Our paper deals with the existence and uniqueness of a solution for a large class of circuit models, specified in Section 2 and described by a system of parabolic linear equations coupled by "crossed" boundary conditions, and also involving ordinary differential equations. Section 3 gives sufficient conditions which ensure that our model is a "consident" one, i.e. it has a variational solution which is unique. Our main tool in proof is the theory of monotone operators ([1], [2], [12]). In Section 4 an engineering example is given, which proves that our conditions ane not very restrictive so that our method can be applied to actual networks. For the same reason of direct applicability we keep all coefficients and parameters as they appear in electrical engineering in spite of a somewhat unpleassant mathematical form of our equations.

The problem under study is a linear one. However, the method used in the proof, as will be clear below, can be used to treat some nonlinear cases which can be easily inferred.

2. Mathematical model

We say that a network belongs to the (G, B) class if its resistive lumped part can be viewed as a linear multiport governed by the equation:

$$j = -G \cdot v + B(t) \tag{2.1}$$

Her $j, v \in \mathbf{R}^{2n+m}$ are respectively the current and voltage vectors at $2n + m$ pairs of terminals, G is a $(2n + m) \times (2n + m)$ matrix (partitioned as in Fig. 2.1) and $B(t) \in \mathbf{R}^{2n+m}$ for all $t > 0$.

To the first $2n$ pairs of terminals of the multipart n distributed parameters elements ($r_k > 0, c_k > 0, g_k \geqslant 0$) are connected, described by one-dimensional degenerate telegraph equations, [4]:

$$\text{(E)} \qquad \begin{cases} c_k \dfrac{\partial u_k}{\partial t} = \dfrac{1}{r_k} \dfrac{\partial^2 u_k}{\partial x^2} - g_k u_k \\ x \in (0, d_k),\ k = \overline{1, n},\ t > 0 \end{cases}$$

where $u_k(t, x)$ is the voltage at the moment t at a point x of the structure. To the last m pairs of terminals capacitors with capacity $c_k > 0$, $k = \overline{1, m}$ are connected, representing (together with some resistors from the multiport) the lumped modelled devices.

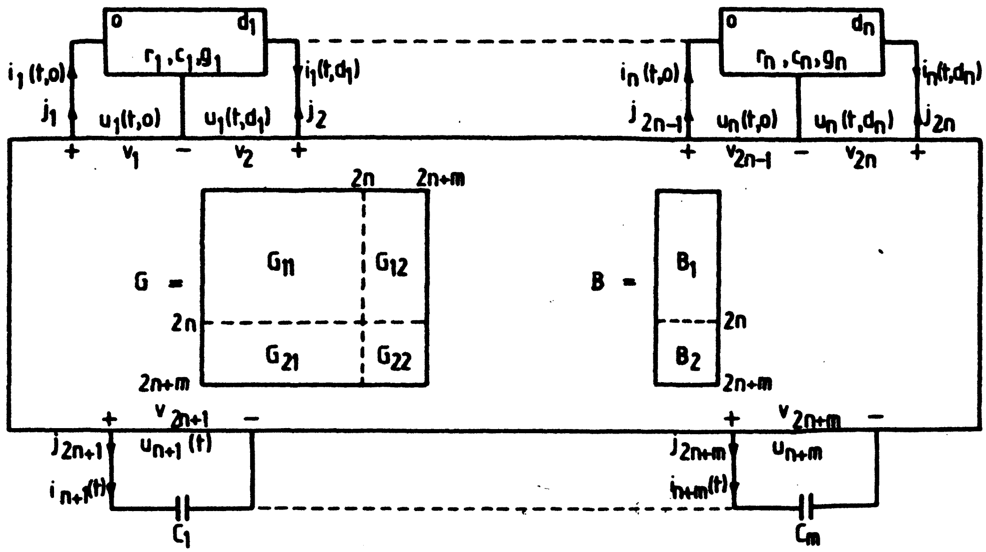

Fig. 2.1

If we take into account that $i_k = -\frac{1}{r_k}\frac{\partial u_k}{\partial x}$, $j_{2k-1} = i_k(t,0)$, $j_{2k} = -i_k(t,d_k)$, $v_{2k-1} = u_k(t,0)$, $v_{2k} = u_k(t,d_k)$ for $k = \overline{1,n}$ and $i_k = C_{k-n}\frac{\mathrm{d}u_k}{\mathrm{d}t}$, $j_{n+k} = i_k(t)$, $v_{n+k} = u_k(t)$ for $k = \overline{n+1, n+m}$, (2.1) yields

