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ON THE GROUP PULSE PROCESSES
III. Power Spectrum of the Processes with Independent Intervals*

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Power spectrum of the nth-order group pulse process with independent intervals among reference points (i.e., of the process LB ... B) is derived. The matrix method introduced in an earlier paper is extended to the processes B and D so that the power spectrum formula of these and of the mixed processes can be easily determined. At the end applications of the pulse processes are summarized.

1. INTRODUCTION

The processes A and B were defined in [1] as the processes with independent occurrence times of reference points and with independent intervals among reference points, respectively. The power spectrum of the nth-order group pulse process LA ... A for arbitrary n was found in [2]. In this paper we want to proceed a bit further. Using similar approach as in [2] the power spectrum of the nth-order group pulse process LB ... B for arbitrary n and of the mixed group pulse processes will be found.

In comparison with the processes A the processes B are more complex and so far only the process of the order zero was studied in the literature. Its power spectrum was derived independently by Churgin [3], Lukes [4], Mazzetti [5] and Banta [6]. Results derived here represent, in a certain sense, a generalization of the mentioned works.

The paper is organized in the following way. First, the power spectrum of the process BB will be derived in Section 2. Examples of several basic functions associated with the process B will be given in Section 3. The procedure used in Section 2 will be generalized in Section 4 to obtain the power spectrum of the process LB ... B. In Section 5 the matrix method introduced in [2] for a fast determination of the power

* This work and the papers referred to here as [1], [2], [8] and [21] are based on author's unpublished habilitation thesis [36].

spectrum formula will be extended to include the processes studied here. Power spectra of some important processes will be given in Section 6. Several basic differences among the processes A, B and D will be discussed in Section 7. At last, applications of the pulse processes will be summarized in Section 8.

2. POWER SPECTRUM OF THE PROCESS BB

According to the definition of the process BB [1] the intervals among the reference points on the level zero, ϑ, and among the reference points on the 1st level, ε, are independent random variables with probability densities $w_1(\vartheta)$ and $w_1(\varepsilon)$. An example of a possible realization $\xi(t)$ of the process BB is shown in Fig. 1.

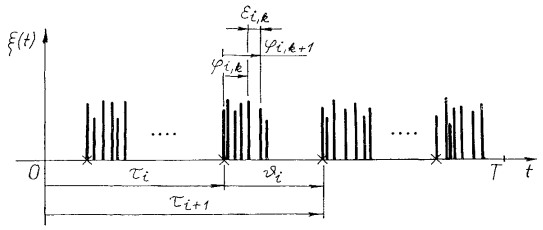

Fig. 1. Example of the group pulse process BB.

The considered group pulse process $\xi(t)$ may be written in the form [1], [2]

$$\xi(t) = \sum_{i=-\infty}^{\infty} \sum_{k=1}^{K_i} f(t - \tau_i - \varphi_{ik}, \boldsymbol{a}_{ik}) . \tag{1}$$

Here $f(t, \boldsymbol{a})$ is a function defining the shape of a pulse, $\boldsymbol{a}$ is a random vector of m random pulse patameters and the number of pulses in the ith group was denoted K_i. Position of the group and pulse reference points was denoted τ_i and φ_{ik}, respectively.

The power spectrum, $\mathscr{W}(\omega)$, will be determined from the formula [2]

$$\mathscr{W}(\omega) = \lim_{T \to \infty} \frac{1}{T} \langle |S_T(\omega)|^2 \rangle , \tag{2}$$

where $S_T(\omega)$ is a spectrum of a realization $\xi_T(t)$, truncated in an principal interval $(0, T)$ and the symbol $\langle\ \rangle$ denotes an ensemble average.

Following the procedure described in detail elsewhere [2] we shall start by considering a truncated realization $\xi_{TI}(t)$ consisting of exactly I pulse groups. The spectrum $S_{TI}(\omega)$ of this truncated realization will be obtained by taking the Fourier transform of $\xi_{TI}(t)$. At the next step the modulus squared of the spectrum, $|S_{TI}(\omega)|^2 = S_{TI}(\omega)\, S_{TI}^*(\omega)$, will be determined. Here the complex conjugate was denoted

by an asterisk. The resulting expression for $|S_{TI}(\omega)|^2$ will be formed by a four-fold sum. From this four-fold sum the terms for which $i = l$ and $k = m$ simultaneously and the terms for which $i = l$ but $k \neq m$ may be taken out. After this rearrangement we obtain [2]

$$(3) \qquad |S_{TI}(\omega)|^2 = \sum_{i=1}^{I} \sum_{k=1}^{K_i} |s(\omega, \boldsymbol{a}_{ik})|^2 +$$

$$+ \sum_{i=1}^{I} \sum_{k=1}^{K_i} \sum_{\substack{m=1 \\ k \neq m}}^{K_i} s(\omega, \boldsymbol{a}_{ik})\, s^*(\omega, \boldsymbol{a}_{im}) \exp\left[-j\omega(\varphi_{ik} - \varphi_{im})\right] +$$

$$+ \sum_{i=1}^{I} \sum_{k=1}^{K_i} \sum_{\substack{l=1 \\ i \neq l}}^{I} \sum_{m=1}^{K_l} s(\omega, \boldsymbol{a}_{ik})\, s^*(\omega, \boldsymbol{a}_{lm}) \exp\left[-j\omega(\tau_i + \varphi_{ik} - \tau_l - \varphi_{lm})\right].$$

Now the expectation of (3) on an ensemble of all possible realizations containing just I pulse groups may be found. It is convenient to find this expectation for each term on the right-hand side of (3) separately. The expectation of the *first term* (i.e., of the two-fold sum) equals simply

$$(4) \qquad \left\langle \sum_{i=1}^{I} \sum_{k=1}^{K_i} |s(\omega, \boldsymbol{a}_{ik})|^2 \right\rangle = I\langle K\rangle \left\langle |s(\omega, \boldsymbol{a})|^2 \right\rangle .$$

In determining the expectation of the *second term* (i.e., of the three-fold sum) we shall consider the variables K, $\boldsymbol{a}$ and φ separately. Since the process is of the type BB, the argument of the exponential function can be arranged as follows. If $k > m$, then

$$\varphi_{ik} - \varphi_{im} = \varepsilon_{i,m} + \varepsilon_{i,m+1} + \ldots + \varepsilon_{i,k-1}$$

and if $k < m$, then

$$\varphi_{im} - \varphi_{ik} = -(\varphi_{ik} - \varphi_{im}) = -(\varepsilon_{i,m} + \ldots + \varepsilon_{i,k-1}) .$$

