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LARGE SAMPLE BEHAVIOUR OF THE t-TRANSFORMATION OF TWO-SAMPLE RANK STATISTICS

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Among important problems in the analysis of signals belongs the testing of mutual shift of two samples. This contribution is devoted to the study of properties of test procedure based on "t-transformation" of a two-sample linear rank statistic $S = \sum_{i=1}^{m} a(R_i)$, where m is a size of a first sample. The authors of the present paper specified the assumptions for this convergence and studied a behaviour of t-transformed rank statistics (Wilcoxon and median) with the help of simulation of their values under the null hypothesis. The results are presented in Tables 1–4. The numerical studies show a good fit of the t-approximation of the distribution.

1. INTRODUCTION

Let $X_1, \ldots, X_m$ and $Y_1, \ldots, Y_n$ be two independent samples from the continuous distribution function F and $F(x - \Delta)$, respectively. For testing the hypothesis $\Delta = 0$ various rank tests based on the linear statistic $S = \sum_{i=1}^{m} a(R_i)$, where $(R_1, \ldots, R_N)$ is the vector of ranks of the pooled sample, $N = m + n$, $a_1, \ldots, a_N$ are scores, are well-known. Nath and Duran [3] studied the asymptotic distribution of the "t-transform" statistic

$$T_R = \frac{\sqrt{(N-2)}\,(S - \mathsf{E}_0 S)}{[(N-1)\operatorname{var}_0 S - (S - \mathsf{E}_0 S)^2]^{1/2}}, \tag{1}$$

where $\mathsf{E}_0 S$, $\operatorname{var}_0 S$ are expectation and variance, respectively, of S under the hypothesis $\Delta = 0$:

$$\mathsf{E}_0 S = \frac{m}{N} \sum_{i=1}^{N} a_i = m\bar{a}, \tag{2}$$

$$\operatorname{var}_0 S = \frac{mn}{N(N-1)} \sum_{i=1}^{N} (a_i - \bar{a})^2 = \frac{mn}{N} \sigma_a^2. \tag{3}$$

They showed that the first four moments of the statistic

$$L = \frac{S - \mathsf{E}_0 S}{\sqrt{((N-1)\operatorname{var}_0 S)}} \tag{4}$$

coincide or converge, under certain conditions which will be stated later, to the moments of the distribution with the density

$$\begin{aligned} g(x) &= \frac{1}{B(1/2, N/2 - 1)} (1 - x^2)^{N/2-2}, \quad -1 \leqq x \leqq 1, \\ &= 0 \qquad \text{otherwise.} \end{aligned} \tag{5}$$

If a random variable X has a density (5) then $\sqrt{(N-2)}\,X/\sqrt{(1-X^2)}$ has the Student distribution with $(N-2)$ degrees of freedom. Under the assumption of symmetry

$$a_i + a_{N-i+1} = c, \quad 1 \leqq i \leqq N, \tag{6}$$

or

$$m = n = N/2, \tag{7}$$

it can be easy derived (utilizing Hájek, Šidák [2], Problems and complements to Chapter II, 24, 25, e.g.) that

$$\mathsf{E}_0(S - \mathsf{E}_0 S)^4 = \frac{m^2 n^2}{N^2(N+1)} \left(3(N-1)\,\sigma_a^4 + \left(\frac{N^2+N}{mn} - 6\right) c_{4a}\right), \tag{8}$$

where

$$c_{4a} = \frac{1}{(N-1)(N-2)(N-3)} \left[N(N+1) \sum_{i=1}^{N} (a_i - \bar{a})^4 - 3(N-1)^3 \sigma_a^4\right].$$

Then,

$$\mathsf{E}_0 L^{2k+1} = 0, \quad k = 0, 1, \ldots, \tag{9}$$

$$\operatorname{var}_0 L = \frac{1}{N-1}, \tag{10}$$

$$\mathsf{E}_0 L^4 = \frac{3}{N^2 - 1} (1 + k_{mn}), \tag{11}$$

where

$$k_{mn} = \frac{1}{3(N-1)} \left(\frac{N^2+N}{mn} - 6\right) \frac{c_{4a}}{\sigma_a^4}. \tag{12}$$

Nath and Duran [4] showed that $k_{mn} \to 0$ under the assumption that

$$\frac{\max (a_i - \bar{a})^2}{\sum (a_i - \bar{a})^2} = o\left(\frac{1}{N}\right),$$

which is stronger than the corresponding assumption for the asymptotic normality of S. Moreover, for most usual rank statistics (including Wilcoxon one) this condition is not satisfied. Following the pattern of the proof of Theorem in Appendix of Nath and Duran [4] we can see $o(1/N)$ is not necessary.

