Kybernetika

Jerry Uche Onwuatu

Null controllability of nonlinear infinite neutral system

Kybernetika, Vol. 29 (1993), No. 4, 325--338

Persistent URL: http://dml.cz/dmlcz/125634

KYBERNETIKA — VOLUME 29 (1993), NUMBER 4, PAGES 325–336

NULL CONTROLLABILITY OF NONLINEAR INFINITE NEUTRAL SYSTEM

JERRY U. ONWUATU

Sufficient conditions are developed for the null controllability of the nonlinear infinite neutral system

$$\frac{\mathrm{d}}{\mathrm{d}t} D(t, x_t) = L(t, x_t) + B(t)\, u(t) + f(t, x_t, u(t)) + \int_{-\infty}^{0} A(\theta)\, x(t+\theta)\, \mathrm{d}\theta$$

when the values of the control function lie in an n-dimensional unit cube C^m of R^m. Conditions are placed on f which guarantee that if the linear control base system is proper and if the uncontrolled linear system is is uniformly asymptotically stable, then the nonlinear perturbed system is null controllable with constraints.

1. INTRODUCTION

Consider the linear difference-differential equation of neutral type

$$\frac{\mathrm{d}}{\mathrm{d}t}(x(t) - A_{-1}x(t-h)) = A_0\, x(t) + B_0\, u(t)$$
$$x(t) = \phi(t), \quad t \in [-h, 0]$$

in which x is an element in the Euclidean n-space, R^n; A_{-1}, A_0 are $n \times n$ constant matrices; B_0 is an $n \times m$ constant matrix and $h > 0$. Neutral functional differential equations, an example of which is given above, are characterized by a delay in the derivative. Equations of this type have applications in the study of electrical networks containing lossless transmission lines [2], electrodynamics [5], variational problems [6] etc. For efficient utilization of power, the stability and *control* of voltage and current fluctuations are of fundamental importance for systems planners. Other authors who have contributed in the study of these systems include Chukwu [4], Arrow [1, p. 184], Knowles [15, p. 3], Kalecki [13] among others. The stability of functional differential equations of retarded type with a finite delay $h > 0$ have been studied by Hale [11]. These results have been extended to systems with infinite delay such as

$$\dot{x}(t) = \sum_{i=1}^{N} B_i\, x(t - h_i) + \int_{-\infty}^{0} B(\theta)\, x(t+\theta)\, \mathrm{d}\theta$$

by the same author [10].

In [17, p. 31], Stéphán have some results on stability for the neutral system with infinite delays such as

$$\dot{x}(t) = \int_{-\infty}^{0} [\mathrm{d}\eta(\theta)]\ \dot{x}(t+\theta) + \int_{-\infty}^{0} [\mathrm{d}\,\eta_0(\theta)]\ x(t+\theta)$$

and

$$\dot{x}(t) = C\,\dot{x}(t-h) + \int_{-\infty}^{0} [\mathrm{d}\eta_0(\theta)]\ x(t+\theta).$$

This last equation can also be written as

$$\frac{\mathrm{d}}{\mathrm{d}t}\,D(x_t) = \int_{\infty}^{0} [\mathrm{d}\eta_0(\theta)]\ x(t+\theta)$$

where

$$D(x_t) = x(t) - C\,x(t-h)$$

and C is an $n \times n$-constant matrix.

With respect to controllability, it is known [12] that if the linear ordinary control system

$$\dot{x}(t) = A(t)\,x(t) + B(t)\,u(t) \tag{1}$$

is proper and if the free system

$$\dot{x}(t) = A(t)\,x(t) \tag{2}$$

is uniformly asymptotically stable, then (1) is null controllable with constraints. An analogous result was obtained by Chukwu [3] for the delay system

$$\dot{x}(t) = L(t, x_t) + B(t)\,u(t) + f(t, x_t, u(t)) \tag{3}$$

where

$$L(t,\phi) = \sum_{k=0}^{\infty} A_k(t)\,\phi(-t_k) + \int_{-\gamma}^{0} A(t,s)\,\phi(s)\,\mathrm{d}s. \tag{4}$$

Shinha [18] studied the nonlinear infinite delay system

$$\dot{x}(t) = L(t, x_t) + B(t)\,u(t) + \int_{-\infty}^{0} A(\theta)\,x(t+\theta)\,\mathrm{d}\theta + f(t, x_t, u(t)) \tag{5}$$

and showed that (5) is Euclidean null controllable if the linear base system

$$\dot{x}(t) = L(t, x_t) + B(t)\,u(t) \tag{6}$$

is proper and the free system

$$\dot{x}(t) = L(t, x_t) + \int_{-\infty}^{0} A(\theta)\,x(t+\theta)\,\mathrm{d}\theta. \tag{7}$$

is uniformly asymptotically stable, provided that f satisfies some growth conditions.