Hence the expectation of the exponential functions in the second term over all possible values of the random variable ε equals

$$\left\langle \sum_{k=1}^{K_i} \sum_{\substack{m=1 \\ k \neq m}}^{K_i} \exp\left[-j\omega(\varphi_{ik} - \varphi_{im})\right] \right\rangle = \sum_{k=1}^{K_i - 1} (K_i - k)\left[\chi_\varepsilon^k(\omega) + \chi_\varepsilon^{*k}(\omega)\right] =$$

$$(5) \qquad = 2\,\mathrm{Re}\left\{\sum_{k=1}^{K_i - 1} (K_i - k)\,\chi_\varepsilon^k(\omega)\right\} = A_\varepsilon(\omega, K_i) .$$

Here $\chi_\varepsilon(\omega)$ is a characteristic function of the random variable ε and the finite arithmetic-geometric series (5) was denoted as $A_\varepsilon(\omega, K_i)$. Let us assume that $|\chi_\varepsilon(\omega)| < 1$ for $\omega \neq 0$. Then with the exception of $\omega = 0$ the sum of the arithmetic-geometric series (5) equals [7]

$$(6) \qquad A_\varepsilon(\omega, K_i) = 2\,\mathrm{Re}\left\{\frac{\chi_\varepsilon(\omega)}{1 - \chi_\varepsilon(\omega)}\left[K_i - B_\varepsilon(\omega, K_i)\right]\right\} .$$

Here we have denoted the sum of the finite geometric series as $B_\varepsilon(\omega, K_i)$, that is

$$B_\varepsilon(\omega, K_i) = \sum_{k=1}^{K_i} \chi_\varepsilon^{(k-1)}(\omega)\,, \tag{7}$$

which may be summed for $\omega \neq 0$ to give

$$B_\varepsilon(\omega, K_i) = \frac{1 - \chi_\varepsilon^{K_i}(\omega)}{1 - \chi_\varepsilon(\omega)}\,. \tag{8}$$

If $\omega = 0$ then $\chi_\varepsilon(\omega) = 1$ and from (5) and (7) it follows that $A_\varepsilon(0, K_i) = K_i^2 - K_i$ and $B_\varepsilon(0, K_i) = K_i$.

So far the expectation of the exponential functions in the second term over all possible values of the random variable ε was found. However, the function $A_\varepsilon(\omega, K_i)$ also depends on the discrete random variable K_i. The expectation of $A_\varepsilon(\omega, K_i)$ over all possible values of K_i may be determined from the formula

$$\langle A_\varepsilon(\omega, K)\rangle = \sum_{K=0}^{\infty} P(K_i = K)\, A_\varepsilon(\omega, K_i)\,, \tag{9}$$

where $P(K_i = K)$ is a probability of an event that there are just K pulses in the ith group. The expectation of $B_\varepsilon(\omega, K_i)$ over all possible values of K_i can be found in the same way.

Using (5) and (9) the expectation of the three-fold sum in (3) equals

$$\begin{aligned} &\Big\langle \sum_{i=1}^{I} \sum_{\substack{k=1 \\ k \neq m}}^{K_i} \sum_{m=1}^{K_i} s(\omega, \boldsymbol{a}_{ik})\, s^*(\omega, \boldsymbol{a}_{im}) \exp\left[-\mathrm{j}\omega(\varphi_{ik} - \varphi_{im})\right]\Big\rangle = \\ &= I\langle |s(\omega, \boldsymbol{a})|\rangle^2 \langle A_\varepsilon(\omega, K)\rangle\,. \end{aligned} \tag{10}$$

In the case of the *third term* (the four-fold sum) it will be proceeded in a similar manner. First of all the argument of the exponential function will be arranged. If $i > l$, then

$$\begin{aligned} \tau_i + \varphi_{ik} - \tau_l - \varphi_{lm} = \vartheta_l + \ldots + \vartheta_{i-1} + \varepsilon_{i,1} + \ldots + \varepsilon_{i,k-1} - \\ - \varepsilon_{l,1} - \ldots - \varepsilon_{l,m-1}\,, \end{aligned}$$

and if $i < l$, then

$$\tau_l + \varphi_{lm} - \tau_i - \varphi_{ik} = -(\tau_i + \varphi_{ik} - \tau_l - \varphi_{lm})\,.$$

Hence

$$\begin{aligned} &\Big\langle \sum_{i=1}^{I} \sum_{k=1}^{K_i} \sum_{\substack{l=1 \\ i \neq l}}^{I} \sum_{m=1}^{K_l} \exp\left[-\mathrm{j}\omega(\tau_i + \varphi_{ik} - \tau_l - \varphi_{lm})\right]\Big\rangle = \\ &= \sum_{i=1}^{I} \sum_{k=1}^{K_i} \sum_{\substack{l=1 \\ i > l}}^{I} \sum_{m=1}^{K_l} \chi_\vartheta^{(i-l)}(\omega)\, \chi_\varepsilon^{(k-1)}(\omega)\, \chi_\varepsilon^{*(m-1)}(\omega) + \\ &+ \sum_{i=1}^{I} \sum_{k=1}^{K_i} \sum_{\substack{l=1 \\ i < l}}^{I} \sum_{m=1}^{K_l} \chi_\vartheta^{*|i-l|}(\omega)\, \chi_\varepsilon^{*(k-1)}(\omega)\, \chi_\varepsilon^{(m-1)}(\omega)\,. \end{aligned} \tag{11}$$

Here $\chi_\vartheta(\omega)$ is a characteristic function of the random variable ϑ. Similarly as in the case of the second term the finite geometric series in (11) will be denoted as $B_\varepsilon(\omega, K_i)$, that is

$$B_\varepsilon(\omega, K_i) = \sum_{k=1}^{K_i} \chi_\varepsilon^{(k-1)}(\omega) \tag{12}$$

and

$$B_\varepsilon^*(\omega, K_l) = \sum_{m=1}^{K_l} \chi_\varepsilon^{*(m-1)}(\omega) . \tag{13}$$