Theorem 1.1. Let $\sum_{i=1}^{N} (a_i - \bar{a})^2 > 0$,

$$\frac{\max_{1 \leq i \leq N} (a_i - \bar{a})^2}{\sum_{i=1}^{N} (a_i - \bar{a})^2} = 0(1/N^{\frac{1}{2}+\delta}), \tag{13}$$

$$\delta > 0\,, \quad \frac{m}{N} \to \lambda\,, \quad 0 < \lambda < 1\,, \quad N \to \infty\,. \tag{14}$$

Then, k_{mn} given by (12) tends to zero as $N \to \infty$.

Proof. According to (12) we can write

$$k_{mn} = \frac{1}{3(N-2)(N-3)} \left[\frac{N(N+1)\sum(a_i - \bar{a})^4}{(\sum(a_i - \bar{a})^2)^2} - 3(N-1)\right]\left(\frac{N^2 + N}{mn} - 6\right).$$

If we estimate

$$\frac{\sum(a_i - \bar{a})^4}{(\sum(a_i - \bar{a})^2)^2} \leqq \frac{N(\max(a_i - \bar{a})^2)^2}{(\sum(a_i - \bar{a})^2)^2}\,,$$

and then use the assumptions (6) and (7), we can see that $[(N^2 + N)/mn] - 6$ tends to a constant and the other part of k_{mn} tends to zero as $N \to \infty$. □

Now, we may reformulate the Nath and Duran's result as:

Theorem 1.2. Let (6) or (7) and the assumptions of Theorem 1.1 be satisfied. Then, for L given by (4), $\mathsf{E}_0 L^{2k+1}$, $\mathsf{E}_0 L^2$ coincide and $\mathsf{E}_0 L^4$ tends to the corresponding moments of the distribution with the density (5); the distribution of

$$T_R = \frac{\sqrt{(N-2)}\, L}{\sqrt{(1 - L^2)}}$$

is, for $N \to \infty$, approximately Student's t with $N - 2$ degrees of freedom.

Proof. The assertion follows from (9)–(11) and Theorem 1.1. □

2. SIMULATIONS

The convergence of k_{mn}, i.e. of the fourth moment $\mathsf{E}_0 L^4$, gives no imagination how fast the convergence of T_R in distribution is. Numerical studies of Nath and Duran [4], concerned mainly a power and efficiency of T_R tests, were based on simulated samples from different distributions (normal, uniform, double exponential and Cauchy).

In contradiction to Nath and Duran [4] we were interested in the accuracy of the approximation of the exact distribution of the statistic L, given by (4), by the distribution with the density (5), which determines the accuracy of the t-approximation of T_R given by (1). For this goal we have not simulated the samples but directly the linear rank statistics (we used Wilcoxon and median ones). For simplification of the procedure we simulated the dual form of linear statistic, i.e.

$$S = \sum_{i=1}^{N} a_i Z_i \,, \tag{15}$$

where $Z_i = 1$ if the ith order statistic $X^{(i)}$ is from the first sample and $Z_i = 0$ otherwise.

It is well known that under hypothesis $\Delta = 0$ the vector of ranks $(R_1, \ldots, R_N)$ has the uniform distribution over the space of all permutations $(1, \ldots, N)$, so that every permutation of m ones and n zeros, $n + m = N$, has the same probability. Now it is obvious that the simplest way for simulating the realizations of the rank statistic S is, in every step, to permute randomly elements of the vector $\mathbf{Z} = (Z_1, \ldots$ $\ldots, Z_N)$ of m ones and n zeros and to compute S using (15).

Below we bring the scheme of our simulations:

1) Type of the rank statistic was chosen. We used Wilcoxon and median ones.
2) Sample sizes m and n were fixed. We used all combinations of $m = 10\cdot60\,(10)$ and $n = 10\cdot60\,(10)$.
3) The vector $\mathbf{a} = (a_1, \ldots, a_N)$ of scores was computed.
4) The vector $\mathbf{Z} = (Z_1, \ldots, Z_N)$ of m ones and n zeros was prepared. The initial form of $\mathbf{Z}$ was chosen in conformity with the null hypothesis. More precisely, we put
   $\mathbf{Z} = (1, 0, 1, 0, \ldots, 1, 0)$ when $n = m$,
   $\mathbf{Z} = (1, 0, 0, 1, 0, 0, \ldots, 1, 0, 0)$ when $2m = n$ etc.
5) Number k of repetitions of the simulation was fixed. We put $k = 1000$ in all cases.
6) In every repetition of the simulation specificated during the steps 1) – 5):
   (i) value of the rank statistic S was computed (using (15)) and stored;
   (ii) vector $\mathbf{Z}$ was randomly permuted.
7) Obtained values of S were evaluated.