$$\text{(BC)} \qquad \begin{bmatrix} -\frac{1}{r_1} & \frac{\partial u_1}{\partial x}(t,0) \\ +\frac{1}{r_1} & \frac{\partial u_1}{\partial x}(t,d_1) \\ & \vdots \\ -\frac{1}{r_n} & \frac{\partial u_n}{\partial x}(t,0) \\ +\frac{1}{r_n} & \frac{\partial u_n}{\partial x}(t,d_n) \\ & C_1\frac{\mathrm{d}u_{n+1}}{\mathrm{d}t}(t) \\ & \vdots \\ & C_m\frac{\mathrm{d}u_{n+m}}{\mathrm{d}t}(t) \end{bmatrix} = -G \begin{bmatrix} u_1(t,0) \\ u_1(t,d_1) \\ \vdots \\ u_n(t,0) \\ u_n(t,d_n) \\ u_{n+1}(t) \\ \vdots \\ u_{n+m}(t) \end{bmatrix} + B(t).$$

In fact, we shall add the initial conditions

$$\text{(IC)} \qquad \begin{cases} u_k(0,x) = u_{k0}(x), & x \in (0,d_k),\ k = \overline{1,n} \\ u_k(0) = u_{k0}, & k = \overline{n+1,n+m} \end{cases}$$

In the sequel we shall use the following notation

$$
\begin{aligned}
u &= (u_1, u_2, \ldots, u_n)^{tr},\\
u^1 &= (u_{n+1}, u_{n+2}, \ldots, u_{n+m})^{tr},\\
u_0 &= (u_{10}, u_{20}, \ldots, u_{n0})^{tr},\\
u_0^1 &= (u_{n+1,0}, u_{n+2,0}, \ldots, u_{n+m,0})^{tr},\\
u_b(s) &= \big(u_1(s,0), u_1(s,d_1), \ldots, u_n(s,0), u_n(s,d_n)\big)^{tr},\\
C &= \operatorname{diag}(C_1, C_2, \ldots, C_m),\\
\bar G_{21} &= C^{-1} G_{21},\\
\bar G_{22} &= C^{-1} G_{22},\\
\bar B_2(t) &= C^{-1} B_2(t),\\
\bar B_1(t) &= B_1(t) - G_{12} \cdot \mathrm{e}^{-t\bar G_{22}} u_0^1 - G_{12} \int_0^t \mathrm{e}^{(s-t)\bar G_{22}} \cdot \bar B_2(s)\mathrm{d}s,\\
K(t) &= G_{12} e^{-t\bar G_{22}} \cdot \bar G_{21},\\
M &= \operatorname{diag}(c_1, c_2, \ldots, c_n),\\
N &= \operatorname{diag}(r_1^{-1}, r_2^{-1}, \ldots, r_n^{-1}),\\
P &= \operatorname{diag}(g_1, g_2, \ldots, g_n),\\
S &= \operatorname{diag}\left(-\frac{1}{r_1}, +\frac{1}{r_1}, \ldots, -\frac{1}{r_n}, +\frac{1}{r_n}\right)
\end{aligned}
$$

If we formally solve the last m equations from (BC), we get

$$
u^1 = \mathrm{e}^{-t\bar G_{22}} u_0^1 - \int_0^t \mathrm{e}^{(s-t)\bar G_{22}} \cdot \bar G_{21} u_b(s)\mathrm{d}s + \int_0^t \mathrm{e}^{(s-t)\bar G_{22}} \cdot \bar B_2(s)\mathrm{d}s. \tag{2.2}
$$

Substituting (2.2) in to (BC) we obtain the boundary conditions (bc) of a problem with only the function u as unknown:

$$
M\frac{\partial u}{\partial t}(t,x) = N\frac{\partial^2 u}{\partial x^2}(t,x) - Pu(t,x) \tag{e}
$$

$$
S\left(\frac{\partial u}{\partial x}\right)_b(t) = -G_{22}u_b(t) + \int_0^t K(t-s)u_b(s)\mathrm{d}s + \bar B_1(t) \tag{bc}
$$

$$
u(0,x) = u_0(x) \tag{ic}
$$

where the derivatives of the vector functions are meant componentwise.