Therefore, if the left-hand side of (11) is denoted as $L(\omega, K_i, K_l)$ we obtain

$$L(\omega, K_i, K_l) = \sum_{\substack{i=1 \\ i>l}}^{I} \sum_{l=1}^{I} \chi_\vartheta^{(i-l)}(\omega)\, B_\varepsilon(\omega, K_i)\, B_\varepsilon^*(\omega, K_l) + \tag{14}$$

$$+ \sum_{\substack{i=1 \\ i<l}}^{I} \sum_{l=1}^{I} \chi_\vartheta^{*|i-l|}(\omega)\, B_\varepsilon^*(\omega, K_i)\, B_\varepsilon(\omega, K_l) .$$

The expectation of $L(\omega, K_i, K_l)$ over all possible values of K_i and K_l may be determined from the formula

$$\langle L(\omega, K, K)\rangle = \sum_{K=0}^{\infty} \sum_{M=0}^{\infty} P(K_i = K)\, P(K_l = M)\, L(\omega, K_i, K_l) . \tag{15}$$

Here $P(K_i = K)$ and $P(K_l = M)$ are probabilities of an event that there are just K and M pulses in the ith and in the lth group, respectively.

After substituting (14) in (15) we obtain

$$\langle L(\omega, K, K)\rangle = \langle |B_\varepsilon(\omega, K)|\rangle^2 \sum_{i=1}^{I-1} (I - i) \left[\chi_\vartheta^i(\omega) + \chi_\vartheta^{*i}(\omega)\right] . \tag{16}$$

The finite arithmetic-geometric series on the right-hand side of (16) may be denoted, in accordance with (5), as $A_\vartheta(\omega, I)$, so that

$$\langle L(\omega, K, K)\rangle = \langle |B_\varepsilon(\omega, K)|\rangle^2\, A_\vartheta(\omega, I) . \tag{17}$$

If $|\chi_\vartheta(\omega)| < 1$ for $\omega \neq 0$ then $A_\vartheta(\omega, I)$ may be expressed in the form similar to '6) with $B_\vartheta(\omega, I)$ given by (8). If $\omega = 0$, then $A_\vartheta(0, I) = I^2 - I$ and $B_\vartheta(0, I) = I$.

Using (11)–(17) the expectation of the third term in (3) may be written at last as

$$\Big\langle \sum_{i=1}^{I} \sum_{\substack{k=1 \\ i \neq l}}^{K_i} \sum_{l=1}^{I} \sum_{m=1}^{K_l} s(\omega, \boldsymbol{a}_{ik})\, s^*(\omega, \boldsymbol{a}_{lm}) \exp\left[-j\omega(\tau_i + \varphi_{ik} - \tau_l - \varphi_{lm})\right] \Big\rangle = \tag{18}$$

$$= \langle |s(\omega, \boldsymbol{a})|\rangle^2 \langle |B_\varepsilon(\omega, K)|\rangle^2\, A_\vartheta(\omega, I) .$$

Now we may return to the equation (3). With respect to (4), (10) and (18) the expectation of (3) equals

$$\langle |S_{TI}(\omega)|^2\rangle = I\langle K\rangle \langle |s(\omega, \boldsymbol{a})|^2\rangle + I\langle |s(\omega, \boldsymbol{a})|\rangle^2 \langle A_\varepsilon(\omega, K)\rangle + \tag{19}$$

$$+ \langle |s(\omega, \boldsymbol{a})|\rangle^2 \langle |B_\varepsilon(\omega, K)|\rangle^2\, A_\vartheta(\omega, I) .$$

The expression (19) represents an average on the ensemble of all truncated realizations that consisted just of I pulse groups. The average on the ensemble of all truncated realizations may be obtained as

$$\langle |S_T(\omega)|^2 \rangle = \sum_{I=0}^{\infty} P(I) \langle |S_{TI}(\omega)|^2 \rangle . \tag{20}$$

Here $P(I)$ is a probability of an event that there are exactly I pulse groups in the interval $(0, T)$. Substituting (19) into (20) we have

$$\begin{aligned} \langle |S_T(\omega)|^2 \rangle = \langle I \rangle \langle K \rangle \langle |s(\omega, \boldsymbol{a})|^2 \rangle + \langle I \rangle \langle |s(\omega, \boldsymbol{a})| \rangle^2 \langle A_\varepsilon(\omega, K) \rangle + \\ + \langle |s(\omega, \boldsymbol{a})| \rangle^2 \langle |B_\varepsilon(\omega, K)| \rangle^2 \langle A_\vartheta(\omega, I) \rangle . \end{aligned} \tag{21}$$

Now, if the length of the interval $(0, T)$ is increased, the number I of the groups included in the interval will also increase. In the limit $T \to \infty, I \to \infty$, and we obtain [2]

$$\lim_{T \to \infty} \frac{\langle I \rangle}{T} = 1/\langle \vartheta \rangle = \langle \nu \rangle . \tag{22}$$

Here $\langle \nu \rangle$ is a mean group density. The geometric series $\langle B_\vartheta(\omega, I) \rangle$ will converge for $I \to \infty$ and $\omega \neq 0$, so that it will also hold that

$$\lim_{T \to \infty} \frac{1}{T} \langle B_\vartheta(\omega, I) \rangle = 0 , \quad \omega \neq 0 . \tag{23}$$

It also follows from (5) that $\langle A_\vartheta(0, I) \rangle = \langle I^2 \rangle - \langle I \rangle$. With respect to (22) we may write therefore

$$\lim_{T \to \infty} \frac{1}{T} \langle A_\vartheta(0, I) \rangle = \lim_{T \to \infty} \frac{1}{T^2} (\langle I \rangle^2 + \sigma_I^2 - \langle I \rangle) T = \langle \nu \rangle^2 \, 2\pi \, \delta(\omega), \quad \omega = 0 . \tag{24}$$

Hence, using (23) and (24) we obtain

$$\lim_{T \to \infty} \frac{1}{T} \langle A_\vartheta(\omega, I) \rangle = \begin{cases} \langle \nu \rangle \, 2 \operatorname{Re} \left\{ \dfrac{\chi_\vartheta(\omega)}{1 - \chi_\vartheta(\omega)} \right\}, & \omega \neq 0 , \\ \langle \nu \rangle^2 \, 2\pi \, \delta(\omega) , & \omega = 0 . \end{cases} \tag{25}$$