Define a function $D: R \times B \to R^n$ by

$$D(t, x_t) = x(t) - g(t, x_t)$$

where,

$$g(t, x_t) = \sum_{n=1}^{\infty} A_n(t)\, x\,(t - w_n(t)) + \int_{-\gamma}^{0} A(t, s)\, x(t + s)\, \mathrm{d}s$$

and where $0 < w_n(t) \le \gamma$ and $A_n(t)$ and $A(t, s)$ are $n \times n$ matrix functions, and let B be an $n \times m$ matrix. We shall consider the system whose state is given by

$$\frac{\mathrm{d}}{\mathrm{d}t} D(t, x_t) = L(t, x_t) + B(t)\, u(t) + \int_{-\infty}^{0} A(\theta)\, x(t + \theta)\, \mathrm{d}\theta + f(t, x_t, u(t)) \quad (8)$$
$$x(t) = \phi(t), \quad t \in (-\infty, 0)$$

where $L(t, \phi)$ is as defined in (4), $A(\theta)$ is an $n \times n$ continuous matrix and $f(t, x, u(t))$ is a nonlinear continuous matrix function. The controls of interest, u, are square integrable with values in the unit cube C^m,

$$C^m = \{u : u \in R^m,\ |u_j| \le 1,\ j = 1, \ldots, m\}\,.$$

Here, we develop sufficient computable criteria for the null controllability of system (8). Our results extend those of [12, 3, 18] to neutral systems with infinite delay.

2. PRELIMINARIES

Let n and m be positive integers, R the real line $(-\infty, \infty)$. Denote by R^n, the space of real n-tuples with the Euclidean norm defined by $|\cdot|$. If $J = [t_0, t_1]$ is any interval of R, the usual Lebesgue space of square integrable (equivalence, classes of) functions from J to R^m will be denoted by $L_2(J, R^m)$. Let $\gamma \ge h \ge 0$ be a given real number and let $B = B([-\gamma, 0],\ R^n)$ be the Banach space of functions which are continuous on $[-\gamma, 0]$ with $\|\phi\| = \sup_{-\gamma \le \theta \le 0} |\phi(s)|$ $\phi \in B([-\gamma, 0],\ R^n)$. If x is a function from $[t_0 - \gamma, \infty)$ to R^n, let x_t, $t \in [0, \infty)$, be a function from $[-\gamma, 0]$ to R^n defined by $x_t(s) = x(t + s)$, $s \in [-\gamma, 0]$. In system (8), assume that $D(\cdot, \cdot)$; $R \times B \to R^n$ is defined by

$$D(t, x_t) = x(t) - g(t, x_t)$$

where

$$g(t, \phi) = \sum_{n=1}^{\infty} A_n(t)\, \phi(-w_n(t)) + \int_{-\gamma}^{0} A(t, s)\, \phi(s)\, \mathrm{d}s \equiv \int_{-\gamma}^{0} \mathrm{d}_\theta\, H(t, \theta)\, \phi(\theta)$$

and where $0 < w_n(t) \le \gamma$ and

$$\int_{-\gamma}^{0} |A(t, s)|\, \mathrm{d}s + \sum_{n=1}^{\infty} |A_n(t)| \le \delta(\varepsilon) < \infty$$

for all t, where $\delta(\varepsilon) \to 0$. We also assume in system (8) that

$$L(t,\phi) = \int_{-\gamma}^{0} \mathrm{d}_\theta \eta(t,\theta)\,\phi(\theta) \tag{9}$$

where

$$\eta(t,0) = 0 \quad \text{for} \quad \theta \geq 0, \quad \eta(t,\theta) = \eta(t-h) \quad \text{for } \theta \leq -h,$$

η is a measurable $n \times n$ matrix-valued function from R^2 into R^{n^2}, of bounded variation in its second argument with

$$\operatorname{Var} \eta(t,\cdot) \leq m(t); \; m(\cdot) \text{ locally integrable on } R.$$

We assume that there exists a continuous, nonnegative, nondecreasing function $h(s)$, $s \in [0,\gamma]$ such that $h(0) = 0$, and

$$\left| \int_{-\gamma}^{0} \mathrm{d}_\theta\, H(t,\theta)\,\phi(\theta) \right| \leq h(s)\,\|\phi\|;$$

so that g is uniformly nonatomic at zero.

A nonautonomous linear homogeneous neutral differential equation is defined to be

$$\frac{\mathrm{d}}{\mathrm{d}t} D(t,x_t) = L(t,x_t). \tag{10}$$

A function x is said to be a solution of (10) if there exists $t_0 \in R$, $a > 0$ such that $x \in B([t_0-\gamma,\, t_0+a], R^n)$, $t \in (t_0, t_0+a)$ and x satisfies (10) on $[t_0, t_0+a]$. Given $t_0 \in R$, $\phi \in B$, we say $x(t_0,\phi)$ is a solution of (10) with initial value (t_0,ϕ) if there exists an $a > 0$ such that $x(t_0,\phi)$ is a solution of (10) on $[t_0-\gamma, t_0+a]$ and $x_{t_0}(t_0,\phi) = \phi$.

