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Statistical Estimation of Deducibility in Polyadic Algebras

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This paper is devoted to the treatment of deducibility testing in formalized theories. A method of testing whether some given formula is or is not a theorem of the treated theory is proposed and some of its principal properties are considered. This method is based on investigating, whether the tested formula holds or does not hold in a sequence of randomly chosen extensions of the theory.

The paper is divided into three parts. In the first part the necessary algebraic and logic apparatus is developed, which is a slight modification of that of Halmos' papers. In the second part the test is designed and some of its properties are considered. The third part contains some remarks about possibilities of practical application of this test.

The present paper is concerned with elementary problems of statistical estimation od deducibility or non-deducibility of propositions. It is well known that the provable propositions of a polyadic axiomatic theory form a polyadic filter, namely that generated by a fixed element. As usual, this fixed element is said to be the axiom of the theory. In order to eliminate degenerate cases we shall always suppose that the theory is consistent, i.e. not every proposition of the polyadic logic is provable, or, in other words, the axiom does not coincide with the zero element of the polyadic algebra. The deducibility or non-deducibility of propositions is to be estimated on the basis of a random sample. Roughly speaking, a random sample in polyadic logic, according to our definition is nothing else but a sequence of extensions of the given polyadic theory chosen at random by an appropriate chance mechanism. The number of terms of this random sequence is itself a random variable.

The problem of statistical estimation of deducibility was for the first time formulated and solved by Antonín Špaček ten years ago. His work deals with the case of Boolean logics. The main presumption on which Špaček's work was based was that of decidability of a given proposition in every randomly chosen extension (see [1], [2]). In the present paper we shall remove this presumption and we shall suppose, that the probability of choosing such a "decidable" extension is "great enough". This generalization will enable us to derive a practical way of exploiting our theoretical results, for example, by using a computer. However this question will be only briefly mentioned in this paper.

Let us describe roughly our results. First of them shows, that our treatment of a given proposition will finish with probability one. The second shows that the probability of proclaiming a given non-theorem to be a theorem can be regulated and can be done smaller than a given positive number. Our last results show that the average number of steps in our treatment and all moments of this random variable are finite and give estimations for moments and quantiles of this random variable.

The logical apparatus used in this paper is the generalization of that of polyadic logics. We shall assume, that the notions of Boolean algebra, Boolean logic, Boolean filter, Boolean endomorfism, existential and universal quantifier and the most simple properties of these notions are known to the reader. All these notions are very carefully and clearly explained in [5]. In this paper we shall continually use the notation and results of this book.

1. GENERALIZED POLYADIC ALGEBRAS AND LOGICS

Let us remark that as a transformation we shall mean a mapping of a given set into itself. Now let us begin with the definition of polyadic algebra. This notion is of basic significance for this paper.

Definition 1. A quadruple $(A, I, \exists, S)$, in which

A is a given Boolean algebra,

I is a given set, the elements of which are called variables,

$\exists$ is a mapping from the set of all subsets of the set I into the set of all existential quantifiers on the given Boolean algebra A,

S is a mapping from the set of all transformations, defined on the set I into the set of all Boolean endomorphisms on the given Boolean algebra A, is called a *polyadic algebra*, if and only if the following conditions hold:

(a) If δ is the identity transformation on the set I, then $S(\delta)$ is the identity mapping on the Boolean algebra A.

(b) If σ and τ are given transformations on the set I, and $\sigma \times \tau$ denotes their product (composed transformation) then for every $p \in A$

$$S(\sigma \times \tau)\, p \equiv S(\sigma)\, S(\tau)\, p$$

where $\equiv$ is the symbol for the equivalence relation of the given Boolean algebra A.

(c) If $\emptyset$ is the empty subset of the set I, then $\exists(\emptyset)$ is the identity mapping on the given Boolean algebra A.

(d) If J, K are given subsets of the set I, then for all $p \in A$:

$$\exists(J \cup K)\, p \equiv \exists(J)\, \exists(K)\, p\,.$$

(e) If σ and τ are given transformations on the set I, if J is a subset of the set I with the property, that for all $i \in I - J$ the relation $\sigma i = \tau i$ holds, then for all $p \in A$:

$$S(\sigma)\, \exists(J)\, p \equiv S(\tau)\, \exists(J)\, p\,.$$

(f) If J is a given subset of the set I and if τ is a transformation on the set I, one-to-one on the set $\tau^{-1}J$, then for all $p \in A$:

$$\exists(J)\, S(\tau)\, p \equiv S(\tau)\, \exists(\tau^{-1}J)\, p\,.$$

An element p of a given polyadic algebra $(A, I, \exists, S)$ is called *independent* on the given subset J of the set I, if $\exists(J)\, p \equiv p$. The element p is called *closed*, if the relation $\exists(J)\, p \equiv p$ holds for every subset J of the set I. A subset K of the set I is called a *support* of the element p, if the element $\exists(K)\, p$ is closed. Polyadic algebra $(A, I, \exists, S)$ is called *locally finite*, if the cardinal number of the set I is equal to $\aleph_0$ and at the same time every element has a finite support.

Definition 2. A subset M of a given polyadic algebra is called a *polyadic filter*, if it has the following three properties:

(a) If $p \in M$, $q \in M$, then $p \wedge q \in M$,
(b) if $p \in M$, $q \in A$, then $p \vee q \in M$,
(c) if $p \in M$ and J is a subset of the set I, then $\forall(J)\, p \in M$, where the symbols $\wedge$ and $\vee$ denote the relations of infimum and supremum of the given Boolean algebra respectively and $\forall(J)$ denotes the universal quantifier, dual according to the existential quantifier $\exists(J)$.

The notion of a polyadic algebra was defined by P. R. Halmos [5] just in the way we have followed in our definition. Now we shall try to generalize this notion as follows:

Definition 3. A sequence $(A, (I_i, \exists_i, S_i))_{i=1}^{\infty}$ is called a *generalized polyadic algebra* or a *type algebra*, if for every index i the quadruple $(A, I_i, \exists_i, S_i)$ is a polyadic algebra and the condition

$$I_j \cap I_k = \emptyset$$

holds for every two indices j, k, $j \neq k$.

This notion will play a fundamental role in the present paper. An element p of a given generalized polyadic algebra $(A, (I_i, \exists_i, S_i))_{i=1}^{\infty}$ is called *relatively closed* with respect to the variables of the type k (i.e. the elements of the set I_k) if, and only, if this element is a closed element of the polyadic algebra $(A, I_k, \exists_k, S_k)$. The element p is called *absolutely closed* if it is relatively closed with respect to the variables of all types k, $k = 1, 2, \ldots,$ A subset J of the set I_k is called a *support of the type* k of the element p, if J is a support of p in the polyadic algebra $(A, I_k, \exists_k, S_k)$. A subset J

of the Cartesian product $\overset{\infty}{\underset{j=1}{\times}} I_j$ is called an *absolute support* of the elements p, if for every $j = 1, 2, \ldots$ the set of all j-th coordinates of the elements of the set J is a support of the type j of the element p. A generalized polyadic algebra is called *locally finite*, if the cardinal number of every $I_j, j = 1, 2, \ldots$ is equal to $\aleph_0$ and if for every element p there exists an index n and at the same time there exist indices $k_1, k_2, \ldots, k_n$ so that p is relatively closed according to the variables of all types different from $k_1, k_2, \ldots, k_n$ and at the same time for every $j = 1, 2, \ldots, n$, p has a finite support of the type k_j.