Remark. For the random permutations we used Durstenfeld's algorithm as it is described in Appendix.

3. RESULTS

Results of our simulations are summarized in Tables 1 – 4.

Tables 1, 2 provide the values of theoretical expectation $\mathsf{E}S$, resp. theoretical dispersion var S, and their empirical counterparts $\mathsf{E}\hat{S}$, resp. var $\hat{S}$, obtained by our simulations for Wilcoxon (Table 1) and median (Table 2) statistics. We can see that the fit is almost absolute for expectation and very good for dispersion, too.

The more detailed information is contained in Tables 3–4. We prefered to present the values of theoretical density (5) and estimates of this density based on simulated values of rank statistic to giving here the results of tests of goodness of fit between the theoretical model and empirical data. We believe that this approach gives to the reader more information about the real shape of empirical distribution of studied statistics. We shall describe Tables 3–4 in more details below.

Table 3 provides the results for the Wilcoxon statistic. Because the Wilcoxon statistic can acquire a lot of values even for small m and n, interval $[-1, +1]$ was divided into 20 subintervals $I_1, \dots, I_{20}$, $\bigcup_{i=1}^{20} I_i = [-1, +1]$, $I_i \cap I_j = \emptyset$, $i \neq j$, and we have counted number of values of statistic S which fell into different subintervals I_j, $1 \leqq j \leqq 20$, during every simulation. The density $g(x)$ was estimated by the frequency of occurrences of S in a given subinterval normed to one over all I_j, $j = 1, \dots, 20$. More precisely, Table 3 contains:

– denotation of subintervals, denoted $I_4 - I_{17}$;
– centres of these subintervals, denoted x;
– values of density $g(x)$ in points x, denoted $g(x)$;
– estimates $g(x)$ in points x, denoted $\hat{g}(x)$.

Results for subintervals $I_1 - I_3$ and $I_{18} - I_{20}$ were not included because $\hat{g}(x) = 0$ in all cases, what corresponds to the fact that for these intervals $g(x) \approx 0$.

Table 4 provides similar results for the median statistic. In contradiction to the Wilcoxon statistic, the median statistic can acquire only a few values even for large m and n. This was the reason why we did not divided $[-1, +1]$ into subintervals like in the previous case, but we estimated $g(x)$ in all points which could be acquired by the median statistic for given m and n. Again, the density $g(x)$ was estimated by the frequency of occurrence of S in these points normed to one.

More precisely, Table 4 contains:

– values which can be acquired by median statistic, denoted x;
– values of density $g(x)$ in points x, denoted $g(x)$;
– estimates $g(x)$ in points x, denoted $\hat{g}(x)$.

As in the previous case we omitted those values of x for which $g(x) \approx 0$.

4. CONCLUSIONS

Results of our simulations show a surprisingly good fit between the empirical distribution for both studied statistics and the theoretical model. The fact, that the empirical density is slightly flatter than the theoretical one corresponds to the fact that k_{mn} is positive.

According to our numerical studies t-transformation of rank statistics seems to be

Table 1. Wilcoxon statistic.

m	n	N	ES	E$\hat{S}$	var S	var $\hat{S}$
10	10	20	105	105·0	175	177·4
10	20	30	155	155·2	516·$\bar{6}$	534·0
20	20	40	410	408·8	1 366·$\bar{6}$	1 403·3
10	30	40	205	202·6	1 025	1 020·8
10	40	50	255	257·1	1 700	1 746·2
20	30	50	510	508·4	2 550	2 550·5
30	30	60	915	913·4	4 575	4 587·2
20	40	60	610	606·9	4 066·$\bar{6}$	4 162·7
10	50	60	305	305·0	2 541·$\bar{6}$	2·665·8
30	40	70	1 065	1 061·8	7 100	6 900·7
10	60	70	355	353·1	3 550	3 752·5
20	50	70	710	708·8	5 916·$\bar{6}$	6 048·9
40	40	80	1 620	1 620·6	10 800	11 007·7
30	50	80	1 215	1 220·2	10 125	10 263·9
20	60	80	810	808·7	8 100	8 021·8
40	50	90	1 820	1 815·1	15 166·$\bar{6}$	14 288·6
30	60	90	1 365	1 362·9	13 650	12 931·1
50	50	100	2 525	2 522·9	21 041·$\bar{6}$	21 564·9
40	60	100	2 020	2 019·0	20 200	21 375·0
50	60	110	2 775	2 773·1	27 750	27 747·5
60	60	120	3 630	3 632·1	36 300	33 057·8

Table 2. Median statistic.