Our objective is to study the controllability of the perturbed system with infinite delay described by

$$\frac{\mathrm{d}}{\mathrm{d}t} D(t,x_t) = L(t,x_t) + C(t)\,u(t) + \int_{-\infty}^{0} A(\theta)\,x(t+\theta)\,\mathrm{d}\theta + f(t,x_t,u(t)) \tag{11}$$

through its linear base control system

$$\frac{\mathrm{d}}{\mathrm{d}t} D(t,x_t) = L(t,x_t) + C(t)\,u(t) \tag{12}$$

and its free system

$$\frac{\mathrm{d}}{\mathrm{d}t} D(t,x_t) = L(t,x_t) + \int_{-\infty}^{0} A(\theta)\,x(t+\theta)\,\mathrm{d}\theta. \tag{13}$$

Here C is a continuous $n \times m$ matrix function, each A_k is a continuous $n \times n$ matrix function for $0 \leq h_k \leq \gamma$, $A(\theta)$ is an $n \times n$ matrix whose elements are square integrable on $(-\infty,0)$. D, L, f satisfy enough smoothness conditions to ensure that a solution of (11) exists through each $(t_0,\emptyset)$ in $I \times B$, is unique, depends continuously on

$(t_0, \emptyset)$ and can be continued to the right as the trajectory remains in a bounded set in $I \times B$. Sufficient conditions to ensure these properties are developed in [9]. If $T(t,t_0) : B \to B, t \geq t_0$ is defined by $T(t,t_0)\phi = x_t(t_0,\phi)$, where $x(t_0,\phi)$ is the solution of (10), then a variation of constant formulae for system (12) is given in Hale [11]. Indeed, there exist an $n \times n$ matrix function $X(t,s)$ defined for $0 \leq s \leq \gamma$, $t \in [0,\infty]$, continuous in s from the right, of bounded variation in s, $X(t,s) = 0$, $t' < s \leq t+\gamma$, such that the solution $x(t_0,\phi)$ of (12) is given by

$$x(t,t_0,\phi,u) = T(t,t_0)\,\phi(0) + \int_{t_0}^{t} X(t,s)\,C(s)\,u(s)\,\mathrm{d}s, \quad t \geq 0. \tag{14}$$

The corresponding solution of (11) is given by

$$\begin{aligned} x(t,t_0,\phi,u,f) &= T(t,t_0)\,\phi(0) + \int_{t_0}^{t} X(t,s)\,C(s)\,u(s)\,\mathrm{d}s + \\ &+ \int_{t_0}^{t} X(t,s)\int_{-\infty}^{0} A(\theta)\,x(t+\theta)\,\mathrm{d}\theta\mathrm{d}s + \int_{t_0}^{t} X(t,s)\,f(s,x_s,u(s))\,\mathrm{d}s. \end{aligned} \tag{15}$$

Observe that the uniqueness of solutions of (10) imply that

$$T(t_2,t_1)\,T(t_1,t_0) = T(t_2,t_0), \quad t_1,\, t_2 \geq t_0 \geq 0.$$

If we let

$$X_0(s) = \begin{cases} 0, & -\gamma \leq s < 0 \\ I, & s = 0 \end{cases}$$

then $T(t,t_0)\,X_0(s) = X(t+s,t_0) = X_t(\cdot,t_0)(s)$. Therefore, $T(t,t_0)\,I = X(t,t_0)$. In (15), introduce the notation $Y(t,s) = X(t,s)\,C(s)$, $t \geq s \geq t_0$ and define the controllability matrix of (12) at time t by

$$W(t_0,t) = \int_{t_0}^{t} Y(t,s)\,Y^*(t,s)\,\mathrm{d}s$$

where the $*$ denotes the matrix transpose.

Definition 1. (Proper system)
The system (12) is said to be proper on an interval $[t_0,t_1]$ if $C^*Y(t,s) = 0$ a.e.; $s \in [t_0,t_1)$ implies $C = 0$, $C \in R^n$. If (12) is proper on each interval $[t_0,t_1]$, $t_1 \geq t_0$, we say (12) is proper in R^n.

Definition 2. (Complete controllability)
The system (12) is completely controllable on $J = [t_0,t_1]$ if for every function ϕ and every $x_1 \in R^n$ there exists an admissible control function u such that a solution of system (12) satisfies $x(t_1) = x_1$.

Definition 3. (Null controllability)
The system (11) is said to be null controllable on $[t_0, t_1]$ if for each $\phi \in B([-\gamma, 0], R^n)$ there exists a $t_1 \geq t_0$, $u \in L_2([t_0, t_1], P)$, P a compact convex subset of R^m, such that the solution $x(t, t_0, \phi, u, f)$ of (11) satisfies $x_{t_0}(t_0, \phi, u, f) = \phi$ and $x(t_1, t_0, \phi, u, f) = 0$.

Following Gabasov and Kirrilova [7] we introduce the determining equations for a simple case of (12), namely

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(x(t) - A_{-1}x(t-h)\right) = A_0\, x(t) + A_1\, x(t-h) + B\, u(t) \tag{15a}$$

where A_{-1}, A_0, A_1 are constant $n \times n$ matrices and B is an $n \times m$ constant matrix.