Now we may substitute (21) to (2) and take the limit for $T \to \infty$. With respect to (22) and (25) and after rearrangement of the terms we finally obtain the expression for the power spectrum in the form

$$\begin{aligned} \mathscr{W}(\omega) = \langle \nu \rangle \langle K \rangle \langle |s(\omega, \boldsymbol{a})|^2 \rangle + \langle \nu \rangle \langle |s(\omega, \boldsymbol{a})| \rangle^2 \langle A_\varepsilon(\omega, K) \rangle - \\ - \langle \nu \rangle \langle |s(\omega, \boldsymbol{a})| \rangle^2 \langle |B_\varepsilon(\omega, K)| \rangle^2 + \langle \nu \rangle \langle |s(\omega, \boldsymbol{a})| \rangle^2 \langle |B_\varepsilon(\omega, K)| \rangle^2 Z_\vartheta(\omega) + \\ + \langle \nu \rangle^2 \langle K \rangle^2 \langle |s(0, \boldsymbol{a})| \rangle^2 \, 2\pi \, \delta(\omega) . \end{aligned} \tag{26}$$

Here we have denoted

$$Z_\vartheta(\omega) = 1 + 2\,\mathrm{Re}\left\{\frac{\chi_\vartheta(\omega)}{1 - \chi_\vartheta(\omega)}\right\}, \quad \omega \neq 0\,. \tag{27}$$

Some properties of the function $Z_\vartheta(\omega)$ were studied in [8]. It may be useful to examine the functions $\langle A_\varepsilon(\omega, K)\rangle$ and $\langle |B_\varepsilon(\omega, K)|\rangle^2$ in a similar way. This will be done in the next section where examples of these functions will also be given.

3. EXAMPLES OF THE FUNCTIONS $A_\varepsilon(\omega, K)$ AND $|B_\varepsilon(\omega, K)|^2$

The functions $A_\varepsilon(\omega, K)$ and $B_\varepsilon(\omega, K)$ are defined by the series (5) and (7), respectively. If $|\chi_\varepsilon(\omega)| < 1$ for $\omega \neq 0$ then the series may be summed according to the formulae (6) and (8). For $\omega = 0$ we obtain from (5) and (7) relations $A_\varepsilon(0, K) = K^2 - K$ and $B_\varepsilon(0, K) = K$ or for the averages $\langle A_\varepsilon(0, K)\rangle = \langle K^2\rangle - \langle K\rangle$ and $\langle |B_\varepsilon(0, K)|\rangle^2 = \langle K\rangle^2$. The functions $A_\varepsilon(\omega, K)$ and $B_\varepsilon(\omega, K)$ are defined for $K \geqq 1$. If $K = 1$, then it follows from (5) and (7) that $A_\varepsilon(\omega, 1) = 0$ and $B_\varepsilon(\omega, 1) = 1$. At last, when comparing the processes A and B we can see that the functions $\langle A_\varepsilon(\omega, K)\rangle$ and $\langle |B_\varepsilon(\omega, K)|\rangle^2$ correspond to the expressions $(\langle K^2\rangle - \langle K\rangle)\,|\chi_\varphi(\omega)|^2$ and $\langle K\rangle^2\,|\chi_\varphi(\omega)|^2$, respectively.

Similarly as on the basic level zero the process B may also be either quasiperiodic, periodic or aperiodic on higher levels [1]. We believe it will be useful to illustrate this behaviour of the process B by several examples.

First let us deal with the *quasiperiodic process* B_Q and let us suppose that the random intervals among neighbouring points, ε, are normally distributed with the probability density

$$w_1(\varepsilon) = \frac{1}{\sigma_\varepsilon \sqrt{(2\pi)}} \exp\left[-\frac{(\varepsilon - \langle\varepsilon\rangle)^2}{2\sigma_\varepsilon^2}\right], \quad \varepsilon \geqq 0\,. \tag{28}$$

In this case the characteristic function, $\chi_\varepsilon(\omega) = \chi_G(\omega)$, has three arguments, namely the variable ω and parameters σ_ε and $\langle\varepsilon\rangle$. By introducing nondimensional variables $V = \sigma_\varepsilon/\langle\varepsilon\rangle$ and $x = \omega\langle\varepsilon\rangle$ the number of arguments may be reduced by one. We obtain

$$\chi_G(x) = \exp\left[\mathrm{j}x - (Vx)^2/2\right]. \tag{29}$$

The characteristic functions $\chi_G(x)$ may be summed according to the formulae (5) and (7). The functions $A_G(x, K)$ and $|B_G(x, K)|^2$ thus obtained are plotted for several values of K and for $V = 0{\cdot}1$ in Fig. 2. The functions $A_G(x, K)$ and $|B_G(x, K)|^2$ have an absolute maximum at $x = 0$ and several local maxima at $x = 2k\pi$, $k = \pm 1$, ± 2, ... The value of all these maxima is directly proportional to K: with increasing K these maxima also grow. The value of the absolute maximum does not depend on V; however, the values of the local maxima depend on V strongly. In fact, these maxima quickly diminish with growing V. Thus, for example, there is hardly any

observable local maximum at $x = 2\pi$ for $V = 0{\cdot}3$ and $K = 10$. On the other hand, if V is sufficiently small the values of the local maxima approach the value of the absolute maximum independently on K. In the limit we obtain

$$\lim_{V \to 0} A_G(x, K) = A_D(x, K), \tag{30}$$

$$\lim_{V \to 0} |B_G(x, K)|^2 = |B_D(x, K)|^2, \tag{31}$$

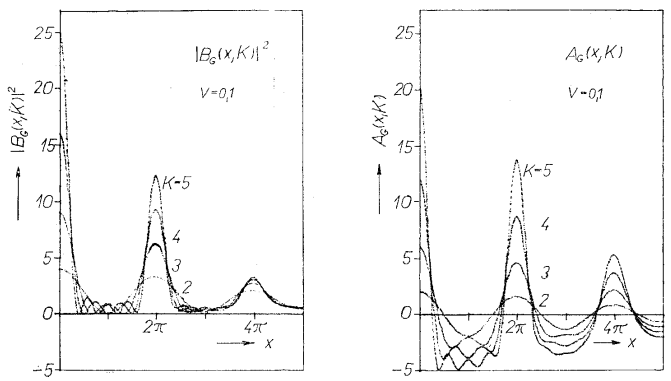

Fig. 2. Examples of the functions $|B_G(x, K)i^2$ and $A_G(x, K)$.