The mathematical core of this paper is the problem of existence and uniqueness of a weak solution of the problem (e)+(bc)+(ic). If these properties are already established, they extend (due of (2.2)) also to the solution $\begin{bmatrix} u \\ u^1 \end{bmatrix}$ of the problem (E)+(BC)+(IC).

Existence and uniqueness

If we denote $V = \prod_{k=1}^{n} H^1(0, d_k)$ and $H = \prod_{k=1}^{n} L^2(0, d_k)$ then identifying H with its own dual, we have $V \subset H \subset V^*$ in both algebraic and topological sense. Also, if we fix $T > 0$ for $\mathcal{V} = L^2(0,T;V)$ and $\mathcal{H} = L^2(0,T;H) = \mathcal{H}^*$ we clearly obtain $\mathcal{V} \subset \mathcal{H} \subset \mathcal{V}^* \equiv L^2(0,T,V^*)$. For two natural numbers $\ell \leqslant n$, we shall denote by $\tau(\ell,n) = \{\tau_1, \tau_2, \ldots, \tau_\ell\}$ an incresing sequence of ℓ elements selected from $\{1,2,\ldots,n\}$ and by $x_{\tau(\ell,n)}$ the vector $(x_{\tau_1}, x_{\tau_2}, \ldots, x_{\tau_\ell})^{\mathrm{tr}} \in \mathbf{R}^\ell$. Let $\tilde{\tau}(\ell,n) = \{1,2,\ldots,n\} \setminus \tau(\ell,n)$ an increasing sequence, as well.

It is well known that the following two norms of $\mathcal{V}$ are equivalent:

$$\|u\|_{\mathcal{V}}^2 = \|u\|_{\mathcal{H}}^2 + \left\| \frac{\partial u}{\partial x} \right\|_{\mathcal{H}}^2$$

and

$$\|u\|_{\mathcal{V}}^2(\xi, \tau(\ell,n)) = \|u_{\tau(\ell,n)}(.,\xi)\|^2_{L^2(0,T,\mathbf{R}^\ell)} + \|u_{\tilde{\tau}(\ell,n)}\|^2_{L^2\left(0,T;\ \prod_{k \in \tilde{\tau}(\ell,n)} L^2(0,dk)\right)} + \left\| \frac{\partial u}{\partial x} \right\|_{\mathcal{H}}^2$$

for any $\tau(\ell,n)$ with $1 \leqslant \ell \leqslant n$ and $\xi \in \prod_{l.c. \in \tau(\ell,n)} [0, d_k]$ provided $u_{\tilde{\tau}(n,n)} = 0$.

Also, for $u \in \mathcal{V}$, if we denote by $u_k(t,0)$ and $u_k(t,d_k)$ the trace values of $u_k(t,.) \in H^1(0,d_k)$ at 0 and d_k respectively, then $u_b(.)$ defined in the previous section is in $L^2(0,T;\mathbf{R}^{2n})$ and due to the above equivalence of the $\mathcal{V}$ norms we have $\|u_b\|^2_{L^2(0,1;\mathbf{R}^{2n})} \leqslant k\|u\|^2_{\mathcal{V}}$.

Let us now define on $\mathcal{V} \times \mathcal{V}$ the form

$$a(u,v) = \left\langle N\frac{\partial u}{\partial x}, \frac{\partial v}{\partial x} \right\rangle_{\mathcal{H}} + \langle Pu, v \rangle_{\mathcal{H}} + \langle \mathcal{T}u_b, v_b \rangle_{L^2(0,T,\mathbf{R}^{2n})}$$

where $\mathcal{T}\colon L^2(0,T,\mathbf{R}^{2n}) \to L^2(0,T,\mathbf{R}^{2n})$ is the linear operator defined by

$$(\mathcal{T}f)(t) = G_{11}f(t) - \int_0^t K(t-s)f(s)\mathrm{d}s.$$

The following properties are rather straightforward.