$Q_k(s) = A_0\, Q_{k-1}(s) + A_1\, Q_{k-1}(s-h) + A_{-1}\, Q_k(s-h)$, $k = 0, 1, 2, \ldots$; $s = 0, h, 2h, \ldots$

$Q_0(0) = I$, the identity matrix; $Q_0(s) = 0$, $s < 0$; and define

$$\overline{Q}_n(t_1) = \{Q_k(s)\, B,\ k = 0, 1, \ldots, n-1,\ s \in [0, t_1]\}.$$

We have

Proposition 1. The system (15a) is proper on the interval $[0, t_1]$ if and only if rank $\overline{Q}_n(t_1) = n$.

P r o o f. To prove this, we use the relation $C^* Y(t, s) = 0$ a.s. $s \in [0, t_1]$, $C \in R^n$ implies $C = 0$, instead of the relative controllability definition of Gabasov and Kirrilova and then proceed as in [7, pp. 53–60]. □

The following proposition on the controllability of system (12) is similar to corresponding results for linear control systems of various types including some with delays and some without. The proof can be obtained from Hermes and LaSalle [12, p. 92].

Proposition 2. The following are equivalent

(i) $W(t_0, t_1)$ is nonsingular,

(ii) System (12) is completely controllable on $[t_0, t_1]$, $t_1 > t_0$,

(iii) System (12) is proper on $[t_0, t_1]$, $t_1 > t_0$.

The next proposition is the result of Gahl [8, Thm. 2] on the complete controllability (and hence properness) of system (12).

Proposition 3. Consider the system

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(x(t) - A\, x(t-1)\right) = B\, x(t-1) + C\, x(t) + D\, u(t) + H\, u(t-h). \tag{15b}$$

If rank $[D, CD] = n$, then the system (15b) is completely controllable.

3. MAIN RESULT

Theorem 1. Assume for system (11) that

(i) the constraint set IU is an arbitrary compact subset of R^n,

(ii) the system (13) is uniformly asymptotically stable so that the solution of (13) satisfies $\|x(t,t_0,\phi,0,0)\| \leq M\, e^{-a(t-t_0)}\|\phi\|$ for some $a > 0,\ M > 0$,

(iii) the linear control system (12) is proper in R^n,

(iv) the continuous function f satisfies $|f(t,x(\cdot),\, u(\cdot))| \leq \exp(-bt)\, \pi(x(\cdot),\, u(\cdot))$, for all $(t, x(\cdot,\, u(\cdot)) \in [t_0,\infty) \times B \times L_2$, where

$$\int_{t_0}^{\infty} \pi(x(\cdot),\, u(\cdot))\, \mathrm{d}s \leq k < \infty \quad \text{and} \quad b - a \geq 0,$$

then system (11) is Euclidean null controllable.

Proof. By (iii), W^{-1} exists for each $t_1 > t_0$. Suppose the pair of functions x, u form a solution pair to the set of integral equations

$$\begin{aligned} u(t) &= -Y^*(t_1,t)\, W^{-1}(t_0,t_1)\, \Big[T(t,t_0)\,\phi_0 + \int_{t_0}^{t} X(t,s) \int_{-\infty}^{0} A(\theta)\, x(t+\theta)\, \mathrm{d}\theta \mathrm{d}s \\ &+ \int_{t_0}^{t} X(t_1,s)\, f(s,x_s,u(s))\, \mathrm{d}s\Big]. \end{aligned} \tag{16}$$

For some suitably chosen $t_1 \geq t \geq t_0$

$$\begin{aligned} x(t) &= T(t,t_0)\,\phi(0) + \int_{t_0}^{t} Y(t,s)\, u(s)\, \mathrm{d}s + \int_{t_0}^{t} X(t,s) \left(\int_{-\infty}^{0} A(\theta)\, x(t+\theta)\, \mathrm{d}\theta\right) \mathrm{d}s \\ &+ \int_{t_0}^{t} X(t,s)\, f\,(s, x(\cdot), u(\cdot))\, \mathrm{d}s \\ x(t) &= \phi(t), \quad t \in [t_0-\gamma, t_0]. \end{aligned} \tag{17}$$

Then u is square integrable on $[t_0,t_1]$ and x is a solution of (11) corresponding to u with initial state $x(t_0) = \phi$. Also,

$$\begin{aligned} x(t_1) &= T(t_1,t_0)\,\phi(0) - \int_{t_0}^{t_1} Y(t_1,s)\, Y^*(t_1,s)\, W^{-1}(t_0,t_1)\, [T(t_1,t_0)\,\phi(0) + \\ &+ \int_{t_0}^{t_1} X(t_1,s) \left(\left(\int_{-\infty}^{0} A(\theta)\, x(t+\theta)\, \mathrm{d}\theta + f(s,x(s),u(s))\right)\right) \mathrm{d}s\Big] + \\ &+ \int_{t_0}^{t_1} X(t,s) \int_{-\infty}^{0} A(\theta)\, x(t+\theta)\, \mathrm{d}\theta \mathrm{d}s + \int_{t_0}^{t_1} X(t,s)\, f(s,x_s,u(s))\, \mathrm{d}s = 0. \end{aligned}$$

