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ON THE LEBESGUE DECOMPOSITION OF THE NORMAL STATES OF A JBW-ALGEBRA

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Summary. In this article, a theorem is proved asserting that any linear functional defined on a JBW-algebra admits a Lebesgue decomposition with respect to any normal state defined on the algebra. Then we show that the positivity (and the unicity) of this decomposition is insured for the trace states defined on the algebra. In fact, this property can be used to give a new characterization of the trace states amongst all the normal states.

Keywords: JBW-algebra; state, normal state, trace state, Lebesgue decomposition of a linear functional with respect to another linear functional, support of linear functional.

AMS classification: 06C15, 46H70, 46L30, 46L50, 81B10

1. INTRODUCTION

The study of Jordan algebras has its origin in the mathematical foundation of quantum physics where one of the natural axioms is that the observables form a Jordan algebra. Furthermore, if we want the observables to satisfy the functional calculus of spectral theory, we would assume them to form a JB-algebra. A JBW-algebra is the Jordan analogue of an "abstract" von Neumann algebra, that is, a C^* algebra which is also a Banach dual space. Von Neumann algebras have been extensively studied in the literature. The main purpose of this paper is to investigate the Lebesgue decomposition of linear functionals with respect to a given linear functional in this setting of a JBW-algebra.

A generalization of the classical theorem on the Lebesgue decomposition of a σ-finite signed measure with respect to a σ-finite measure on a σ-algebra is obtained

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for the normal states of certain JBW-algebras A such as shown in [6] and [8]. In [6], we suppose that A is the JBW-algebra of all self adjoint operators on a separable Hilbert space of dimension at least three; in [8], we suppose that A is associative. In the latter, the decomposition is positive. In this paper, we establish, in particular, a similar theorem for any JBW-algebra.

This paper is divided into two main sections. The first presents facts and definitions needed later. Then, we state and prove the main theorem on the Lebesgue decomposition of the linear functionals defined on a JBW-algebra A with respect to a normal state on A. Then, we show that the positivity and unicity of this Lebesgue decomposition is ensured in the case of the trace states. Moreover, this last property can be used to give a new characterization of the trace states amongst all the normal states on A. As an application of the aforementioned result, we obtain a characterization of associative JBW-algebras in terms of the "bigness" of the trace space of the algebra.

2. Preliminaries and notation

Let us begin with the definitions and basic facts from the theory of JBW-algebras pertaining to this paper. For greater details, we refer the reader to Hanche-Olsen and Størmer [5] ([1] and [9] are also examples).

A real algebra A, not necessarily associative, equipped with the product $(a, b) \to a \circ b$ is said to be a *Jordan algebra* if the identities

$$a \circ b = b \circ a$$
$$a \circ (b \circ a^2) = (a \circ b) \circ a^2$$

hold true for any a, $b \in A$.

A Jordan algebra A is said to be a *JB-algebra* if it is also a Banach space with respect to a norm $\|\cdot\|$ satisfying, for any $a, b \in A$,

$$\|a \circ b\| \leqslant \|a\| \, \|b\|$$
$$\|a^2\| = \|a\|^2$$
$$\|a^2\| \leqslant \|a^2 + b^2\|.$$

We denote by A_+

$$A_+ := \{a^2;\, a \in A\}$$

the set of all the positive elements in A. A_+ is a generating cone in A, called the positive cone of A. For our purposes, A is considered equipped with the partial vector space order, denoted by $\leqslant$, induced by the cone A_+. As usual, a linear functional $f: A \to \mathbf{R}$ will be said to be a ***positive linear functional*** if $f(A_+) \subseteq \mathbf{R}_+$. A positive linear functional such that $\|f\| = 1$ is called a ***state*** on A.

A JB-algebra A is said to be ***monotone complete*** if each bounded increasing net (a_α) in A has a least upper bound a in A. A bounded linear functional f on A is called ***normal*** if $f(a_\alpha) \to f(a)$ for each net (a_α)as above. We will denote by $S(A)$ the set of the normal states on A. $S(A)$ is referred to as the ***normal state space*** of A. A is said to be a ***JBW-algebra*** if A is monotone complete and $S(A)$ is a separating family. According to theorem 4.4.16 of [5], this is equivalent to the fact that A is a Banach dual space. This predual of A is unique and consists of the normal linear functionals on A.

In all that follows, A will be a JBW-algebra A^* and A_* will denote, respectively, the dual and the predual of A; $\sigma(A, A_*)$ will denote the w^*-topology on A determined by A_*. We will also consider A_* to be imbedded into A^* under the evaluation map.