Lemma 3.1.

1. *For each $u \in \mathcal{V}$, $v \to a(u,v)$ is a continuous linear functional on $\mathcal{V}$.*
2. *The operator $\mathcal{A}\colon \mathcal{D}(\mathcal{A}) = \mathcal{V} \to \mathcal{V}^*$ defined by $\langle \mathcal{A}u, v\rangle = a(u,v)$ is demicontinuous.*

Note that, here and below, $\langle y, z\rangle$ denotes the value of $y \in \mathcal{V}^*$ for $z \in \mathcal{V}$, with $\langle y, z\rangle = \langle y, z\rangle_{\mathcal{H}}$ (the inner product in $\mathcal{H}$) provided $y \in \mathcal{H}^*$.

Let us define a linear operator $\mathcal{B}\colon \mathcal{D}(\mathcal{B}) \subset \mathcal{V} \to \mathcal{V}^*$ by $\mathcal{D}(\mathcal{B}) = \{u \in \mathcal{V}\colon \dfrac{\partial u}{\partial t} \in \mathcal{V}^*$ and $u(0) = u_0 \in H\}$ and $\mathcal{B}u = M\dfrac{\partial u}{\partial t}$.

Lemma 3.2. *$\mathcal{B}$ is maximal monotone.*

The proof is essentialy known ([1], p. 167). □

Definition. A function u is called a variational solution of the problem (e) + (bc) + (ic) if $u \in \mathcal{D}(\mathcal{B})$ and

$$\left\langle M\frac{\partial u}{\partial t}, v\right\rangle + a(u,v) = \langle \bar{B}_1, v_b\rangle_{L^2(0,T;\mathbf{R}^{2n})}$$

for all $v \in \mathcal{V}$.

Let us remark that a sufficiently smooth classical solution of (e)+(bc)+(ic) is a variational solution, as an elementary calculus shows. Conversely, the fact that u is a variational solution having in addition certain regularity properties (for example $\dfrac{\partial u}{\partial t} \in \mathcal{H}$, $\dfrac{\partial^2 u}{\partial x^2} \in \mathcal{H}$) implies that u verifies (e)+(bc)+(ic) a.e. on $[0,T]$ with $u_0 \in H$.

The following hypotheses will be used below:

A$_1$. G is a symmetric matrix and G_{22} is a positive definite one with $\gamma > 0$ as its smallest eigenvalue.

A$_2$. 1. If $g_1, g_2, \ldots, g_\ell = 0$ and $g_{\ell+1}, g_{\ell+2}, \ldots, g_n > 0$, then there are $\alpha, \beta > 0$ with $\alpha - \frac{\beta^2}{\gamma} > 0$ and $\tau(\ell, 2n)$ with $\tau_k \in \{2k-1, 2k\}$ for $k = \overline{1,\ell}$ such that

$$(3.2)\qquad \begin{cases} \langle G_{11}x, x\rangle_{\mathbf{R}^{2n}} \geqslant \alpha\|x_{\tau(\ell,2n)}\|^2_{\mathbf{R}^\ell} & \text{and} \\ \|G_{21}x\|_{\mathbf{R}^m} \leqslant \beta\|x_{\tau(\ell,2n)}\|_{\mathbf{R}^\ell} & \text{for all } x \in \mathbf{R}^{2n}. \end{cases}$$

2. If $g_1, g_2, \ldots, g_n > 0$, then there exist $\alpha, \beta > 0$ with $\alpha - \frac{\beta^2}{\gamma} \geqslant 0$ and $\tau(\ell, 2n)$ such that (3.2) is valid.

A$_3$. $B_1 \in L^2(0,T;\mathbf{R}^{2n})$ and $B_2 \in L^1(0,T;\mathbf{R}^m)$.