where the functions $A_D(x, K)$ and $B_D(x, K)$ correspond to the deterministic distribution of intervals ε (see later). It may also be easily verified that

$$\lim_{x \to \infty} A_G(x, K) = 0 \tag{32}$$

and

$$\lim_{x \to \infty} |B_G(x, K)|^2 = 1. \tag{33}$$

In the case of the *periodic process* D the intervals among neighbouring points, ε, are deterministically distributed with the probability density

$$w_1(\varepsilon) = \delta(\varepsilon - T_p), \quad T_p > 0. \tag{34}$$

In this case the characteristic function, $\chi_\varepsilon(\omega) = \chi_D(\omega)$, has two arguments, namely the variable ω and a parameter T_p. By introducing a nondimensional variable $x = \omega T_p$ the number of arguments will be reduced by one. The characteristic function equals now

$$\chi_D(x) = \exp(\mathrm{j}x) \tag{35}$$

and the functions $|B_D(x, K)|^2$ and $A_D(x, K)$ attain a very simple form

$$|B_D(x, K)|^2 = \frac{\sin^2(Kx/2)}{\sin^2(x/2)}, \tag{36}$$

$$A_D(x, K) = |B_D(x, K)|^2 - K. \tag{37}$$

The functions $|B_D(x, K)|^2$ and $A_D(x, K)$ are plotted for several values of K in Fig. 3. The functions $|B_D(x, K)|^2$ and $A_D(x, K)$ have maxima at $x = 2k\pi$, $k = 0, \pm 1, \pm 2, \ldots$,

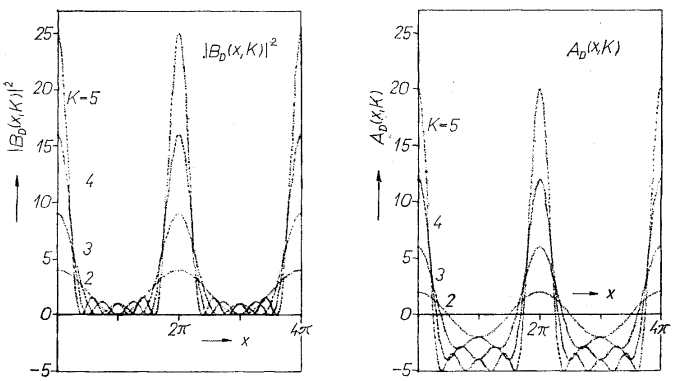

Fig. 3. Examples of the functions $|B_D(x, K)|^2$ and $A_D(x, K)$.

and at these maxima they attain the values K^2 and $(K^2 - K)$, respectively. In the limit we obtain an important relation [9]

$$\lim_{K \to \infty} \frac{1}{K} |B_D(\omega, K)|^2 = \omega_p \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_p). \tag{38}$$

Here $\omega_p = 2\pi/T_p$.

Finally, let us consider the *aperiodic process* $\mathrm{B_A}$ and let us suppose that the random variable ε is exponentially distributed

$$w_1(\varepsilon) = \frac{1}{\langle \varepsilon \rangle} \exp\left(-\varepsilon/\langle \varepsilon \rangle\right), \quad \varepsilon \geqq 0. \tag{39}$$

In this case the characteristic function, $\chi_\varepsilon(\omega) = \chi_E(\omega)$, has two arguments again, namely the variable ω and a parameter $\langle \varepsilon \rangle$. By introducing a nondimensional variable $x = \omega\langle \varepsilon \rangle$ the number of arguments will be reduced by one. We obtain

$$\chi_E(x) = \frac{1}{1 - \mathrm{j}x}. \tag{40}$$

The characteristic functions $\chi_E(x)$ may be summed to give the functions $A_E(x, K)$ and $|B_E(x, K)|^2$. Examples of these functions are plotted for several values of K in Fig. 4.

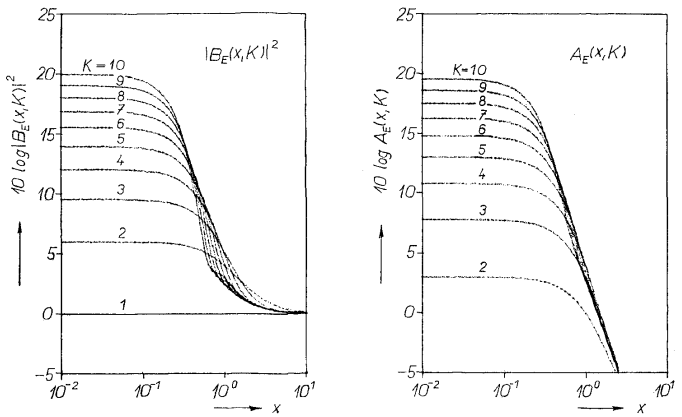

Fig 4. Examples of the functions $|B_E(x, K)|^2$ and $A_E(x, K)$.

Before concluding this section let us mention four limits that may be useful in characterizing the behaviour of the functions $A_E(x, K)$ and $|B_E(x, K)|^2$:

$$\lim_{x\to\infty} A_E(x, K) = 0, \tag{41}$$

$$\lim_{x\to\infty} |B_E(x, K)|^2 = 1, \tag{42}$$

$$\lim_{K\to\infty} A_E(x, K) = 2/x^2, \qquad x \neq 0, \tag{43}$$

$$\lim_{K\to\infty} |B_E(x, K)|^2 = 1 + 1/x^2, \quad x \neq 0. \tag{44}$$

4. POWER SPECTRUM OF THE PROCESS LB ... B

In Section 2 the power spectrum of the process BB was derived almost step by step. Similarly as in the previous paper [2] the procedure will be generalized now to obtain the power spectrum of the process LB ... B. Again, simplified notation will be adopted here. Beside the symbols defined in [2] the following simplifications will be used

$\varepsilon_{i,k_1,\ldots,k_p}$ will be replaced by ε_p,

$A_{\varepsilon_p}(\omega, K)$ will be replaced by A_p,

and $B_{\varepsilon_p}(\omega, K)$ will be replaced by B_p.