For any $a \in A$, the two operators T_a and U_a are defined for any $b \in A$ by:

$$T_a(b) = a \circ b \quad .$$
$$U_a(b) = \{aba\}$$

where, for elements a, b, and c in A, the Jordan triple product is defined by

$$\{abc\} = a \circ (b \circ c) - b \circ (c \circ a) + c \circ (a \circ b).$$

If the operators T_a and T_b commute, the two elements a, b in A are said to ***operator commute***.

We denote by $I(A)$ the collection of all ***idempotents*** of A,

$$I(A) := \{p \in A\,; p^2 = p\}$$

and, for any element q in $I(A)$, we write

$$q^\perp = 1 - q$$

where 1 denotes the unit in algebra A whose existence is guaranteed by lemma 4.1.7 of [5].

Two elements p and q in $I(A)$ are said to be ***orthogonal*** if $p \leqslant q^\perp$ (or equivalently if $p \circ q = 0$). It follows from lemmas 4.2.6. and 4.2.8. of [5] that $\langle I(A), \leqslant, \perp, 0, 1\rangle$ is a complete orthomodular lattice (see [8], section two for the definition). If D is a nonempty subset of $I(A)$, we denote by $\vee D$ (resp. by $\wedge D$) the supremum of D (resp. the infimum of D). When $D = \{p, q\}$, we will simply write $\vee D = p \vee q$ and

$\wedge D = p \wedge q$. We also denote by $\mathcal{F}(D)$ the collection of all finite subsets of D directed by set inclusion. Given that D is an orthogonal subset of $I(A)$ (that is, D is a family of pairwise orthogonal idempotents of A), then

$$\vee D = \sigma(A, A_*) - \lim_{F \in \mathcal{F}(D)} \sum_{p \in F} p.$$

Without a doubt, the most important example of a JBW-algebra is the algebra $B(H)_{sa}$ of all self-adjoint bounded linear operators on a Hilbert space H equipped with the operator norm and the product $(a, b) \to a \circ b = \frac{ab+ba}{2}$, where ab is the usual operator composition. In this case, $I(B(H)_{sa})$ is the famous lattice of the closed subspaces of H; this lattice plays a significant role, in particular in quantum mechanics. A profound, celebrated theorem by Gleason [4] asserts that if H is separable and at least tridimensional, then there is a one-to-one correspondance between nonnegative finite signed states on $I(B(H)_{sa})$ [such a state is called a Gleason measure] and the positive semidefinite self adjoint operators of the trace class on H (see [4] for details).

3. The Lebesgue decomposition

The purpose of this section is to show, in particular, that any linear functional on a JBW-algebra admits a Lebesgue decomposition with respect to any normal state on the algebra. We begin with the necessary definitions.

Let A be a JBW-algebra. For any linear functional f on A, we let

$$N(f) := \{p \in I(A);\ f(q) = 0 \quad \text{for any } q \text{ in } I(A) \text{ such that } q \leqslant p\}.$$

Note that if f is positive, $\{p \in I(A);\ f(p) = 0\} = N(f)$. Given f and g, two linear functionals on A, we say, as in the classic measure theory, that f is ***absolutely continuous*** (resp. ***singular***) with respect to g, and we write $f \ll g$ (resp. $f \perp g$), if $N(g) \subseteq N(f)$ (resp. if there exists $p \in N(g)$ such that $p^{\perp} \in N(f)$). Then we say that f ***admits a Lebesgue decomposition with respect to*** g if there exist two linear functionals f_1 and f_2 on A such that:

$$f = f_1 + f_2 \text{ where } f_1 \ll g \text{ and } f_2 \perp g.$$

We say that this decomposition is ***bounded*** (resp. ***normal***) (resp. ***positive***) if f_1 and f_2 belong to A^* (resp. A_*) (resp. f_1 and f_2 are positive linear functionals on A).

We can now proceed to the main results.

Theorem 1. *Let A be a JBW-algebra and let g be a normal state on A. Then any linear functional f on A admits a Lebesgue decomposition with respect to g.*

This decomposition is bounded (resp. normal) if belongs to A^* (resp. if f belongs to A_*).

Proof. Let $p := \vee N(g)$. It is obvious that $f = f \circ T_{p^\perp} + f \circ T_p$, $f \circ T_{p^\perp} \ll g$ and $p^\perp \in N(f \circ T_p)$. To complete the proof of the first part of the theorem, there remains only to show that $p \in N(g)$.