Lemma 1. *Let $(A, I, \exists, S)$ be a given polyadic algebra. Let us denote by $\bar{A}$ the subset of all closed elements of this polyadic algebra. Then $\bar{A}$ is a Boolean algebra with respect to the same relations, operations and zero and unity elements as the Boolean algebra A.*

Proof. From the definition of the existential quantifier (see [5]) it follows:

$$\exists(J)\, 0 \equiv 0\,, \quad J \subset I\,,$$

$$\exists(J)\, 1 \geqq 1\,, \quad J \subset I\,, \quad \text{therefore} \quad \exists(J)\, 1 \equiv 1\,,$$

$$\exists(J)\,(p') \equiv \exists(J)\,(\exists(J)\,p)' \equiv (\exists(J)\,p)' \equiv p'\,, \quad p \in \bar{A}\,, \quad J \subset I\,,$$

$$\exists(J)\,(p \vee q) \equiv \exists(J)\,p \vee \exists(J)\,q \equiv p \vee q\,.$$

We can see that set $\bar{A}$ is closed according to the operations of negation and supremum and that the set $\bar{A}$ contents the elements 0 and 1. Since $\bar{A} \subset A$, the conditions of the Boolean algebra hold for all elements of $\bar{A}$. QED.

Remark. If k is a given index, then the set of all elements of a given generalized polyadic algebra, that are relatively closed with respect to the variables of the type k forms the Boolean algebra according to the relations, operations, zero and unity elements of the Boolean algebra A.

Lemma 2. *Let $(A, (I_i, \exists_i, S_i))_{i=1}^{\infty}$ be a given generalized polyadic algebra. Let us denote by $\bar{A}$ the subset of all absolutely closed elements of this generalized polyadic algebra. Then $\bar{A}$ is a Boolean algebra to the same relations, operations, zero and unity elements as the Boolean algebra A.*

Proof. In the same way as in the proof of Lemma 1 we can derive

$$\exists_j(J)\, 0 \equiv 0\,, \quad \exists_j(J)\, 1 \equiv 1\,, \quad j = 1, 2, \ldots, J \subset I_j\,, \quad \text{therefore} \quad 0 \in \bar{A}\,,\ 1 \in \bar{A}\,,$$

$$\exists_j(J)\, p' \equiv p'\,, \quad \exists_j(J)\,(p \vee q) \equiv p \vee q\,, \quad p, q \in \bar{A}\,, \quad j = 1, 2, \ldots, J \subset I_j\,.$$

If we follow again the argumentation of the proof of Lemma 1 we find that the statement of Lemma 2 is valid. QED.

Definition 4. A subset M of a given generalized polyadic algebra $(A, (I_i, \exists_i, S_i))_{i=1}^{\infty}$ is called a *generalized polyadic filter*, if the following three conditions hold:

(a) If $p \in M$, $q \in M$, then $p \wedge q \in M$,
(b) if $p \in M$, $q \in A$, then $p \vee q \in M$,
(c) if $p \in M$, then $\forall_j(J)\, p \in M$ for every $j = 1, 2, \ldots$ and every subset J of the set I_j.

It is well known from the theory of Boolean algebras that the subset $I(a)$ of the given Boolean algebra A, defined for a given element a of the set A defined by the relation

$$I(a) = \{x : x \in A,\, x \geqq a\}$$

is a filter of this Boolean algebra. We now try to generalize this result as follows:

Lemma 3. *Let $(A, (I_i, \exists_i, S_i))_{i=1}^{\infty}$ be a given generalized polyadic algebra, let a be an element of the set A. Then the set $I_*(a)$, defined by the relation:*

$$I_*(a) = \{x : x \in A,\, a \leqq x,\, a \leqq \forall_{j1}(I_{j1}) \ldots \forall_{jk}(I_{jk})\, x\}$$

for every $k = 1, 2, \ldots$ and every sequence $(j_1, \ldots, j_k)$ of indices, is a generalized polyadic filter of the given generalized polyadic algebra.

Proof. (a) Let us suppose that $p \in I_*(a)$, $q \in A$. It follows: $(p \vee q) \wedge a \equiv (p \wedge a) \vee \vee\, (q \wedge a) \equiv a \vee (q \wedge a) \equiv a$, therefore $p \vee q \geqq a$. From the fact that $p \vee q \geqq p$ and from the properties of the universal quantifier (see [5]) it follows for all indices $k, j_1, \ldots, j_k$:

$$\forall_{j1}(I_{j1}) \ldots \forall_{jk}(I_{jk})\,(p \vee q) \geqq \forall_{j1}(I_{j1}) \ldots \forall_{jk}(I_{jk})\, p \geqq a\,.$$

The definition of $I_*(a)$ gives then the result: $p \vee q \in I_*(a)$.

(b) Let us suppose that $p, q \in I_*(a)$. It follows:

$$(p \wedge q) \wedge a \equiv (p \wedge a) \wedge (q \wedge a) \equiv a\,,$$

therefore $p \wedge q \geqq a$. Since $\forall_{j1}(I_{j1}) \ldots \forall_{jk}(I_{jk})\, p$ as well as $\forall_{j1}(I_{j1}) \ldots \forall_{jk}(I_{jk})\, q$ are elements of $I_*(a)$, we have:

$$\forall_{j1}(I_{j1}) \ldots \forall_{jk}(I_{jk})\,(p \wedge q) \equiv \forall_{j1}(I_{j1}) \ldots \forall_{jk}(I_{jk})\, p \wedge \forall_{j1}(I_{j1}) \ldots \forall_{jk}(I_{jk})\, q \geqq a$$

according to what we have just proved. Therefore $p \wedge q \in I_*(a)$.

(c) The fact that the set $I_*(a)$ is closed with respect to the universal quantifier of any type follows directly from the definition of $I_*(a)$. QED.

Definition 5. A pair (A, M) where A is a given generalized polyadic algebra and M is a given generalized polyadic filter of this algebra will be called *generalized polyadic logic*. If $M = I_*(a)$ for some $a \in A$ we shall call this *a axiom* of our generalized polyadic logic. A generalized polyadic logic (A, M) will be called *consistent* if, and only if, there is no element p of the set A for which the two sentences $p \in M$ and $p' \in M$ hold simultaneously.