As the derivation is very similar to the one given in [2] only more significant departures will be mentioned here. Thus, following the usual procedure the expression for $|S_{TI}(\omega)|^2$ can be found. This expression contains $2(n+1)$-fold sum. From this sum $(n+1)$ simpler units may be taken out. The structure of these units was given in [2]. Hence, for example, the pth unit will contain those terms from the $2(n+1)$-fold sum for which it holds simultaneously that

$$i = l\,, \quad k_1 = m_1, \ \ldots, \ k_{n-p} = m_{n-p}, \quad k_{n-p+1} \neq m_{n-p+1}\,.$$

The pth unit will be formed by the $(n+p+1)$-fold sum of the terms having the form

$$ss^{*\prime} \exp\left[-j\omega(\varphi_n - \varphi_n')\right] \ldots \exp\left[-j\omega(\varphi_{n-p+1} - \varphi_{n-p+1}')\right]. \tag{45}$$

In contrast to the process A where the random variables φ_p and φ_p' were mutually independent now a certain statistical dependence among them exists. However, with respect to the definition of the process B, the difference $\varphi_p - \varphi_p'$ equals the sum of independent variables ε_p [1] (*cf* also Section 2 of this paper). Hence, when determining the expectation of $|S_{TI}(\omega)|^2$ (first on a set of the truncated realizations that contain just I clusters of the order zero and then on an ensemble of all truncated realizations) we obtain

$$\begin{aligned}\langle|S_T(\omega)|^2\rangle &= \langle I\rangle \langle K_1\rangle \ldots \langle K_n\rangle \langle|s|^2\rangle + \\ &+ \langle I\rangle \langle K_1\rangle \ldots \langle K_{n-1}\rangle \langle|s|^2\rangle \langle A_n\rangle + \\ &+ \langle I\rangle \langle K_1\rangle \ldots \langle K_{n-2}\rangle \langle|s|^2\rangle \langle|B_n|^2\rangle \langle A_{n-1}\rangle + \ldots \\ &\ldots + \langle I\rangle \langle|s|^2\rangle \langle|B_n|^2\rangle \ldots \langle|B_2|^2\rangle \langle A_1\rangle + \\ &+ \langle|s|^2\rangle \langle|B_n|^2\rangle \ldots \langle|B_1|^2\rangle \langle X_0(\omega, I)\rangle\,.\end{aligned} \tag{46}$$

Here the function corresponding to the level zero was denoted $\langle X_0(\omega, I)\rangle$. In the case of a homogeneous process this function equals [2]

$$\langle X_0(\omega, I)\rangle = \langle I^2\rangle \, |\chi_\tau(\omega)|_T^2\,,$$

in the case of a periodic process

$$\langle X_0(\omega, I)\rangle = \langle A_D(\omega, I)\rangle\,,$$

and in the case of a quasiperiodic or aperiodic process

$$\langle X_0(\omega, I)\rangle = \langle A_B(\omega, I)\rangle\,.$$

After substituting (46) into (2) the power spectrum of the process LB ... B will be obtained in the form

$$\begin{aligned}\mathcal{W}(\omega) &= \langle\nu\rangle \langle K_1\rangle \ldots \langle K_n\rangle \langle|s|^2\rangle + \langle\nu\rangle \langle K_1\rangle \ldots \langle K_{n-1}\rangle \langle|s|^2\rangle \langle A_n\rangle + \\ &+ \langle\nu\rangle \langle K_1\rangle \ldots \langle K_{n-2}\rangle \langle|s|^2\rangle \langle|B_n|^2\rangle \langle A_{n-1}\rangle + \ldots + \\ &+ \langle\nu\rangle \langle|s|^2\rangle \langle|B_n|^2\rangle \ldots \langle|B_2|^2\rangle \langle A_1\rangle + \langle\nu\rangle^2 \langle|s|^2\rangle \langle|B_n|^2\rangle \ldots \langle|B_1|^2\rangle \, Y_0(\omega)\,.\end{aligned} \tag{47}$$

The form of the function $Y_0(\omega)$ for the processes A, B and D was given in [2].

If the process is of the type A on the pth level, then it is not difficult to see from (45) that in (46) and (47) all $\langle A_p \rangle$ s and $\langle |B_p| \rangle^2$ s will be replaced by $(\langle K_p^2 \rangle - \langle K_p \rangle) |\chi_p|^2$ s and $\langle K_p \rangle^2 |\chi_p|^2$ s, respectively.

5. PROCEDURE FOR FAST DETERMINATION OF THE POWER SPECTRUM

Due to its generality the formula (47) is impractical for determination of the power spectrum of a certain process. It was shown in [2] that "matrices" can be used conveniently for this purpose. When comparing the power spectrum formulae of the processes A and B it can be seen immediately that if the process is of the type B on the pth level then a'_p and a''_ps in the pth column of the matrix will be replaced by b'_p and b''_ps, respectively, where the symbols b'_p and b''_p have the following meaning

$$b'_p = \langle A_p \rangle ,$$

$$b''_p = \langle |B_p| \rangle^2 .$$

The procedure can be illustrated best by an *example*. Let us consider a process of the type DA^PB, which can be considered as an alternative to the process DA^PA^P in modelling the Barkhausen voltage, and let us determine its power spectrum.

We shall start by writing down the matrix of the process DAB. It has the form

	D	A	B	
	0	1	2	3
0	v_0	$\langle K_1 \rangle$	$\langle K_2 \rangle$	$\langle \|s\|^2 \rangle$
1	v_0	$\langle K_1 \rangle$	b'_2	$\langle \|s\| \rangle^2$
2	v_0	a'_1	b''_2	$\langle \|s\| \rangle^2$
3	v_0^2	a''_1	b''_2	$\langle \|s\| \rangle^2 \, Y_D(\omega)$

After mutual multiplying the elements in each row and adding all rows together we obtain

$$\mathcal{W}(\omega) = v_0 \langle K_1 \rangle \langle K_2 \rangle \langle |s|^2 \rangle + v_0 K_1 b'_2 \langle |s| \rangle^2 + v_0 a'_1 b''_2 \langle |s| \rangle^2 + \\ + v_0^2 a''_1 b''_2 \langle |s| \rangle^2 \, Y_D(\omega) .$$

Now the symbols a'_1, a''_1, b'_2, b''_2 and $Y_D(\omega)$ can be replaced by the respective functions

so that we have

$$\mathcal{W}(\omega) = v_0\langle K_1\rangle\langle K_2\rangle\langle|s|^2\rangle + v_0\langle K_1\rangle\langle|s|\rangle^2\langle A_2\rangle +$$
$$+ v_0(\langle K_1^2\rangle - \langle K_1\rangle)\langle|s|\rangle^2\langle|B_2|\rangle^2|\chi_1|^2 +$$
$$+ v_0^2\langle K_1\rangle^2\langle|s|\rangle^2\langle|B_2|\rangle^2|\chi_1|^2\left[-1/v_0 + 2\pi\sum_{k=-\infty}^{\infty}\delta(\omega - k\omega_0)\right].$$