The following property of the operator $\mathcal{A}$ defined in Lemma 3.1 will play a prominent part in the sequel:

Lemma 3.3. *Let us suppose A_1 and A_2 to be valid. Then $\mathcal{A}$ is strongly monotone.*

P r o o f. Let us consider the case $g_1, g_2, \ldots, g_\ell = 0$ and $g_{\ell+1}, g_{\ell+2}, \ldots, g_n > 0$. Because $\mathcal{A}$ is linear and everywhere defined, the only fact we have to prove is the existence of $\delta > 0$ such that for any $u \in \mathcal{V}$

$$\langle \mathcal{A}u, u\rangle \geqslant \delta\|u\|^2 \tag{3.3}$$

Let p be a complex number with $\lambda = \operatorname{Re} p > 0$. It follows by A_1 that for every $x \in \mathbf{C}^m$ we have

$$\operatorname{Re}\big\langle (G_{22} + pC)x, x\big\rangle_{\mathbf{C}^m} \geqslant (\gamma + \lambda \min_{i=\overline{1,m}} C_i)\|x\|^2_{\mathbf{C}^m}.$$

So the Laplace transform

$$\hat{K}(p) = G_{12}(pI + C^{-1}G_{22})^{-1}C^{-1}G_{21} = G_{12}(pC + G_{22})^{-1}G_{21}$$

exists and

$$\begin{aligned}\operatorname{Re}\big\langle \hat{K}(p)\hat{u}_b(p), \hat{u}_b(p)\big\rangle_{\mathbf{C}^{2n}} &= \operatorname{Re}\big\langle (pC + G_{22})^{-1}G_{21}\hat{u}_b(p), G_{21}\hat{u}_b(p)\big\rangle_{\mathbf{C}^m} \\ &\leqslant \frac{1}{\gamma + \lambda \min\limits_{i=\overline{1,m}} C_i}\|G_{21}\hat{u}_b(p)\|^2_{\mathbf{C}^m}\end{aligned}$$

where $\hat{u}_b$ is the Laplace transform of $u_b \in L^2(0, T; \mathbf{R}^{2n})$ extended with zero values on $[T, \infty)$.

In view of the assumption $A_2.1$ we obtain

$$\operatorname{Re}\big\langle \hat{K}(p), \hat{u}_b(p), \hat{u}_b(p)\big\rangle_{\mathbf{C}^{2n}} \leqslant \frac{\beta^2}{\gamma + \lambda \min\limits_{i=\overline{1,m}} C_i}\|\hat{u}_b(p)\|^2_{\mathbf{C}^\ell}$$

for any $p \in \mathbf{C}$ with $\lambda > 0$.

A slight change in the proof of a known theorem regarding the positivity of kernels (see [1], p. 236 for the scalar case) shows taht the last inequality implies

$$\begin{aligned}\int_0^T \Big\langle \int_0^t K(t-s)u_b(s)\mathrm{d}s, u_b(t)\Big\rangle_{\mathbf{R}^{2n}} \mathrm{d}t &\leqslant \frac{\beta^2}{\gamma}\|u_b\|^2_{L^2(0,T;\mathbf{R}^\ell)} \cdot \langle \mathcal{T}u_b, u_b\rangle_{L^2(0,T;\mathbf{R}^{2n})} \\ &\geqslant \left(\alpha - \frac{\beta^2}{\gamma}\right)\|u_{\tau(\ell,n)}(.,\xi)\|^2_{L^2(0,T;\mathbf{R}^\ell)}\end{aligned} \tag{3.4}$$

where $\xi_k = 0$ for $\tau_k = 2k - 1$ and $\xi_k = d_k$ if $\tau_k = 2k$, $k = \overline{1,\ell}$.

On the other hand, we easily observe that

$$\left\langle N\frac{\partial u}{\partial x}, \frac{\partial u}{\partial x}\right\rangle_{\mathcal{H}} \geqslant \left(\min_{k=\overline{1,n}} \frac{1}{r_k}\right)\left\|\frac{\partial u}{\partial x}\right\|^2_{\mathcal{H}}$$

and

$$\langle Pu, u\rangle \geqslant \left(\min_{k=\overline{\ell+1,n}} g_k\right)\|u_{\bar{\tau}(\ell,n)}\|^2_{L^2\left(0,T;\ \prod\limits_{k\in\bar{\tau}(\ell,n)} L^2(0,d_k)\right)}.$$

These inequalities together with (3.4) yield the positive constant δ in the required inequality (3.3).