We now show that $u : [t_0, t_1] \to IU$ is in the arbitrary compact constraint subset of R^m, that is $|u| \le a_1$, for some constant $a_1 > 0$. By (ii), $|Y^*(t_1, t_0)\, W^{-1}(t_0, t_1| \le k_1$ for some $k_1 > 0$, and $|T(t_1, t_0)\, \phi(0)| \le k_2 \exp(-a(t_1 - t_0))$ for some $k_2 > 0$. Hence,

$$|u(t)| \le k_1 \left[k_2 \exp(-a(t_1 - t_0))\right] \int_{t_0}^{t_1} k_3 \exp\left[-a(t_1 - s) \exp(-bs)\, \pi(x(\cdot,\, u(\cdot))\, \mathrm{d}s\right].$$

Thus

$$|u(t)| \le k_1 \left[k_2 \exp(-a(t_1 - t_0))\right] + k\, k_3 \exp(-at_1) \tag{18}$$

since $b - a \ge 0$ and $s \ge t_0 \ge 0$. Hence, by taking t_1 sufficiently large, we have $|u(t)| \le a_1,\ t \in [t_0, t_1]$, showing that u is an admissible control. We now prove the existence of a solution pair of the integral equations (16) and (17).

Let B be the Banach space of all function (x, u): $[t_0 - h, t_1] \times [t_0 - h, t_1] \to R^n \times R^m$ where $x \in B([t_0 - h, t_1], R^n)$; $u \in L_2(t_0 - h, t_1], R^m)$ with the norm defined by $||(x, u)|| = ||x||_2 + ||u||_2$, where

$$||x||_2 = \left\{\int_{t_0-h}^{t_1} |x(s)|^2\, \mathrm{d}s\right\}^{\frac{1}{2}}; \quad ||u||_2 = \left\{\int_{t_0-h}^{t_1} |u(s)|^2\, \mathrm{d}s\right\}^{\frac{1}{2}}.$$

Define the operator $T : B \to B$ by $T(x, u) = (y, v)$, where

$$\begin{aligned} v(t) \ &= -Y^*(t_1, t)\, W^{-1}(t_0, t_1) \Bigg[T(t_1, t_2)\, \phi(0) + \int_{t_0}^{t} \int_{-\gamma}^{0} X(t, s)\, A(\theta)\, x(t + \theta)\, \mathrm{d}\theta \mathrm{d}s + \\ &+ \int_{t_0}^{t_1} X(t, s)\, f(s, x_s, u(s))\, \mathrm{d}s\Bigg] \quad \text{for } t \in J = [t_0, t_1] \end{aligned} \tag{19}$$

and $v(t) = w(t)$ for $t \in [t_0 - \gamma, t_0]$.

$$\begin{aligned} y(t) \ &= T(t, t_0)\, \phi(0) + \int_{t_0}^{t} Y(t, s)\, v(s)\, \mathrm{d}s + \int_{t_0}^{t} \int_{-\gamma}^{0} X(t, s)\, A(\theta)\, x(t + \theta)\, \mathrm{d}\theta \mathrm{d}s + \\ &+ \int_{t_0}^{t} X(t, s)\, f(s, x_s, u(s))\, \mathrm{d}s \quad \text{for } t \in J \end{aligned} \tag{20}$$

and $y(t) = \phi(t)$ for $t \in [t_0 - \gamma, t_0]$.
We have already shown that $|v(t)| \le a_1,\ t \in J$ and also $v : [t_0 - h, t_0] \to IU$, we have $v(t) \le a_1$. Hence, $||v(t)||_2 \le a_1(t_1 + h - t_0)^{\frac{1}{2}} = b_0$. Again

$$|y(t)| \le k_2 \exp\left[-a(t - t_0)\right] + k_4 \int_{t_0}^{t} |v(s)|\, \mathrm{d}s + k\, k_3 \exp(-at_1)$$

$k_4 = \sup |Y(t, s)|$. Since $a > 0,\ t \ge t_0 \ge 0$, we deduce that

$$|y(t) \le k_2 + k_4\, a(t_1 - t_0) + k\, k_3 = b_1, \quad t \in J$$

and $|y(t)| \le \sup |\phi(t)| = d,\ t \in [t_0 - \gamma, t_0]$.
Hence, if $\lambda = \max[b_1, d]$, then $||y||_2 \le \lambda(t_1 + h - t_0)^{\frac{1}{2}} = b_2 < \infty$. Let $r = \max\{b_0, b_2\}$.

Then if we let $Q(r) = \{(x,u) \in B : \|x\|_2 \leq r,\ \|u\|_2 \leq r\}$, we have thus shown that $T : Q(r) \to Q(r)$. Since $Q(r)$ is closed, bounded and convex, by Riesz theorem [14, p. 297], it is relatively compact under the transformation T. Hence, the Schauder's fixed point theorem implies that T has a fixed point. Hence, system (11) is Euclidean null controllable.