After taking into account that the points are Poisson distributed on the first level we finally obtain

$$\mathcal{W}(\omega) = v_0\langle K_1\rangle\langle K_2\rangle\langle|s|^2\rangle + v_0\langle K_1\rangle\langle|s|\rangle^2\langle A_2\rangle + \tag{48}$$
$$+ v_0^2\langle K_1\rangle\langle|s|\rangle^2\langle|B_2|\rangle^2|\chi_1|^2\,2\pi\sum_{k=-\infty}^{\infty}\delta(\omega - k\omega_0).$$

6. POWER SPECTRUM OF SOME SPECIAL PROCESSES

The structural formulae of several special processes were mentioned in [1]. In this section we want to give their power spectrum.

Let us first consider a *pure periodic process* $DD^D \dots D^D$. Using relation (37) we obtain from (47)

$$\mathcal{W}(\omega) = v_0K_1 \dots K_n\langle|s|^2\rangle - v_0K_1 \dots K_n\langle|s|\rangle^2 + \tag{49}$$
$$+ v_0^2\langle|s|\rangle^2|B_n|^2 \dots |B_1|^2\,2\pi\sum_{k=-\infty}^{\infty}\delta(\omega - k\omega_0).$$

The functions $|B_p|^2$ are given by (36). In the case of the *deterministic process* it holds that $\langle|s|^2\rangle = \langle|s|\rangle^2 = |s|^2$ and the expression (49) will take the form

$$\mathcal{W}(\omega) = v_0^2|s|^2|B_n|^2 \dots |B_1|^2\,2\pi\sum_{k=-\infty}^{\infty}\delta(\omega - k\omega_0). \tag{50}$$

The power spectrum of a *pure homogeneous process* $AA^P \dots A^P$ has the form [2]

$$\mathcal{W}(\omega) = \langle v\rangle\langle K_1\rangle \dots \langle K_n\rangle\langle|s|^2\rangle + \tag{51}$$
$$+ \langle v\rangle\langle K_1\rangle \dots \langle K_{n-1}\rangle\langle K_n\rangle^2\langle|s|\rangle^2|\chi_n|^2 + \dots$$
$$\dots + \langle v\rangle\langle K_1\rangle^2 \dots \langle K_n\rangle^2\langle|s|\rangle^2|\chi_n|^2 \dots |\chi_1|^2 +$$
$$+ \langle v\rangle^2\langle K_1\rangle^2 \dots \langle K_n\rangle^2\langle|s|\rangle^2\,2\pi\,\delta(\omega).$$

The process $AA^P \dots A^P$ represents a certain generalization of the *homogeneous Poisson process* A, the power spectrum of which can be readily obtained from (51) by putting $n = 0$:

$$\mathcal{W}(\omega) = \langle v\rangle\langle|s|^2\rangle + \langle v\rangle^2\langle|s|\rangle^2\,2\pi\,\delta(\omega). \tag{52}$$

The power spectrum of the *Poisson periodic process* $DD^D \dots D^DA^P$ can be

obtained from (47) using (37) in the form

$$\mathcal{W}(\omega) = v_0 K_1 \dots K_{n-1} \langle K_n \rangle \langle |s|^2 \rangle + \tag{53}$$

$$+ v_0^2 \langle K_n \rangle^2 \langle |s| \rangle^2 |\chi_n|^2 |B_{n-1}|^2 \dots |B_1|^2 2\pi \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_0) .$$

The functions $|B_p|^2$ are given by (36) again. For $n = 1$ we readily obtain from (53) the power spectrum of the process DA^P

$$\mathcal{W}(\omega) = v_0 \langle K_1 \rangle \langle |s|^2 \rangle + v_0^2 \langle K_1 \rangle^2 \langle |s| \rangle^2 |\chi_1|^2 2\pi \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_0) . \tag{54}$$

The expression (54) was derived in [10].

7. COMPARISON OF THE PROCESSES A, B AND D

When comparing the processes A, B and D it can be seen that the process A is the most simple. This is a consequence of mutual independence of reference points. The simplicity of the process A manifests itself, for example, in the power spectrum formula where the numbers of points in groups, K_p, and the characteristic functions, χ_p, are present in the form of a product in each unit. As there is a certain dependence among points of the process B, this process is more complex. This greater complexity is also reflected in the power spectrum formula, in which the numbers K_p and the characteristic functions χ_p appear in the form of finite series now. However, there are two important exceptions. The first exception regards the process B_D, which is a very simple process again. It was shown elsewhere [8] that the process B_D equals the process D on the level zero. In Section 3 this equivalence was extended even on higher levels. Thus on any level the processes B_D and D are equivalent. The second exception regards the process B_E. According to the definition [1] the process A is always homogeneous Poisson on the level zero. An interesting feature of the homogeneous Poisson process is that it is both the process with independent and uniformly distributed points and the process with independent and exponentially distributed intervals among neighbouring points [11]. It follows then that on the level zero the processes A and B_E are equivalent. However, on higher levels the processes A and B are always different. This can be illustrated, for example, by the *mean group length* $\langle \mu_p \rangle$. The mean group length of the process B equals $\langle \mu_p \rangle = \langle K_p \rangle \langle \varepsilon_p \rangle$. On the other hand the mean group length of the process A does not depend on $\langle K_p \rangle$ at all.

Another difference between the processes A and B concerns the *point density in groups*. On the level zero the point density was denoted $\langle v \rangle$ ($\langle v \rangle = 1/\langle \vartheta \rangle$) and it is always a constant for stationary processes. Let us denote the point density in groups on higher levels as $\langle \lambda_p \rangle$. From the definition of the process B it follows [1] that $\langle \varepsilon_p \rangle = \text{const.}$ and hence $\langle \lambda_p \rangle = 1/\langle \varepsilon_p \rangle$ is always constant for processes B.

As for the processes A, the point density in groups equals $\langle \lambda_p \rangle = \langle K_p \rangle w_1(\varphi_p)$ and thus it is constant only in a special case when φ_p is uniformly distributed. It follows then that the process A can be conveniently used to model a variable point density.