The case $g_1, g_2, \ldots, g_n > 0$ can be treated similarly.

Now we are ready to establish the main result. □

Theorem 3.1. *If A_1, A_2, A_3 hold, then there is a unique variational solution of the problem (e)+(bc)+(ic).*

Proof. According to Lemma 3.1, 3.2, 3.3 and to a well known perturbation result of Rockafellar ([1] p. 48) the operator $\mathcal{A}+\mathcal{B}$ is maximal monotone and coercive. Therefore this operator is also surjective from $\mathcal{D}(\mathcal{B})$ in $\mathcal{V}^*$. This means if we take $f \in \mathcal{V}^*$ with $\langle f, v\rangle = \langle \bar{B}_1, v_b\rangle_{L^2(0,T;\mathbf{R}^{2n})}$, the equation $\mathcal{A}u + \mathcal{B}u = f$ has a solution in $\mathcal{D}(\mathcal{B})$ and due to the monotony this solution is unique. But this equation is equivalent to (3.1) and the proof is complete. □

Remarks.

1. Our solution belongs to $C(0,T;H)$—see [2] p. 62.
2. If u is already found, (2.2) implies $u^1 \in \mathrm{cap}^{1,1}(0,T;\mathbf{R}^m)$.
3. The above theorem can be paraphrased as follows: If A_1, A_2 and A_3 are valid, the model (E)+(BC)+(IC) of the (G,B) class of circuits is a consistent one.

4. An example

Let us consider the simplest fan out circuit appearing in MOS interconnections, [17], and presented in Fig. 4.1.

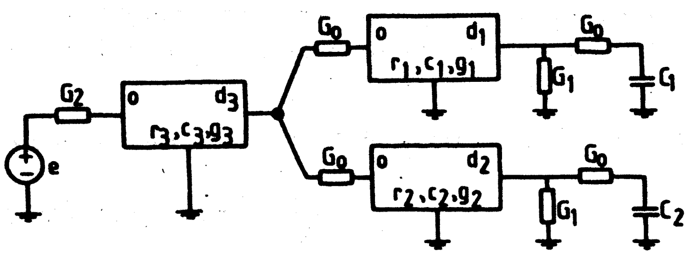

Fig. 4.1

This network is in the (G, B) class with

$$G_{11} = \begin{bmatrix} G_0 & 0 & 0 & 0 & 0 & -G_0 \\ 0 & G_0+G_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & G_0 & 0 & 0 & -G_0 \\ 0 & 0 & 0 & G_0+G_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & G_2 & 0 \\ -G_0 & 0 & -G_0 & 0 & 0 & 2G_0 \end{bmatrix},$$

$$G_{12} = G_{21}^{tr} = \begin{bmatrix} 0 & 0 \\ -G_0 & 0 \\ 0 & 0 \\ 0 & -G_0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix},$$

$$G_{22} = \mathrm{diag}(G_0, G_0), \ B(t) = (0,0,0,0,G_2 e(t),0,0,0)^{tr}.$$

Considering the initial condition $u_0 \in \prod_{k=1}^{3} L^2(0, d_k)$ and the source $e \in L^2(0,T)$, let us introduce two cases:

1. $g_1 = g_2 = 0$, $g_3 > 0$, $G_0, G_1, G_2 > 0$.

The hypotheses A_1, $A_2 1$, A_3 are fulfilled with $\ell = 2$, $\tau_{(2,6)} = \{2,4\}$, $\alpha = G_0 + G_1$ and $\beta = \gamma = G_0$. Therefore the model is consistent.

2. $g_1, g_2, g_3 > 0$, $G_0, G_2 > 0$, $G_1 = 0$.

Again the assumptions A_1, $A_2 2$, A_3 are fulfilled with $\ell = 2$, $\tau_{(2,6)} = \{2,4\}$, $\alpha = \beta = \gamma = G_0$ and the model is consistent in this case as well.