To this end, let M be a maximal orthogonal subset of $N(g)$ whose existence is ensured by Zorn's lemma and set $q := \vee M$. This yields $q = \sigma(A, A_*) - \lim_{F \in \mathcal{F}(M)} \sum_{s \in F} s$ and, since g is positive and $\sigma(A, A_*)$-continuous, we obtain $q \in N(g)$, so $q \leqslant p$.

We now want to establish that for any element s of $N(g)$ we have $s \leqslant q$. Indeed, for $n = 1, 2, 3, \ldots$, if we let $a_n := 1 - \left(1 - \frac{q+s}{2}\right)^n$, the sequence (a_n) is increasing and bounded. Therefore, $a := \vee\{a_n : n \geqslant 1\}$ exists and $a = \sigma(A, A_*) - \lim_{n\to\infty} a_n$. By corollary 3.6.3 of [5], we have

$$g(a) = \lim_{n\to\infty} g(a_n) = \lim_{n\to\infty} \sum_{k=1}^{n} \binom{n}{k} \left(-\frac{1}{2}\right)^k g((q+s)^k) = 0.$$

On the other hand, a evidently belongs to $I(A)$ and the equality $a \circ (q+s) = q+s$ holds true. Therefore, $q \vee s \leqslant a$ (lemmas 4.2.8 and 4.2.6 of [5]), so $q \vee s \in N(g)$ and $q^\perp \wedge (q \vee s) \in N(g)$. But $M \cup \{q^\perp \wedge (q \vee s)\}$ is an orthogonal subset of $N(g)$ containing M and therefore, by the maximality of M, we conclude that $q^\perp \wedge (q \vee s) = 0$. From this equality, we get $q \vee s = q$, so $s \leqslant q$ for any element s of $N(g)$.

We have thus shown that $p = q \in N(g)$, proving the first part of the theorem. The remaining part follows directly from the fact that, for any $b \in A$, the operator T_b is bounded and $\sigma(A, A_*)$-continuous (see corollary 4.1.6 of [5]). □

Remarks.

(i) When $A = B(H)_{sa}$, H being a separable Hilbert space of dimension at least three, theorem 1 includes theorem 1 of [6]. Also, example 1 of [6] can be used to show that the positivity of the Lebesgue decomposition does not follow, in general, from the positivity of the linear functional.

(ii) In the course of the proof of theorem 1, we have shown that any normal and positive linear functional on A admits a *support*. This means that $\vee N(g)$ belongs to $N(g)$ and this support is defined as $(\vee N(g))^\perp$. In fact, the result of theorem 1 remains true under the sole hypothesis that g is a linear functional on A with a support.

(iii) Recall that a function $\mu : I(A) \to \mathbf{R}$ is said to be *additive* if $\mu(p+q) = \mu(p) + \mu(q)$ for any p, q in $I(A)$ such that $p \circ q = 0$ and it is said to be positive if $\mu(I(A)) \subseteq \mathbf{R}_+$. In [3], it is shown that if A is a JBW-algebra without a type I_2 part (see [5] for the definition) and if $\mu : I(A) \to \mathbf{R}$ is additive and positive, then there exists a linear functional f on A which extends μ. Therefore, if $\lambda\colon I(A) \longrightarrow \mathbf{R}$ is positive and additive with a support, then, again by theorem 1, we have that any additive and positive function μ on $I(A)$ admits a Lebesgue decomposition on $I(A)$ with respect to λ.

We now turn our attention to an important subspace of the normal state space of A, the so-called *trace space* of A. We say that a normal state f on A is a *trace state* if $f \circ U_s = f$ for any $s \in A$ such that $s^2 = 1$. This definition is taken from [2]. We set:

$$T(A) = \{f \in S(A);\ f \text{ is a trace state on } A\}$$

The next theorem gives, in particular, a characterization of the trace states on A in term of their Lebesgue decomposition with respect to each normal state on A.