It is well known and can be easily shown that in the case $M = I_*(a)$ the condition $a \neq 0$ necessary and sufficient for the consistency of the is generalized polyadic logic $(A, I_*(a))$.

The elements of a given generalized polyadic logic (A, M), which are at the same time elements of the set M will be called *theorems* of this logic.

Definition 6. Let $(A, I_*(a))$ be a given generalized polyadic logic. Any generalized polyadic logic $(A, I_*(x))$ where $x \leqq a$ will be called *extension* of the given generalized polyadic logic $(A, I_*(a))$.

Let us introduce one result from [1]:

Lemma 4. *Let A a be given Boolean algebra. For given $x \in A$ let us denote by $I(x)$ the set*

$$\{y : y \in A,\ y \geqq x\}\ .$$

For any $x \in A$ $I(x)$ is a Boolean filter of the Boolean algebra A and if a is not an atom of A and $a \neq 0$ (and only in this case) holds the relation:

$$I(a) = \cap\{I(x) : x \in A,\ x > a\}\ .$$

We now try to generalize this lemma as follows:

Lemma 5. *Let $(A, I_*(a))$ be a given consistent generalized polyadic logic. Let $\bar{A}$ be the set of all absolutely closed elements of this logic, let a be a non-zero elements, which is not an atom of A. Then the following relation holds:*

$$\{p : p \in A,\ p \in I_*(a) \cap \bar{A}\} = \cap(I_*(x) \cap \bar{A}\colon x < a,\ x \in \bar{A})\ .$$

Proof. Let us denote by $\bar{I}(x)$ the set $I_*(x) \cap \bar{A}$ for any $x \in A$. We know that $\bar{A}$ forms a Boolean algebra with respect to the operations of the Boolean algebra A. Therefore we can state:

(a) If $p, q \in \bar{I}(a)$, then $p, q \in I_*(a)$, $p, q \in \bar{A}$. Therefore $p \wedge q \in I_*(a)$, $p \wedge q \in \bar{A}$ and it follows: $p \wedge q \in \bar{I}(a)$.

(b) If $p \in \bar{I}(a)$, $q \in \bar{A}$, then $p \in I_*(a) \cap \bar{A}$, therefore $p \vee q \in I_*(a)$, $p \vee q \in \bar{A}$ and it follows: $p \vee q \in \bar{I}(a)$.

We can therefore state that $\bar{I}(a)$ is a filter of the Boolean logic A. From lemma 4 it follows:

$$\bar{I}(a) = \cap\{\bar{I}(x) : x < a,\ x \in \bar{A}\}$$

what is just the assertion of lemma 5. QED.

Let us explain this result in another way:

An absolutely closed element of the given generalized polyadic logic is a theorem of this logic if, and only, if, it is a theorem of all extensions of this logic.

We can generalize the last result by admitting the "degenerate" extensions which are equal to $I_*(a)$ and we obtain:

$$\bar{I}(a) = \cap(\{\bar{I}(x), x \leqq a, x \in \bar{A}\}) .$$

If $p \in \bar{A}$, then $p \wedge a \in \bar{A}$, $p \wedge a \leqq a$. Therefore

$$\bar{I}(a) = \cap(\{\bar{I}(x \wedge a), x \in \bar{A}\}) .$$

We shall say that a given generalized polyadic logic $(A, I_*(a))$ is *complete* with respect to the set $\mathcal{Q} \subset A$ if and only if

$$I_*(a) \cap \bar{A} = \cap(I_*(x \wedge a) \cap \bar{A} : x \in \mathcal{Q}) .$$

From the foregoing explanation we can see: *any generalized polyadic logic is complete at least with respect to the set of all absolutely closed elements.* It is possible in concrete cases to find proper subsets of the set $\bar{A}$ with the same property but we shall not investigate this matter in this paper.

2. STATISTICAL MODEL FOR ESTIMATION OF DEDUCIBILITY

Let $(A, I_*(a))$ be a given generalized polyadic logic with an absolutely closed axiom a. Let us consider the following situation: Let $p \in \bar{A}$. We choose at random an extension of our logic which is of the previously described type. We try to test whether in this extension p is true (i.e. whether p is a theorem), or whether non p is true. If we can decide that p is a theorem of this extension we ascribe to this extension the number 1. If we can decide that non p is a theorem of this extension we ascribe to it the number -1. If we can decide that p as well as non p are theorems of this extension we ascribe to it the number 0. If we cannot decide about p or about non p we choose step by step at random another one, two, ... but at most K elements of our logic (where K is a given positive integer) and try to decide about p or non p supposing that these elements are theorems of our extension. If we than can decide that p is a theorem of our extension, we ascribe to this extension one of the numbers α_i, $i = 1, 2, \ldots, K$ with respect to the number of formulas we have used. If we can decide that non p is a theorem of our extension we ascribe to it the number -1, no matter how many formulas we have used. It we can derive in our extension p as well as non p we ascribe to it the number 0 no matter how many elements we have used. If we cannot decide either in presence of all those K formulas (elements) we ascribe to it the the number 0. In this way we can ascribe a real number to any randomely chosen extension of our logic. We than choose at random a second extension, a third extension and so on and ascribe to them real numbers in the way just described. We shall finish our procedure in two cases:

First: when the number -1 is reached. In this case we proclaim:

"*p is not a theorem of our logic*".

Second: when the sum of the numbers we have ascribed to the randomely chosen extensions reaches or exceeds a formerly given positive number N_0. In this case we proclaim:

"*p is a theorem of our logic*".

The rest of this paper will be devoted to correct description and study of this testing procedure.

Theorem 1. a) *Let* $(\Omega, \mathscr{S}, \mu)$ *be a probability space.*

b) *Let* $(A, I_*(a))$ *be a countable consistent generalized polyadic logic which is complete with respect to a subset* $\mathscr{Q}$ *of the set* A, *let us suppose that* card $\mathscr{Q} = \aleph_0$.

c) *Let the elements of the set* $\mathscr{Q}$ *be numbered and ordered into the sequence* $\{a(1), a(2), \ldots\}$.

d) *Let* $N_0 > 0$ *and* $K \geqq 0$ *be two given integers. Let for any pair* $(x, \omega) \in (\mathscr{Q} \times \Omega)$ *be a given vector* $[y^1(x, \omega), y^2(x, \omega), \ldots, y^{K+1}(x, \omega)]$ *of the elements of the set* $\mathscr{Q}$, *which has the following properties:*

$$\{\omega : y^j(x, \omega) = y\} \in \mathscr{S} \quad \textit{for every} \quad y \in \mathscr{Q}\,, \quad j = 1, 2, \ldots, K + 1\,,$$

$$0 \leqq y^1(x, \omega) \leqq y^2(x, \omega) \leqq \ldots \leqq y^K(x, \omega) \leqq y^{K+1}(x, \omega) = x\,.$$

e) *Let* p *be a random variable defined on the space* $(\Omega, \mathscr{S}, \mu)$ *with values in the metric space* $(\bar{A}, \{A : A \subset \bar{A}\})$.