The above result is quite natural from an intuitive engineer's point of view: when the distributed conductances g_1 and g_2 are absent from the model, we must add two lumped conductances G_1 to the ground and the consistency is ensured. So, our main theorem 3.1 seems to be quite encouraging in an area where is a great need for results of similar type concerned with models of semiconductor circuits.

References

[1] *V. Barbu*: Nonlinear Semigroups and Differential Equations in Banach Spaces, Noordhoff, Leyden, 1976.

[2] *V. Barbu, T. Precupanu*: Convexity and Optimization in Banach Spaces, D. Reidel Publ. Co., Dordrecht, Boston, Lancaster, 1987.

[3] *K. L. Cooke, D. W. Krumme*: Differential-difference equations and nonlinear initial boundary value problems for hyperbolic partial differential equations, J. Math. Anal. Appl. *24* (1968), 372–387.

[4] *M. S. Ghausi, J. J. Kelly*: Introduction to Distributed Parameter Networks with Applications to Integrated Circuits, Holt, Rinehart Winston, New-York, 1968.

[5] *C. A. Marinov*: Qualitative properties of ℓ^p solutions of infinite differential systems via dissipativity, Nonlinear Analysis T.M.A. *8* (1984), 441–456.
[6] *C. A. Marinov*: The delay time for a rcg line, Int. J. Circuit Theory Appl. *15* (1987), 79–83.
[7] *C. A. Marinov, A. Lehtonen*: Mixed-type circuits with distributed and lumped parameters, IEEE Trans. Circ. Syst. *CAS-36* no. 8 (1989), 1080–1086.
[8] *C. A. Marinov, P. Neittaanmaki*: Delay time for general distributed-networks with applications to timing analysis of digital MOS integrated circuits, Simulation of Semiconductor Devices and Processes, vol. 2 (K. Board, D. R. J. Owen, eds.), Pineridge Press, 1986, pp. 322–336.
[9] *C. A. Marinov, P, Neittaanmaki*: A theory of electrical circuits with resistively coupled distributed structures. Delay time pradicting, IEEE Trans. Circ. Syst. *CAS-35* no. 2 (Feb. 1988), 166–175.
[10] *C. A. Marinov, P, Niettaanmaki*: Asymptotical convergence evalution for a parabolic problem arising in circuits theory, ZAMM, Z. Angew. Math. Mech. *70* no. 8 (1990), 344–347.
[11] *C. A. Marinov, P. Neittaanmaki*: A delay time bound distributed parameter circuits with bipolar transistors, Int. J. Circ. Th. Appl. *18* (1990), 99–106.
[12] *G. Morosanu*: Nonlinear Evolution Equations and Applications, D. Riedel Publ. Co., Dordrechr, Boston, Lancaster, 1987.
[13] *G. Moroşanu*: Mixed problems for a class of nonlinear differential hyperbolic systems, J. Math. Anal. Appl. *33* (1971), 470–485.
[14] *G. Moroşanu, C. A. Marinov, P. Neittaanmaki*: Well-posed nonlinear problems in integrated circuits modelling, Circ. Syst. Sign. Proc. *10* (1991), 53–69.
[15] *G, Prada, T. A. Bickart*: Stability of electrical network containing distributed RC components, J. Math. Anal. Appl. *33* (1971), 367–401.
[16] *R. E. Showalter, C. H. Snyder*: A distributed RC network model with dielectric loss, IEEE Trans. Circ. Syst. *CAS-33* (1986), 707–710.
[17] *J. Rubinstein, P. Penfield, M. Horowitz*: Signal delay in RC tree networks, IEEE Trans. Comp. Aided Design *CAD-2* (1983), 202–211.
[18] *J. L. Wyatt Jr.*: Monotone sensitivity of nonlinear nonuniform RC transmission lines with applications to timing analysis of digital MOS integrated circuits, IEEE Trans. Circ. Syst. *CAS-32* (1985), 28–33.

Authors' addresses: *C. A. Marinov*, Associate Professor, Faculty of Electrotechnics, Polytechnical Institute of Bucharest, 77206 Bucharest, Romania; *G. Moroşanu*, Associate Professor, Faculty of Mathematics, University of Iaşi, 6600 Iaşi, Romania.