Theorem 2. *Let A be a JBW-algebra and f be a bounded linear functional on A: (i)If g is a linear functional on A with a support, then f admits at most one bounded and positive Lebesgue decomposition with respect to g. (ii)f is a trace state on A if and only if f is a normal state admitting a unique bounded and positive Lebesgue decomposition with respect to any normal state g.*

P r o o f. (i) Let g be a linear functional on A with a support p and $f = f_1 + f_2$ a bounded and positive Lebesgue decomposition of the linear functional f with respect to g. Since $f_1 \ll g$, $f_2 \perp g$, we have $f_1(p^{\perp}) = f_2(p) = 0$. Therefore, by the positivity of f_1 and f_2 and again by corollary 3.6.3 of [5], we get $f_1 \circ T_{p^{\perp}} = f_2 \circ T_p = 0$. We then have $f \circ T_{p^{\perp}} = f_1 \circ T_{p^{\perp}} + f_2 \circ T_{p^{\perp}} = f_2 \circ T_{p^{\perp}} = f_2$ and similarly $f \circ T_p = f_1$. The unicity of such a decomposition is thus established.

(ii) First, let us assume that $f \in T(A)$ and $g \in S(A)$. By theorem 1, we have that $f = f \circ T_p + f \circ T_{p^{\perp}}$, where p is the support of g, is a bounded Lebesgue decomposition of f with respect to g. By [7, p. 371] this decomposition is positive (and hence unique).

Now we intend to establish that the condition is sufficient. Let us assume that $f \in S(A)$ and that, for any $g \in S(A)$, there exist f_1, $f_2 \in A^*$, positive linear functionals, such that $f = f_1 + f_2$, $f_1 \ll g$, and $f_2 \perp g$. The proof of part (i) indicates that $f_1 = f \circ T_p$ and $f_2 = f \circ T_{p^{\perp}}$ where p is the support of g. So far we have shown that $f(p \circ a) \geqslant 0$ for any $a \in A_+$ and any support p of an element of the normal state space of A.

Now let q be any element of $I(A)\backslash\{0\}$. $S(A)$ being a separating family, there exists $g \in S(A)$ such that $g(q) > 0$. It follows from corollary 4.1.6 and proposition 3.3.6 of [5] that the linear functional $h = \frac{1}{g(q)}\, g \circ U_q$ is a normal state on A. Therefore:

$$D_q = \{s : s \text{ is a support of an element } h \text{ in } S(A) \text{ with } h(q) = 1\}$$

is nonempty. We take M_q, a maximal orthogonal subset of D_q, and assume that $r := (\vee M_q)^\perp \wedge q \neq 0$. Then, there exists an element h_1 of $S(A)$ such that $h_1(r) = 1$, hence $h_1(q) = 1$. If s denotes the support of h_1, we have $s \leqslant r \leqslant (\vee M_q)^\perp$, so that $M_q \cup \{s\}$ is an orthogonal subset of D_q including, strictly, M_q. This is in contradiction with the maximality of M_q and so $r = 0$. Since $\vee M_q \leqslant q$, we have $q = \vee M_q$, so $q = \sigma(A, A_*) - \lim_{F \in \mathcal{F}(M_q)} \sum_{s \in F} s$, $q \circ a = \sigma(A, A_*) - \lim_{F \in \mathcal{F}(M_q)} \sum_{s \in F} s \circ a$ for any $a \in A_+$ (again by the $\sigma(A, A_*)$-continuity of the operator T_a). It follows, from the preceeding paragraph, that $f(q \circ a) \geqslant 0$ for any idempotent element q and any positive element a of A.

Now let $a \in A_+$, $a \neq 0$, $b \in A_+$ and $\varepsilon > 0$. By proposition 4.2.3 of [5], there exist $n \in \mathbf{N}$, $p_1, p_2, \ldots, p_n \in I(A)$ and $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbf{R}$ such $\left\| b - \sum_{i=1}^{n} \alpha_i p_i \right\| < \frac{\varepsilon}{\|a\|}$. An analysis of the proof of proposition 4.2.3 of [5] shows that, since $b \in A_+$, it is possible to choose the α_i in $\mathbf{R}_+$. From what preceeds and the inequality $\left\| a \circ b - \sum_{i=1}^{n} \alpha_i p_i \circ a \right\| < \varepsilon$, we conclude that $f(a \circ b) \geqslant -\varepsilon$. $\varepsilon > 0$ being arbitrary, we have $f(a \circ b) \geqslant 0$ for any $a \in A_+$ and $b \in A_+$. It follows from [7, p. 371] that $f \in T(A)$. The proof is complete. □

We conclude this paper with an application of the obtained result to a characterization of associative JBW-algebras.

We say that Δ, a subset of $S(A)$, is *unital* if, for any $p \in I(A) \backslash \{0\}$, there exists $f \in \Delta$ such that $f(p) = 1$. In the proof of theorem 2, we have shown that $S(A)$ is unital.