f) *Let* $\{\tau_i\}_{i=1}^{\infty}$ *be a sequence of random variables, defined on the space* $(\Omega, \mathscr{S}, \mu)$ *with values in the set* N *of all positive integers.*

Let for any $x \in \bar{A}$ *there exist a set* $T(x) \subset \mathscr{Q}$ *and a real positive number* $\sigma_0(x)$ *such that*

$$\text{(1)} \qquad \textit{for all } y \in T(x) \quad x \textit{ is not an element of } H(y)\,,$$

$$\text{(2)} \qquad \mu(\{\omega : a(\tau_k(\omega)) \in T(x)\}/\{\omega : \tau_0(\omega) = j_0, \ldots, \tau_{k-1}(\omega) = j_{k-1}\}) \geqq \sigma_0(x)/k$$

for every $k \in N$, $(j_0, \ldots, j_{k-1}) \in N^k$.

g) *For any* $x \in \mathscr{Q}$ *let be given a subset* $H(x)$ *of the set* $\bar{A}$ *with the following properties:*

$$\textit{If } x \leqq y \textit{ then } H(x) \subset H(y)\,, \quad x \notin H(x) \quad \textit{for all} \quad x \in \bar{A}\,.$$

h) *Let* A *be the set of the sums of all finite sequences, numbers of which are the given numbers*

$$0 \leqq \alpha_K \leqq \alpha_{K-1} \leqq \ldots \leqq \alpha_1 \leqq \alpha_0 = 1\,.$$

i) *Let* $\{R_n\}_{n=0}^{\infty}$ *be a sequence of transformations of the set* Ω *into* Λ *defined in the following way:*

$$\{\omega : R_0(\omega) = 0\} = \Omega ;$$

$$\{\omega : R_{n+1}(\omega) = -1\} = \{\omega : R_n(\omega) \in \langle 0, N_0) ,$$

$$p(\omega) \in \{(A - H(a(\tau_n(\omega)))) \cap (A - I(a(\tau_n(\omega))))\} \cup$$

$$\cup \left[\bigcap_{j=0}^{K-1} [H(y^{K-j+1}(a(\tau_n(\omega)), \omega) - H(y^{K-j}(a(\tau_n(\omega)), \omega)))] \cap \right.$$

$$\left. \cap (A - I(y^{K-j}(a(\tau_n(\omega)), \omega)))] \right\} ;$$

$$\{\omega : R_{n+1}(\omega) = R_n(\omega) + \alpha_j\} = \{\omega : R_n(\omega) \in \langle 0, N_0), p(\omega) \in$$

$$\in (H(y^{K+1-j}(a(\tau_n(\omega)), \omega)) - H(y^{K-j}(a(\tau_n(\omega)), \omega)))) \cap I(y^{K-j}(a(\tau_n(\omega)), \omega)) ,$$

$$\text{non } p \in [H(y^{K-j}(a(\tau_n(\omega)), \omega)) \cup (A - I(y^{K-j}(a(\tau_n(\omega)), \omega)))]\} , \quad j = 1, 2, \ldots, K ;$$

$$\{\omega : R_{n+1}(\omega) = R_n(\omega)\} = \{\omega : R_n(\omega) \in \langle 0, N_0) ,$$

$$R_{n+1}(\omega) \notin \{-1, R_n(\omega) + \alpha_j, j = 0, \ldots, K\}\} .$$

Then $\{R_n\}_{n=0}^{\infty}$ *is the sequence of random variables which with probability one reach a value from the set* $\langle N_0, \infty) \cup \{-1\}$.

Remark. The ses $H(x)$ represents for $x \in \mathcal{Q}$ the set of all elements of $\bar{A}$, about which we cannot decide in the extension $(A, I_*(x))$. y^j represents the random mechanism by which we find "auxiliary" elements in our procedure. R_n then represents the sum of numbers ascribed to the extensions we have already considered. The statement of our theorem can be reformulated in the following manner: ***With probability one a decision will be taken about any randomly chosen closed formula p.***

Proof. It can be easily seen that R_n reaches only the values from Λ for any positive integer n and card $\Lambda = \aleph_0$. If $\beta \in \Lambda$, then measurability of the set

$$\{\omega : R_n(\omega) = \beta\}$$

is sufficient for measurability of the mapping R_n.

But it can be easily that every set of the just described type can be expressed by sets of the type

$$\{\omega : p(\omega) \in H(y^j(a(\tau_n(\omega)), \omega))\} , \quad \{\omega : p(\omega) \in I(y^j(a(\tau_n(\omega)), \omega))\}$$

which can be expressed as a countable union of the sets

(1) $$\{\omega : p(\omega) \in H(x)\} \cap \{\omega : y^j(a(\tau_n(\omega)), \omega)) = x\} ,$$

(analogously for $I(y^j(a(\tau_n(\omega)))))$.

Measurability of the sets of the type (1) follows from the fact that $H(x)$ is a countable subset of the set A and p, y^j, τ_n are random variables. As only countable unions and intersections are used in order to describe the sets of the foregoing types the measurability of every mapping R_n is proved.

$$\mu(\{\omega : R_{N_0}(p(\omega), \omega) \in \langle 0, N_0)\}) =$$
$$= \sum_{x \in A} \frac{\mu(\{\omega : R_{N_0}(p(\omega), \omega) \in \langle 0, N_0)\} \cap \{\omega : p(\omega) = x\})}{\mu(\{\omega : p(\omega) = x\})} \mu(\{\omega : p(\omega) = x\}) .$$

If for every $x \in \bar{A}$ will hold

$$\frac{\mu(\{\omega : R_{N_0}(p(\omega), \omega) \in \langle 0, N_0)\} \cap \{\omega : p(\omega) = x\})}{\mu(\{\omega : p(\omega) = x\})} \to 0 , \quad N_0 \to \infty ,$$

clearly all the weighted sums will tend to 0. For the convergention of the expression

$$\frac{\mu(\{\omega : R_{N_0}(p(\omega), \omega) \in \langle 0, N_0)\} \cap \{\omega : p(\omega) = p\})}{\mu(\{\omega : p(\omega) = p\})}$$

to 0 the following condition is sufficient:

$$\mu(\{\omega : R_{N_0}(\bar{p}(\omega), \omega) \in \langle 0, N_0)\}) \to 0$$

for every $p \in \bar{A}$ with the property $\mu(\{\omega : p(\omega) = p\}) > 0$ where $\bar{p}$ denotes for every $p \in \bar{A}$ the random variable on $(\Omega, \mathcal{S}, \mu)$ equal to p for every $\omega \in \Omega$.