4. STABILITY RESULTS

Consider the n-dimensional neutral functional differential equation

$$\dot{x}(t) = C\,\dot{x}(t-r) + \int_{-\infty}^{0} [\mathrm{d}\eta(\theta)]\,x(t+\theta) \tag{21}$$

where the discrete delay r is positive;

$$\int_{-\infty}^{0} \mathrm{e}^{-w\theta}[\mathrm{d}\eta_0(\theta)] < +\infty, \quad j,\, k = 1, \ldots, n$$

for some scalar $w > 0$; and c is a constant $n \times n$ matrix. Let $\rho_1 \geq \cdots \geq \rho_r \geq 0$ and $\sigma_1 \geq \cdots \geq \sigma_s = 0$ denote the non-negative real zeros of R and S respectively, where

$$R(w) = \operatorname{Re}\underline{D}(iw); \quad S(w) = \operatorname{Im}\underline{D}(iw)$$

and

$$\underline{D}(\lambda) = \det(\lambda I - \lambda \mathrm{e}^{-\lambda r} C - \int_{-\infty}^{0} \mathrm{e}^{\lambda\theta}\,\mathrm{d}\eta_0(\theta))$$

is the characteristic function of equation (21).

Definition. Let C be an $n \times n$ constant matrix. The coefficient C_k $(k = 1, \ldots, n)$ in the polynomial

$$\det(\lambda I + C) = \sum_{k=0}^{n} C_k\,\lambda^{n-k}$$

is called the kth scalar invariant of C; e. g.

$$C_0 = 1, \quad C_1 = T_r\ C, \ldots,\ C_n = \det C.$$

Proposition 1. The characteristic function of (21) is stable if

$$\sum_{k=1}^{n} |c_k| < 1,$$

and the following conditions hold

We now show that $u : [t_0, t_1] \to IU$ is in the arbitrary compact constraint subset of R^m, that is $|u| \leq a_1$, for some constant $a_1 > 0$. By (ii), $|Y^*(t_1, t_0) W^{-1}(t_0, t_1| \leq k_1$ for some $k_1 > 0$, and $|T(t_1, t_0)\, \phi(0)| \leq k_2 \exp(-a(t_1 - t_0))$ for some $k_2 > 0$. Hence,

$$|u(t)| \leq k_1 \left[k_2 \exp(-a(t_1 - t_0))\right] \int_{t_0}^{t_1} k_3 \exp\left[-a(t_1 - s) \exp(-bs)\, \pi(x(\cdot, u(\cdot))\, \mathrm{d}s\right].$$

Thus

$$|u(t)| \leq k_1 \left[k_2 \exp(-a(t_1 - t_0))\right] + k\, k_3 \exp(-a t_1) \tag{18}$$

since $b - a \geq 0$ and $s \geq t_0 \geq 0$. Hence, by taking t_1 sufficiently large, we have $|u(t)| \leq a_1$, $t \in [t_0, t_1]$, showing that u is an admissible control. We now prove the existence of a solution pair of the integral equations (16) and (17).

Let B be the Banach space of all function (x, u): $[t_0 - h, t_1] \times [t_0 - h, t_1] \to R^n \times R^m$ where $x \in B([t_0 - h, t_1], R^n)$; $u \in L_2(t_0 - h, t_1], R^m)$ with the norm defined by $\|(x, u)\| = \|x\|_2 + \|u\|_2$, where

$$\|x\|_2 = \left\{\int_{t_0-h}^{t_1} |x(s)|^2\, \mathrm{d}s\right\}^{\frac{1}{2}}; \quad \|u\|_2 = \left\{\int_{t_0-h}^{t_1} |u(s)|^2\, \mathrm{d}s\right\}^{\frac{1}{2}}.$$

Define the operator $T : B \to B$ by $T(x, u) = (y, v)$, where

$$\begin{aligned} v(t) &= -Y^*(t_1, t)\, W^{-1}(t_0, t_1) \left[T(t_1, t_2)\, \phi(0) + \int_{t_0}^{t} \int_{-\gamma}^{0} X(t, s)\, A(\theta)\, x(t + \theta)\, \mathrm{d}\theta \mathrm{d}s + \right. \\ &\quad \left. + \int_{t_0}^{t_1} X(t, s)\, f(s, x_s, u(s))\, \mathrm{d}s\right] \quad \text{for } t \in J = [t_0, t_1] \end{aligned} \tag{19}$$

and $v(t) = w(t)$ for $t \in [t_0 - \gamma, t_0]$.

$$\begin{aligned} y(t) &= T(t, t_0)\, \phi(0) + \int_{t_0}^{t} Y(t, s)\, v(s)\, \mathrm{d}s + \int_{t_0}^{t} \int_{-\gamma}^{0} X(t, s)\, A(\theta)\, x(t + \theta)\, \mathrm{d}\theta \mathrm{d}s + \\ &\quad + \int_{t_0}^{t} X(t, s)\, f(s, x_s, u(s))\, \mathrm{d}s \quad \text{for } t \in J \end{aligned} \tag{20}$$

and $y(t) = \phi(t)$ for $t \in [t_0 - \gamma, t_0]$.
We have already shown that $|v(t)| \leq a_1$, $t \in J$ and also $v : [t_0 - h, t_0] \to IU$, we have $v(t) \leq a_1$. Hence, $\|v(t)\|_2 \leq a_1 (t_1 + h - t_0)^{\frac{1}{2}} = b_0$. Again

$$|y(t)| \leq k_2 \exp\left[-a(t - t_0)\right] + k_4 \int_{t_0}^{t} |v(s)|\, \mathrm{d}s + k\, k_3 \exp(-a t_1)$$

$k_4 = \sup |Y(t, s)|$. Since $a > 0$, $t \geq t_0 \geq 0$, we deduce that

$$|y(t) \leq k_2 + k_4\, a(t_1 - t_0) + k\, k_3 = b_1, \quad t \in J$$

and $|y(t)| \leq \sup |\phi(t)| = d$, $t \in [t_0 - \gamma, t_0]$.
Hence, if $\lambda = \max[b_1, d]$, then $\|y\|_2 \leq \lambda(t_1 + h - t_0)^{\frac{1}{2}} = b_2 < \infty$. Let $r = \max\{b_0, b_2\}$.