Corollary 3. *Let A be a JBW-algebra. Then, A is associative if and only if its trace space is unital.*

Proof. If A is associative, $T(A) = S(A)$ is unital.

Conversely, assume that $T(A)$ is unital. Let $p \in I(A) \backslash \{0\}$, s the support of an element g of $S(A)$, and suppose that $r := p \wedge (p \wedge s \vee p \wedge s^\perp)^\perp \neq 0$. By hypothesis, there exists $f \in T(A)$ such that $f(r) = 1$ and, by the preceeding theorem, there exist $\alpha_1, \alpha_2 \in \mathbf{R}_+$, $f_1, f_2 \in S(A)$ such that $f = \alpha_1 f_1 + \alpha_2 f_2$, $\alpha_1 f_1 \ll g$ and $\alpha_2 f_2 \perp g$. If $\alpha_1 \neq 0$ we have $f_1 \ll g$ and $r^\perp \in N(f_1)$. Let r_1 denote the support of f_1. Then $r_1 \leqslant r$ and $r_1 \leqslant s$. This implies that $r_1 \leqslant (p \wedge s)^\perp$ and $r_1 \leqslant p \wedge s$. It follows that $r_1 = 0$, a contradiction, and so $\alpha_1 = 0$ and $f \perp g$. If ϱ denotes the support of f, we then have $\varrho \leqslant r$ and $\varrho \leqslant s^\perp$, so $\varrho \leqslant (p \wedge s^\perp)^\perp$ and $\varrho \leqslant p \wedge s^\perp$. We thus have $\varrho = 0$, again a contradiction. Therefore, $r = 0$ and $p = p \wedge s \vee p \wedge s^\perp = p \wedge s + p \wedge s^\perp$. So far, we have shown that if the set $T(A)$ is unital, then $U_p s = T_p s$ for any $p \in I(A) \backslash \{0\}$ and any s, the support of an element of $S(A)$.

Now let q denote any element of $I(A)\backslash\{0\}$. Exactly as in the proof of theorem 2, we have $q = \sigma(A, A_*) - \lim_{F\in\mathcal{F}(M_q)} \sum_{s\in F} s$ where M_q is an orthogonal subset of $I(A)$ whose elements are the supports of elements of $S(A)$. We deduce that, for any $p \in I(a)\backslash\{0\}$, we have:

$$U_p q = \sigma(A, A_*) - \lim_{F\in\mathcal{F}(M_q)} \sum_{s\in F} U_p s = \sigma(A, A_*) - \lim_{F\in\mathcal{F}(M_q)} \sum_{s\in F} T_p s = T_p q.$$

It follows, by lemma 2.5.5 of [5], that any $p \in I(A)$ and $q \in I(A)$ operator commute and, by lemma 4.2.5 of [5], that p and a operator commute (for any $p \in I(A)$ and any $a \in A$).

To complete the proof, we now let $a \in A$, $a \neq 0$, $b \in A$ and $\varepsilon > 0$. Let $n \in \mathbf{N}$, α_1, $\alpha_2, \ldots, \alpha_n \in \mathbf{R}$, $p_1, \ldots, p_n \in I(A)$ such that $\left\| b - \sum_{i=1}^{n} \alpha_i p_i \right\| \leqslant \frac{\varepsilon}{2\|a\|}$. Based on the preceding:

$$\begin{aligned}
\|T_a T_b - T_b T_a\| &= \left\| T_a T_b - T_a\Big(\sum_{i=1}^{n} \alpha_i T_{p_i}\Big) + \Big(\sum_{i=1}^{n} \alpha_i T_{p_i}\Big) T_a - T_b T_a \right\| \\
&\leqslant \left\| T_a\Big(T_b - \sum_{i=1}^{n} \alpha_i T_{p_i}\Big) \right\| + \left\| \Big(T_b - \sum_{i=1}^{n} \alpha_i T_{p_i}\Big) T_a \right\| \\
&\leqslant 2\|T_a\| \left\| T_{b - \sum_{i=1}^{n} \alpha_i p_i} \right\| \leqslant 2\|a\| \, \|b - \sum_{i=1}^{n} \alpha_i p_i\| \leqslant \varepsilon.
\end{aligned}$$

$\varepsilon > 0$ being arbitrary, we have $T_a T_b = T_b T_a$ for any $a \in A$, $b \in A$ and so, A being commutative, A is associative. □

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