Let $p \in \bar{A}$, let us denote by A_k the set:

$$A_k = \{\omega : a(\tau_k(\omega)) \in T(p)\} , \quad k = 0, 1, 2, \ldots$$

Measurability of every set A_k can be proved in the same way as for the mappings R_n. We can easily see that the set A_k represents the event of choosing a decidable extension by the variable τ_k.

Let us consider the expression $\mu(\bigcap_{k=m}^{n} A'_k)$. It is equal to

$$\mu(\bigcap_{k=m}^{n} A'_k) = \mu(\{\omega : a(\tau_m(\omega)) \notin T(p), \ldots, a(\tau_n(\omega)) \notin T(p)\}) .$$

By decomposition of this measure into the sum of products of the conditional measures and by using the condition f) we can state:

$$\mu(\bigcap_{k=m}^{n} A'_k) \leqq \prod_{k=m}^{n} \left(1 - \frac{\sigma_0(p)}{k}\right).$$

It is well known that for any real $a \in \langle 0,1 \rangle$ there holds: $e^{-a} \geqq 1 - a$. Therefore for any $k = 1, 2, \ldots$ we have:

$$e^{-\sigma_0(p)/k} \geqq 1 - \frac{\sigma_0(p)}{k},$$

$$1 - \exp\left[-\sum_{k=m}^{n} \frac{\sigma_0(p)}{k}\right] \leqq 1 - \prod_{k=m}^{n}\left(1 - \frac{\sigma_0(p)}{k}\right) \leqq 1 - \mu(\bigcap_{k=m}^{n} A'_k)$$

and it follows:

$$\mu(\limsup A_k) = \mu(\bigcap_{m=1}^{\infty} \bigcup_{k=m}^{\infty} A_k) = \lim_{m\to\infty} \mu(\bigcup_{k=m}^{\infty} A_k) =$$

$$= \lim_{m\to\infty} \lim_{n\to\infty} \mu(\bigcup_{k=m}^{n} A_k) = \lim_{m\to\infty} \lim_{n\to\infty} (1 - \mu(\bigcap_{k=m}^{n} A'_k)) = 1,$$

because

$$1 = \lim_{m\to\infty} \lim_{n\to\infty} \left(1 - \exp\left[-\sum_{k=m}^{n} \frac{\sigma_0(p)}{k}\right]\right) \leqq \lim_{m\to\infty} \lim_{n\to\infty} (1 - \mu(\bigcap_{k=m}^{n} A'_k)) \leqq 1.$$

But this result means that with probability one a decidable extension will be chosen infinitely many times. As N_0-times choosing of a decidable extension is a sufficient (but not a necessary) condition for reaching the set $\langle N_0, \infty) \cup \{-1\}$, the last result proves our theorem. QED.

Theorem 2. *Let the conditions* a), b), c), d), e), g), h), i) *of the theorem* 1 *hold, and the condition* f) *be modified in the following way:*

f) *Let* $\{\tau_i\}_{i=0}^{\infty}$ *be a sequence of random variables defined on the space* $(\Omega, \mathcal{S}, \mu)$, *taking their values in the set of all positive integers* N.

Let for any $x \in \bar{A}$ *there exist a subset* $U(x)$ *of the set* $\mathcal{Q}$ *and a real positive number* $\sigma_0(x)$ *so that the following conditions hold:*

(1) *for every* $y \in U(x)$ x *is not an element of* $H(y)$ *and at the same time* x *is not an element of* $I(y)$.

(2) $\mu(\{\omega : a(\tau_k(\omega)) \in U(x)\} \mid \{\omega : \tau_0(\omega) = j_0, \ldots, \tau_{k-1}\}) \geqq \dfrac{\sigma_0(x)}{k}$

for all indices k *and all sequences* $(j_0, j_1, \ldots, j_{k-1})$ *of indices.*

Let $\mu(\{\omega : p(\omega) \in A - I_*(a)\}) > 0$. *Then for* $N_0 \to \infty$:

$$\mu(\bigcap_{n=N_0}^{\infty} \{\omega : R_n(\omega) \in \langle N_0, \infty)\} \mid \{\omega : p(\omega) \in A - I_*(a)\}) \to 0.$$

Remark. Let us denote the conditional probability from the assertion of this theorem by PE_1. PE_1 represents the probability of reaching or overtaking the value N_0 under the condition that the tested element is not a theorem. It means that PE_1 represents the probability of proclaiming a non-theorem to be a theorem. If $\mu(\{\omega : p(\omega) \in A - I_*(a)\}) = 0$ then PE_1 is not defined but in this case only theorems can be chosen for testing and therefore the probability of proclaiming a non-theorem to be a theorem is trivially equal to 0. The probability of refusing a theorem will not be treated in this paper generally. But it can be easily seen that if the set of extensions formed from the set $\mathcal{Q}$ does not contain an inconsistent extension then the probability of refusing a theorem is equal to 0.

Proof.

$$PE_1 = \mu(\bigcup_{n=N_0}^{\infty} \{\omega : R_n(\omega) \in \langle N_0, \infty)\} \mid \{\omega : p(\omega) \in A - I_*(a)\}) =$$

$$= \mu(\bigcup_{n=N_0}^{\infty} \{\omega : R_n(\omega) \in \langle N_0, \infty) , \quad p(\omega) \in A - I_*(a)\}) \mid \mu(\{\omega : p(\omega) \notin I_*(a)\}) .$$

For all $n = N_0, N_0 + 1, \ldots$

$$\{\omega : R_n(\omega) \in \langle N_0, \infty)\} \subset \{\omega : R_{N_0}(\omega) \in \langle 0, N_0)\} ,$$

therefore

$$\bigcup_{n=N_0}^{\infty} \{\omega : R_n(\omega) \in \langle N_0, \infty), p(\omega) \in A - I_*(a)\} \subset \{\omega : R_{N_0}(\omega) \in \langle 0, N_0\rangle\}$$

and from this relation it follows:

$$PE_1 \leqq \mu(\{\omega : R_{N_0}(\omega) \in \langle 0, N_0\rangle\}) \mid \mu(\{\omega : p(\omega) \in A - I_*(a)\}) =$$

$$= \frac{\mu(\{\omega : a(\tau_0(\omega)) \notin U(p(\omega)), \ldots, a(\tau_{N_0}(\omega)) \notin U(p(\omega))\})}{\mu(\{\omega : p(\omega) \in A - I_*(a)\})} .$$