8. APPLICATIONS OF THE PULSE PROCESSES

Several applications of the random pulse processes were mentioned in the previous papers [1], [2], [8] and [10]. In this section all these applications and some new ones will be briefly summarized.

Applications of the random pulse processes may be roughly divided into three groups. The first two groups concern signals and noises, respectively, the third one includes all the remaining cases. Let us now pay attention to each group separately.

The first area where the pulse processes are extensively used is in the *signal theory*. Here they are used to model pulse signals. Thus, for example, the signals in the pulse-amplitude-modulation (PAM) and pulse-width-modulation (PWM) systems can be modelled by the process D, the signals in the pulse-position-modulation(PPM) systems by the process DA^1 and at last in the case of the pulse-code-modulation (PCM) the process DD^D is adequate. In pulse-communication systems the signals from several sources are usually transmitted over one common chanel on the princip of time-division. If the synchronizing pulses are omitted from analyzes and only information transmitting pulses are considered, then these signals may be modelled by the processes DD^DA^1 (PPM) or DD^DD^D (PCM). To model the whole signal more complex processes must be used [12].

In applications concerning the signal theory the inner description of the pulse processes is usually supposed to be known. It is given by the statistics of the information source and by the modulation system under consideration. What is sought are the outer characteristics (usually the power spectrum) so that efficiency, noise immunity etc. of the system could be evaluated analytically. It is often desirable to optimize the coding schema according to some criteria, e.g., to minimize the signal dc component. This can be done best by analyzing the power spectrum formula [13], [14]. Another peculiarity of the pulse signals (as compared with pulse noises to be discussed later) concerns pulse overlapping. To avoid lost of transmitted information it is demanded that the neighbouring pulses do not overlap.

It may perhaps be interesting to note here that even such common signals as *speech* and *music* are impulsive in nature. Both these signals are formed by pulse clusters of a rather high order and of a very complex and heavily correlated structure. The impulsive character of music produced by some instruments such as a drum and piano is evident at the first sight. It may not be so obvious for other instruments (e.g., a violin) or speech. However, the existence of the music scales and of the alphabet is the best evidence of this assertment. Then risking some oversimplification and thus possible rejection by specialists we may say that in speech each vowel

or consonant represents a sound pulse, a word represents a pulse group, a sentence represents a cluster of pulse groups and so on. In the same way music can be resolved. However, it is highly improbable that the group pulse processes will ever be used in speech and music research and this comment was meant to demonstrate the widespread occurrence of pulse signals only.

The second broad area where the random pulse processes are extensively used is in the *noise research*. It is typical for this kind of applications that pulses overlap without restriction and sometimes very densely. Therefore, direct measurement of inner characteristics is usually impossible and only estimates of some outer characteristics can be found. Thus the task in the noise research is just reverse to that in the signal theory. Now the outer characteristics are taken as a point of departure and one tries to estimate the form of the inner ones. This and related questions were discussed in greater detail in paper [1].

Noises can be classified according to a number of criteria. One criterion is whether the noise is *impulsive* (formed by a random pulse sequence) or *continuous* (not formed by a random pulse sequence). Another criterion is whether the noise is generated by micro-events or by macro-events.

With respect to the corpuscular nature of the microworld it is evident that all *micro-event* generated noises have impulsive character with pulses typically overlapping very densely. Perhaps the best known examples of such noises are shot and thermal noises that are usually modelled by the process A. The process A was also applied by Bevan et al. to a cell membrane noise [15].

Another example of a micro-event generated noise can be provided by cathodoluminescence [16], [17] where a beam of electrons impinges on a luminescent material (e.g., a phosphor), each electron generating a shower of photons. If the incident electrons form a homogeneous Poisson point process and each electron generates an exponentially decaying flux of photons, the number of which is governed by the Poisson distribution, then the resulting flux of photons can be modelled by the process AA_E^P. The same model can also be applied to the scintillation photon counting [16] or to Lorentzian flicker noise [17].

We believe that a noise in photomultipliers represents further example where the group puse processes can be successfully used. If the primary flux is considered to be the Poisson homogeneous process and if there are n multiplication stages, then the output current can be modelled by the nth-order group pulse process AA ... A.

Majority of *macro-event* generated noises are impulsive again. Noise generated by turbulent flow can be given as an example of a continuous noise [18]. Cavitation noise (models A, AB [19] and DA^P [20]), Barkhausen noise (models A, B_{A+}, DA^PA^P [21] and DA^PB), impulsive noise in communication [22], [23], impulsive noise due to current pulses of cloud-to-ground lightning discharges [24], [25] and [26], radio interference produced by corona pulses on a.c. lines [27] and noise due to

combustine engines (model D [28] and a model similar to the process DD with unequally spaced points on the 1st level [29]) are just a few examples of macro-event generated pulse noises.

The *third group* includes models of such diverse phenomena as telephone traffic [30], [31], number of fibers in a cord [31], an electric power system [32], sampled signals [33], [34], river streamflows [35] and many others.

9. CONCLUSION

In [1] a classification of several basic processes was submitted. This classification was also supplemented by a notation that made it possible to describe the structure of a process in the concise form, similarly as a chemical formula describes the structure of a compound. In [2] and in the present work the power spectra of the processes included in the classification were determined. As the general formulae obtained are impractical for direct use more suitable matrix method was introduced. This matrix method represents literally a recipe that makes it possible to find the desired formula in a few minutes even by an uninitiate and thus saves precious time for other work.

As mentioned in the previous section the processes A, B and D can be used to model a large number of signals, noises and other phenomena. Thanks to their relative simplicity they are easy to work with. Naturally, in some cases these simple models may represent only a first approximation to the studied phenomena and therefore processes with a more complex structure, e.g., with a correlation among different pulses or different pulse parameters, may be more appropriate. Though several such more complex processes are described in literature (e.g., in [12]–[14]), there is still a lack of general theory.

We tried to develop a theory that would cover both the signal and noise models. We believe such a unifying approach will be even more important with more complex processes as it makes it possible to transfer the results of research from one area to another. A better insight into the structure of the respective processes is also gained. Let us illustrate this with two simple examples and let us first consider the process B. After specifying the distribution of intervals among pulses we can obtain (among others) either a typical signal model B_D or a typical noise model B_E [8]. The process DA can serve as another example. When the distribution of pulses on the 1st level is suitably specified, either the process DA^1, which is a PPM signal model or the process DA^P, which is a cavitation noise model, result [10].

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