Then if we let $Q(r) = \{(x,u) \in B : \|x\|_2 \leq r,\ \|u\|_2 \leq r\}$, we have thus shown that $T : Q(r) \to Q(r)$. Since $Q(r)$ is closed, bounded and convex, by Riesz theorem [14, p. 297], it is relatively compact under the transformation T. Hence, the Schauder's fixed point theorem implies that T has a fixed point. Hence, system (11) is Euclidean null controllable.

4. STABILITY RESULTS

Consider the n-dimensional neutral functional differential equation

$$\dot{x}(t) = C\,\dot{x}(t-r) + \int_{-\infty}^{0} [\mathrm{d}\eta(\theta)]\, x(t+\theta) \tag{21}$$

where the discrete delay r is positive;

$$\int_{-\infty}^{0} \mathrm{e}^{-w\theta}[\mathrm{d}\eta_0(\theta)] < +\infty, \quad j,\, k = 1,\ldots,n$$

for some scalar $w > 0$; and c is a constant $n \times n$ matrix. Let $\rho_1 \geq \cdots \geq \rho_r \geq 0$ and $\sigma_1 \geq \cdots \geq \sigma_s = 0$ denote the non-negative real zeros of R and S respectively, where

$$R(w) = \operatorname{Re} \underline{D}(iw); \quad S(w) = \operatorname{Im} \underline{D}(iw)$$

and

$$\underline{D}(\lambda) = \det(\lambda I - \lambda \mathrm{e}^{-\lambda r} C - \int_{-\infty}^{0} \mathrm{e}^{\lambda\theta}\, \mathrm{d}\eta_0(\theta))$$

is the characteristic function of equation (21).

Definition. Let C be an $n \times n$ constant matrix. The coefficient C_k $(k = 1,\ldots,n)$ in the polynomial

$$\det(\lambda I + C) = \sum_{k=0}^{n} C_k\, \lambda^{n-k}$$

is called the kth scalar invariant of C; e. g.

$$C_0 = 1, \quad C_1 = T_r\, C, \ldots,\ C_n = \det C.$$

Proposition 1. The characteristic function of (21) is stable if

$$\sum_{k=1}^{n} |c_k| < 1,$$

and the following conditions hold

$$n = 2m$$

$$S(\rho_k) \neq 0, \quad k = 1, \ldots, r \tag{22}$$

$$\sum_{k=1}^{r} (-1)^k \operatorname{sgn} S(\rho_k) = (-1)^m\, m \tag{22a}$$

or

$$n = 2m + 1$$

$$R(\sigma_k) \neq 0; \quad k = 1, \ldots, s-1 \tag{23a}$$

$$R(0) > 0 \tag{23b}$$

$$\sum_{k=1}^{s-1} (-1)^k \operatorname{sgn} R(\sigma_k) + \tfrac{1}{2}\left((-1)^s + (-1)^m\right) + (-1)^m\, m = 0 \tag{23c}$$

where m is an integer.

Remark 1. This is Theorem 2.25 of [17].

Remark 2. Consider the following scalar neutral functional differential equation

$$\sum_{j=0}^{m} a_j \frac{\mathrm{d}^j}{\mathrm{d}t^j}\, x(t) = b\, \frac{\mathrm{d}^n}{\mathrm{d}t^n}\, x(t-r) + \sum_{i=0}^{n-1} \int_{-\infty}^{0} x(\theta)\, \mathrm{d}\eta_1(\theta) \tag{24}$$

where $x \in R$, $\gamma > 0$, $\eta > 0$, $a_n = 1$, the scalar η_ℓ $(\ell = 0, \ell, \ldots, n-1)$ are functions of bounded variation which satisfy the condition

$$\int_{-\infty}^{0} \mathrm{e}^{-w\theta} |\mathrm{d}\, \eta_{jk}(\theta)| < \infty \tag{25}$$

$j = 1, \ldots, n$; for some $w > 0$.

If we transform equation (24) into an n-dimensional system of first order neutral functional differential equation (21), then the matrix C in equation (21) has the form

$$C = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & & \\ 0 & \cdots & b \end{pmatrix} .$$

which means that the eigenvalues of C are 0 with multiplicity $(n-1)$, and b. Because of this and the fact that exponential stability is implied by conditions (22) or (23) and

$$\sum_{k=1}^{n} |c_k| = |\mathrm{T}_r\, C| = |b| < 1,$$

we infer that the zero solution of system (24) is exponentially asymptotically stable if and only if the conditions of Proposition 1 holds.