By the same arguments we have used in the proof of theorem 1 it is sufficient for our scopes to prove that the right hand side of the last equality converges to 0 for every "degenerate" random variable $\bar{p}$ defined for every $p \in \bar{A}$ in the same way as in the proof of theorem 1. Let us denote by $PE_1(p)$ the result of the substitution of the random variable $\bar{p}$ for the random variable p in PE_1. By writing the right hand side of the last relation as a product of conditional probabilities and use the condition (2) we obtain:

$$PE_1(p) \leqq \prod_{j=1}^{N_0} \mu(\{\omega : a(\tau_j(\omega)) \notin U(\bar{p}(\omega))\} \mid \{\omega : a(\tau_0(\omega)) \notin U(\bar{p}(\omega)), \ldots$$

$$\ldots, a(\tau_{j-1}(\omega)) \notin U(\bar{p}(\omega))\}) \, \mu(\{\omega : a(\tau_0(\omega)) \notin U(\bar{p}(\omega))\}) \leqq$$

$$\leqq 1 \left(1 - \frac{\sigma_0(p)}{1}\right), \ldots, \left(1 - \frac{\sigma_0(p)}{N_0}\right) = \prod_{k=1}^{N_0} \left(1 - \frac{\sigma_0(p)}{k}\right).$$

By well-known criterion of mathematical analysis we have that $PE_1(p) \to 0$ for $N_0 \to \infty$ because $\sum_{k=1}^{\infty} \sigma_0(p)/k = \infty$ and the proof of theorem 2 is finished. QED.

The conditions concerning the sequence of variables $\{\tau_i\}_{i=0}^{\infty}$ may seem to be rather complicated. We shall see that it is not quite true. Let p be an element of our logic $(A, I_*(a))$ about which we cannot decide whether it is or is not a theorem of this logic. In the extension $(A, I_*(a \wedge \text{non } p))$ (non $p \in \bar{A}$) we can decide that non p is a theorem of this extension but p is not. Therefore the set $U(p)$ contains at least the element non p. If the random variables $\{\tau_i\}_{i=0}^{\infty}$ satisfy the conditions:

$$\tau_j = \tau_0\,,\; j = 0, 1, \ldots, \quad c_j = \mu(\{\omega : \tau_0(\omega) = j\}) > 0\,,\; j = 0, 1, \ldots$$

and the random variables $\{\tau_i\}_{i=0}^{\infty}$ are independent, then

$$\mu(\{\omega : a(\tau_k(\omega)) \in U(x)\} \mid \{\omega : \tau_0(\omega) = j_0, \ldots, \tau_{k-1}(\omega) = j_{k-1}\}) \geqq$$
$$\geqq \mu(\{\omega : a(\tau_k(\omega)) = \text{non } p\}) = \mu(\{\omega : \tau_k(\omega) = j_0\}) = c_{j_0} > 0$$

where j_0 is the index of the element non p. We see that the conditions (1) and (2) of theorem 2 are in this case satisfied.

The number of steps in our procedure is not an a priori given number but it is a random variable. The following two theorems will deal with moments and quantiles of this random variable.

Theorem 3. *Let the conditions of theorem* 2 *be satisfied except the condition* f(2). *Let the condition* f(2′) *holds:*

$$\text{f}(2') \qquad \mu(\{\omega : a(\tau_k(\omega)) \in T(x)\}/\{\omega : \tau_0(\omega) = j_0, \ldots, \tau_{k-1}(\omega) = j_{k-1}\}) \geqq \sigma_0(x)$$

for any index k and any sequence $\{j_0, j_1, \ldots, j_{k-1}\}$ of indices.

Let for any $p \in \bar{A}$ $V(p)$ be the random variable defined by the relation:

$$\{\omega : V(p)(\omega) = n\} = \{\omega : R_n((\omega), \bar{p}) \in \langle N_0, \infty) \cup \{-1\}\}\,.$$

Then the random variable $V(p)$ has for every $p \in \bar{A}$ finite moments of all orders.

Remark. It follows from the relations of theorem 1 that the random variable $V(p)$ is by the last relation completely defined. The random variable $V(p)$ is sometimes (for example in [1], [2]) called *the length of the heuristic reasoning about the element p.*

Proof. Let p be an element of the set $\bar{A}$. It follows from the proof of theorem 1 that the probability of choosing an undecidable extension in fixed j_0 steps is smaller than or equal to $(1 - \sigma_0(p))^{j_0}$. The probability of choosing just j undecidable extensions in the first n steps is therefore smaller than or equal to $\binom{n}{j}(1 - \sigma_0(p))^j$. The probability of choosing at most $N_0 - 1$ decidable extensions or, what is the same, at least $n - N_0 - 1$ undecidable ones is than smaller than or equal to $\sum_{j=0}^{N_0-1}\binom{n}{n-j}\,.\,(1 - \sigma_0(p))^{n-j}$. But choosing at least $n - N_0 - 1$ undecidable extensions in the first n steps is a necessary (but not a sufficient) condition for finishing our process $\{R_n\}$ in the step n. Thus we have:

$$\mu(\{\omega : R_n(\omega) \in \langle N_0, \infty) \cup \{-1\}\}) \leqq \sum_{j=0}^{N_0-1} \binom{n}{n-j} (1 - \sigma_0(p))^{n-j}.$$

Let us denote by $M_k(p)$ the moment of the k-th order of the random variable $V(p)$. It follows:

$$(2) \qquad M_k(p) = \sum_{n=0}^{\infty} n^k \mu(\{\omega : V(p)(\omega) = n\}) \leqq$$

$$\leqq \sum_{n=0}^{N_0} n^k + \sum_{n=N_0+1}^{\infty} n^k \Big(\sum_{j=0}^{N_0-1} \binom{n}{j} (1 - \sigma_0(p))^{n-j} \Big) =$$

$$= \sum_{n=0}^{N_0} n^k + \sum_{j=0}^{N_0-1} \left(\sum_{n=N_0+1}^{\infty} n^k \binom{n}{j} (1 - \sigma_0(p))^{n-j} \right) = \mathrm{OM}_k(p).$$

Let us denote by $A(j)$ the sum

$$\sum_{n=N_0+1}^{\infty} n^k \binom{n}{j} (1 - \sigma_0(p))^{n-j}.$$

It can be easily seen that for

$$n < \max \left\{ \left[\frac{N_0 - 1}{\sqrt{\left(\frac{1 - \sigma_0(p)/2}{1 - \sigma_0(p)} \right)} - 1} + N_0 - 1 \right], \right.$$

$$\left. N_0, \left[\left(\sqrt[2k]{\left(\frac{1 - \sigma_0(p)/2}{1 - \sigma_0(p)} \right)} - 1 \right)^{-1} \right] \right\} + 1$$

the quotient of two successive members of any sum $A(j)$ is smaller then $1 - \sigma_0(p)/2 <$ < 1 and therefore by D'Alembert's criterion any sum $A(j)$ is finite. It follows that $M_k(p)$ is finite for any index k. QED.

Theorem 4. *Let the conditions* a), b), c), d), e), g), h), i) *of the theorem* 1 *hold.*

Let $\{\tau_i\}_{i=0}^{\infty}$ *be a sequence of random variables defined on* $(\Omega, \mathscr{S}, \mu)$ *taking their values in the set of all positive integers* N.

Let for all $x \in \bar{A}$ *there exist subsets* $U(x) \subset T(x) \subset \mathscr{Q}$ *and positive real numbers* $0 < \sigma_0(x) \leqq \varepsilon_0(x) \leqq 1$ *so that the following four conditions hold:*

(1) *For all* $y \in U(x) : x \notin H(y),\ x \notin I(y)$.

(2) $\mu(\{\omega : a(\tau_k(\omega)) \in U(x)\} / \{\omega : \tau_0(\omega) = j_0, \ldots, \tau_{k-1}(\omega) = j_{k-1}\}) \geqq \sigma_0(x)$ *for all* $(j_0, j_1, \ldots, j_{k-1}) \in N^k$ *and all indices* k.

(3) *For* $y \in T(x) : x \notin H(y)$.
(4) $\mu(\{\omega : a(\tau_k(\omega)) \in T(x)\} / \{\omega : \tau_0(\omega) = j_0, \ldots, \tau_{k-1}(\omega) = j_{k-1}\}) \geqq \varepsilon_0(x)$ *for all indices k and all sequences* $\{j_0, j_1, \ldots, j_{k-1}\}$ *of indices.*

Then for any pair of real positive numbers $\delta_1, \delta_2, \delta_1 < 1, \delta_2 < 1$, *there exists a positive integer* $S(\delta_1, \delta_2)$ *which satisfies the following two conditions:*

(a) $$\mu(\{\omega : R_{S(\delta_1,\delta_2)}(\bar{p}, \omega) \in \langle 0, N_0)\}) \leqq \delta_2 ,$$

(b) $$\mu(\bigcap_{n=N_0}^{\infty} \{\omega : R_n(\bar{p}, \omega) \in \langle N_0, \infty)\} / \{\omega : p(\omega) \in A - I_*(a)\}) \leqq \delta_1 , \quad p \in \bar{A} .$$

Remark. In other words this theorem states with probability at least $1 - \delta_2$ a decision about p will be taken in the first $S(\delta_1, \delta_2)$ steps (see (*a*)) and at the same time the probability of proclaiming a non-theorem to be a theorem will be at most equal to δ_1.

Proof. All the conditions of theorem 2 are satisfied and therefore we can state: we can choose a positive integer N_0 in such a way that the condition (b) holds. Such a N_0 will be denoted by $N_0(\delta_1)$.

It follows from the well-known Tchebysheff's inequality that

$$\mu(\{\omega : V(p)(\omega) > M_1(p) + \varepsilon\}) \leqq \frac{D(V(p))}{\varepsilon^2}$$

for any $\varepsilon > 0$, where $D\,V(p)$ is the dispersion of the random variable $V(p)$. By using the inequality (2) we obtain:

$$\mu(\{\omega : V(p)(\omega) > O\,M_1(p) + \varepsilon\}) \leqq \mu(\{\omega : V(p)(\omega) > M_1(p) + \varepsilon\}) \leqq$$
$$\leqq \frac{D(V(p))}{\varepsilon^2} \leqq \frac{M_2(p)}{\varepsilon^2} \leqq \frac{O\,M_2(p)}{\varepsilon^2} .$$

If we choose

$$\varepsilon = \left[\sqrt{\left(\frac{O\,M_2(p)}{\delta_2}\right)}\right] + 1$$

we have

$$\mu\left(\left\{\omega : V(p)(\omega) > O\,M_1(p) + \left[\sqrt{\left(\frac{O\,M_2(p)}{\delta_2}\right)}\right] + 1\right\}\right) \leqq \delta_2 .$$

For any index n

$$\{\omega : R_n(\bar{p}, \omega) \in \langle 0, N_0)\} \subset \{\omega : V(p)(\omega) > n\} ,$$

therefore if we define $S(\delta_1, \delta_2)$ by

$$O\,M_1(p) + \left[\sqrt{\left(\frac{O\,M_2(p)}{\delta_2}\right)}\right] + 1$$

 we have

$$\mu(\{\omega : R_{S(\delta_1,\delta_2)}(\omega) \in \langle 0, N_0)\}) \leqq \delta_2 ,$$

QED.

3. SOME REMARKS ABOUT POSSIBILITIES OF APPLICATION OF THE PREVIOUS TESTING PROCEDURE

This last part of our paper will be devoted to a brief account of the possibilities of practical application of our procedure for testing of the deducibility. Let us consider a formalized mathematical theory based on the simple theory of logical types (definition of the simple theory of logical types see for example in [6]). It can be easily shown that we can define on the set of all well-formed formulas (w.f.f.) of this theory all the necessary relations, operations, quantifiers and endomorphisms in very simple and natural way so that we obtain the structure which has all the properties of generalized polyadic logic according to our previous definition. We shall ommit all the details of this construction and the proof, because it is only the matter of routine.

After what we have just said we shall try to apply the procedure of testing the deducibility, which we have theoretically described in the previous paragraph, to our formalized theory. There are only two things we have to solve, namely how to precise the sense of the expression "*we can* (*we cannot*) *decide about an element* x_1" or, in other words, how to define the sets $H(x)$ and how to define the vectors $[y^1(x, \omega), \ldots, y^k(x, \omega)]$.

Let us begin with this definition:

Definition 7. Let R, s be two positive integers, let $P_1, P_2, \ldots, P_s$ be well-formed formulas of a given formalized theory Λ. Then as a *deductive neighbourhood of the size* R of the formulas $P_1, P_2, \ldots, P_s$ we shall call and by $\mathcal{O}_R(P_1, P_2, \ldots, P_s)$ denote the smallest set of the well-formed formulas of our theory, which satisfies the following conditions:

(a) If p is a w.f.f. and if there exists a proof of the formula p based on the axioms of our theory and the formulas $P_1, P_2, \ldots, P_s$, which contains at most R formulas, then $p \in \mathcal{O}_R(P_1, \ldots, P_s)$.

(b) For any w.f.f. A of our theory the formula $A \supset A$ belongs to $\mathcal{O}_R(P_1, P_2, \ldots, P_s)$.