5. EXAMPLE

Consider the system

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left(x(t) - A_{-1}\, x(t-h)\right) = \qquad (26)\\ = \quad & A_0\, x(t) + A_1\, x(t-h) + B\, u(t) + \mathrm{e}^{-at}\, \mathrm{Sin}\left(x(t) + x(t-h)\right)\cos u\end{aligned}$$

where $A_i,\ i = 0, \pm 1$ are constant matrices given by

$$A_{-1} = \begin{pmatrix} 0 & \frac{1}{4} \\ \frac{1}{4} & 0 \end{pmatrix}, \quad A_0 = \begin{pmatrix} -3 & 1 \\ 3 & -4 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0 & \frac{3}{4} \\ 0 & -\frac{1}{4} \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$
$$f = \begin{pmatrix} 0 \\ \mathrm{e}^{-at}\sin\left(x(t) + x(t-h)\right)\cos u(t) \end{pmatrix}, \quad a > 0.$$

The author [16] has shown that the free system

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(x(t) - A_{-1}\, x(t-h)\right) = A_0\, x(t) + A_1\, x(t-h), \dots \qquad (27)$$

where A_i is given as above, is uniformly asymptotically stable. We now show that the linear base system

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(x(t) - A_{-1} x(t-h)\right) = A_0\, x(t) + A_1\, x(t-h) + B\, u(t) \qquad (28)$$

is proper. By Proposition 3, we show that rank $[B, A_0 B] = n$. But rank $[B, A_0 B] =$ rank $\begin{pmatrix} 0 & 1 \\ 1 & -4 \end{pmatrix} =$ rank $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = 2 = n$. Therefore the system (23) is proper. Moreover,

$$|f(t, x(t),\, x(t-h),\, u(t))| = \left|\mathrm{e}^{-at}\sin(x(t) + x(t-h))\,\cos u(t)\right| \le \mathrm{e}^{-at}\cdot 1.$$

Hence by Theorem 1, system (21) is null controllable.

6. CONCLUSION

Sufficient conditions for the controllability of perturbed nonlinear systems with infinite delays have been derived. These conditions are given with respect to the stability of the free linear base system and the controllability of the linear controllable base system, with the assumption that the perturbation f satisfies some smoothness and growth conditions. Computable criteria for all these are reported and an example is given. These results extended known results in the literature.

(Received June 21, 1991.)

REFERENCES

[1] K. J. Arrow: Production and Capital. Collected papers of Kenneth Arrow, the Beknop Prec. of Harvard University Press, Cambridge, Mass. 1985.

[2] R. Brayton: Nonlinear oscillations in a distributed network. Quart. Appl. Math. *24* (1976), 239–301.

[3] E. N. Chukwu: On the null controllability of nonlinear systems with retarded control. J. Math. Anal. Appl. *76* (1980), 283–396.

[4] E. N. Chukwu: Global asymptotic behaviour of functional differential equations of the neutral type. Nonlinear Analysis, Theory, Methods and Appl. *5* (1981), 5, 853–872.

[5] R. D. Driver: A functional differential system of neutral type in a two-body problem of classical electrodynamics. In: Nonlinear Differential Equations and Nonlinear Mechanics, Academic Press, New York 1963.

[6] L. E. Elsgoltz: Qualitative Methods in Mathematical Analysis. Trans. Math. Mono, 12, American Math. Soc. 1964.

[7] R. Gabasov and F. M. Kirrilova: The Quantitative Theory of Optimal Processes. Marcel Dekker, New York 1976.

[8] R. D. Gahl: Controllability of nonlinear systems of neutral type. J. Math. Anal. Appl. *63* (1978), 33–42.

[9] J. K. Halle and M. A. Cruz: Existence, uniqueness and continuous dependence for hederitary systems. Ann. Mat. Pura Appl. *85* (1970), 63–82.

[10] J. K. Hale: Functional differential equations with infinite delays. J. Math. Anal. Appl. *48* (1974), 276–283.

[11] J. K. Hale: Theory of Functional Differential equations. Springer–Verlag, New York 1977.

[12] H. Hermes and J. P. LaSalle: Functional Analysis and Time Optimal Control. Academic Press, New York 1969.

[13] M. Kalecki: A macrodynamic theory of business cycles. Econometrica *3* (1935), 327–344.

[14] L. V. Kantorovich and G. P. Akilov: Functional Analysis. Pergamon Press, Oxford 1982.

[15] G. Knowles: An Introduction to Applied Optimal Control. Academic Press, New York 1981.

[16] J. U. Onwuatu: On the null controllability in function-space of nonlinear system of neutral functional differential equations with limited control. J. Optim. Theory Appl. *42* (1984), 397–420.

[17] G. Stépán: Retarded Dynamical Systems: Stability and Characteristic Functions. John Wiley, New York 1989.

[18] A. S. C. Sinha: Null controllability of nonlinear infinite delay systems with restrained controls. Internat. J. Control *42* (1985), 735–741.

Dr. Jerry Uche Onwuatu, Federal University of Technology, Department of Mathematics and Computer Science, P. M. B. 1 526, Owerri. Nigeria.