(c) If w.f.f. $Q \supset T \in \mathcal{O}_R(P_1, \ldots, P_s)$, then for every w.f.f. P

$$((Q \wedge P) \supset T) \in \mathcal{O}_R(P_1, \ldots, P_s) ,$$

$$(Q \supset (T \vee P)) \in \mathcal{O}_R(P_1, P_2, \ldots, P_s) .$$

Now let us suppose that we have for every pair (i, j), $i = 0, 1, \ldots, j = 0, 1, \ldots, K$ a random variable τ_i^j, defined on the given probability space $(\Omega, \mathcal{S}, \mu)$ with the values in the set of all positive integers. Let us define for every w.f.f. x:

$$H(x) = A - \mathcal{O}_R((A_1, \ldots, A_n), x)$$

where A is the set of all closed w.f.f. of our theory and $A_1, A_2, \ldots, A_n$ are its axioms. Further let us define for every $x \in A$ and every pair of positive integers (i, j), $j \leqq K$:

$$y^{K-j+1}(a(\tau_i(\omega)), \omega) = a(\tau_i(\omega)) \wedge a(\tau_i^K(\omega)) \wedge \ldots \wedge a(\tau_i^{K+1-j}(\omega)) \,.$$

It can be shown that if, for example, the random variables $\{\tau_i^j\}_{i=0,i,\ldots}^{j=0,\ldots,K}$ are mutually independant, equally distributed and if at the same time:

$$\mu(\{\omega : \tau_0^j(\omega) = i\}) > 0$$

for every index i, then all the conditions of all the theorems of part 2 are satisfied. Therefore all the statements of those theorems hold and our testing procedure, applied to our formalized theory has the following properties:

(a) The length of the heuristic reasoning about any element is finite with probability 1. All the moments and quantiles of this random variable are finite.

(b) The probability of proclaiming a non-theorem to be a theorem decreases to 0 with increasing parameter N_0.

(c) The probability of proclaiming a theorem to be a non-theorem decreases to 0 with increasing parameter R of the used deductive neighbourhood. This last statement is based on the fact that for every theorem p of our theory there exists at least one positive integer $R(p)$ (namely the number of formulas in some proof of p) that in the case we use this $R(p)$ as the parameter of our deductive neighbourhood we cannot come to a mistake for the testing procedure applied to p.

Let us remark that by what we have just said we have reduced our testing of deducibility of sentences into two priciple steps; the random choice of positive integers by some random mechanism and the investigation whether the tested sentence belongs to some deductive neighbourhood or not. If we limit the number of all variables in our tested theory to a finite number, then the number of all sentences of the same length as the tested formula will be finite in every deductive neighourhood of the type we use.

Therefore the decision procedure about belonging or not belonging of the tested formula to a randomly chosen neighourhood can be reduced to the question, whether the tested formula belongs or does not belong to a randomly chosen finite set. Therefore the answer to this question can be effectively found. It is possible to compile a programme in ALGOL or in another programming language which would formally describe our decision procedure. The possibility of practical use of such a programme would depend on many other circumstances, of course, for example on the cost of such a programme or on the time necessary for its realization.

Last year the author considered the case of the formalized theories, which can be expressed in so called Centzen's calculus. The testing procedure considered in this paper was slightly modified and applied to such theories. The following results were proved:

(a) The length of heuristic reasoning about any element is bounded and its upper bound can be chosen in such a way that the probability of proclaiming a non-theorem to be a theorem will be smaller than an a priori given positive number.

(b) Let us consider some formalized theory based on the first order functional calculus. Let us express this theory in two ways, first by means of "*classical*" first order functional calculus (with modus ponens as one of its deductive rules) and second by means of *Gentzen's* calculus. If we apply to the "classical" formalization of our theory the testing procedure considered in this paper and if we apply to the Gentzen's formalization the modification of our testing procedure mentioned above then, under the condition that the probability of proclaiming a non-theorem to be a theorem is in both cases smaller than a given number, the upper bound of the length of the heuristic reasoning about a sentence p in the "*Gentzen's*" formalization will be smaller than the expected value of this random variable in the case of the "*classical*" formalization. The difference between the two quantities can be rather essential in some cases, because if we denote as P_G the probability of choosing such an extension of the considered Gentzen's formalization that we can decide about the tested sentence and if we denote as P_K the probability of choosing an extension with the same properties in the case of the "classical" formalization, then the expression P_K/P_G will tend to 0 when the number of the extensions which can be chosen with a positive probability increases or when the number of the variables of our tested theory increases.

In the present time the author is concerned with the questions of the reduction of this decision procedure into numerical form and elaboration of a programme in ALGOL in order to judge the possibilities of a practical use of this testing procedure.

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Statistický odhad dokazatelnosti v polyadických algebrách

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První část práce je věnována konstrukci algebraického modelu formalizovaných teorií, založených na prosté teorii logických typů. Základním pojmem je zde pojem zobecněné polyadické algebry, která je definována jako systém polyadických algeber nad danou Booleovou algebrou s disjunktními množinami proměnných.

Práce pokračuje zavedením pojmů absolutně uzavřeného elementu, zobecněného filtru a pojmů zobecněné polyadické logiky a jejího rozšíření. Přes několik pomocných tvrzení dostáváme se k základnímu tvrzení této části práce (lemma 5), které praví, že množina všech absolutně uzavřených výroků, které jsou přitom teorémy dané zobecněné polyadické logiky, je průnikem (přes množinu všech rozšíření) množin teorémů těchto rozšíření.

V druhé části práce je konstruována rozhodovací procedura pro testování dokazatelnosti. Princip metody spočívá v prověřování pravdivosti testovaného výroku p v náhodně vybraných rozšířeních. Jestliže v náhodně vybraném rozšíření umíme p dokázat, přiřadíme mu hodnotu 1, umíme-li v něm dokázat p jen za určitých předpokladů, přiřadíme mu jinou hodnotu z intervalu $\langle 0,1 \rangle$ a neumíme-li v něm p dokázat ani tehdy, přiřadíme mu nulu. Hodnoty, které jsou takto náhodně vybíraným rozšířením přiřazovány, sečítáme a proceduru končíme, když:

buď narazíme na rozšíření, ve kterém umíme dokázat negaci p; pak prohlásíme p za *neteorém*,

nebo součet hodnot rozšířením přiřazených přesáhne předem stanovenou mez a pak prohlásíme p za *teorém*.

Všechna další tvrzení zabývají se pak zkoumáním vlastností právě popsané procedury. Tak věta 1 uvádí systémy podmínek postačujících k tomu, aby procedura s pravděpodobností 1 skončila, věta 2 pak uvádí, že za podmínek, které vzniknou částečným zesílením některých podmínek věty 1 konverguje pravděpodobnost, že neteorém bude uvedenou procedurou přijat chybně za teorém, k nule s rostoucí horní hranicí, nutnou k přijetí formule za teorém.

Věty 3 a 4 se pak zabývají počtem pokusů, nutných k vyslovení rozhodnutí o zkoumané formuli, chápaným jako náhodná veličina a nazývaným *heuristická délka úvahy*. Uvádějí se podmínky, za kterých má tato náhodná veličina konečné momenty a kvantily jakož i obecně platné, ale v konkrétních případech dosti hrubé meze pro tyto momenty a kvantily.

Důkazem těchto vět končí část 2. V části 3 je pak naznačeno, že při vhodné volbě postupu, na jehož základě budeme rozhodovat o pravdivosti rozšířeních budou splněny podmínky vět z části 2. Tyto otázky nejsou ale v této práci podrobněji rozvíjeny a některé z nich budou námětem dalších prací.

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