m	n	N	ES	E$\hat{S}$	var S	var $\hat{S}$
10	10	20	5	5·01	1·32	1·32
10	20	30	5	4·91	1·72	1·73
20	20	40	10	9·93	2·56	2·59
10	30	40	5	5·01	1·92	1·80
10	40	50	5	4·96	2·04	2·07
20	30	50	10	10·0	3·06	2·99
30	30	60	15	15·0	3·81	3·54
20	40	60	10	10·1	3·39	3·52
10	50	60	5	5·09	2·12	2·03
30	40	70	15	14·93	4·35	4·12
10	60	70	5	4·95	2·17	2·20
20	50	70	10	9·95	3·62	3·36
40	40	80	20	20·09	5·06	5·11
30	50	80	15	15·06	4·75	4·29
20	60	80	10	10·11	3·80	3·58
40	50	90	20	19·99	5·62	5·81
30	60	90	15	15·03	5·06	4·79
50	50	100	25	24·95	6·31	6·19
40	60	100	20	20·12	6·06	6·18
50	60	110	25	25·02	6·88	7·42
60	60	120	30	30·14	7·56	7·59

Table 3. Wilcoxon statistic.

		I_4	I_5	I_6	I_7	I_8	I_9	I_{10}	I_{11}	I_{12}	I_{13}	I_{14}	I_{15}	I_{16}	I_{17}
	x	−0·65	−0·55	−0·45	−0·35	−0·25	−0·15	−0·05	0·05	0·15	0·25	0·35	0·45	0·55	0·65
$N = 20$	$g(x)$	0·05	0·19	0·55	1·17	1·99	2·78	3·27	3·27	2·78	1·99	1·17	0·55	0·19	0·05
$m = 10$, $n = 10$	$\hat{g}(x)$	0·04	0·22	0·46	1·26	2·00	2·80	3·12	3·18	2·70	2·26	1·16	0·52	0·22	0·06
$N = 30$	$g(x)$	0·00	0·04	0·22	0·77	1·81	3·11	4·05	40·5	3·11	1·81	0·77	0·22	0·04	0·00
$m = 10$, $n = 20$	$\hat{g}(x)$	0·00	0·04	0·34	0·68	1·78	3·06	3·96	3·98	3·02	2·00	0·86	0·22	0·06	0·00
$N = 40$	$g(x)$	0·00	0·01	0·08	0·47	1·53	3·24	4·67	4·67	3·24	1·53	0·47	0·08	0·01	0·00
$m = 20$, $n = 20$	$\hat{g}(x)$	0·00	0·00	0·08	0·64	1·52	3·54	4·56	4·46	2·88	1·80	0·42	0·10	0·00	0·00
$m = 10$, $n = 30$	$\hat{g}(x)$	0·00	0·00	0·08	0·62	1·66	3·78	4·46	4·52	2·92	1·40	0·44	0·12	0·00	0·00
$N = 50$	$g(x)$	0·00	0·00	0·03	0·27	1·25	3·26	5·19	5·19	3·26	1·25	0·27	0·03	0·00	0·00
$m = 10$, $n = 40$	$\hat{g}(x)$	0·00	0·00	0·04	0·20	1·30	3·52	4·54	4·80	3·52	1·70	0·38	0·00	0·00	0·00
$m = 20$, $n = 30$	$\hat{g}(x)$	0·00	0·00	0·00	0·28	1·36	3·86	4·74	4·68	3·66	1·06	0·36	0·00	0·00	0·00
$N = 60$	$g(x)$	0·00	0·00	0·01	0·16	0·99	3·20	5·64	5·64	3·20	0·99	0·16	0·01	0·00	0·00
$m = 30$, $n = 30$	$\hat{g}(x)$	0·00	0·00	0·02	0·12	1·20	3·30	5·62	5·20	3·24	1·14	0·16	0·00	0·00	0·00
$m = 20$, $n = 40$	$\hat{g}(x)$	0·00	0·00	0·02	0·26	1·08	3·40	5·60	5·58	2·90	0·88	0·26	0·02	0·00	0·00
$m = 10$, $n = 50$	$\hat{g}(x)$	0·00	0·00	0·02	0·22	1·08	3·20	5·34	5·60	3·30	1·06	0·18	0·00	0·00	0·00

$N = 70$	$g(x)$	0·00	0·00	0·00	0·09	0·78	3·09	6·04	6·04	3·09	0·78	0·09	0·00	0·00	0·00
$m = 30$, $n = 40$	$\hat{g}(x)$	0·00	0·00	0·02	0·12	0·86	3·12	6·10	5·86	3·02	0·80	0·10	0·00	0·00	0·00
$m = 10$, $n = 60$	$\hat{g}(x)$	0·00	0·00	0·00	0·12	0·06	3·24	5·74	5·76	3·08	0·84	0·16	0·00	0·00	0·00
$m = 20$, $n = 50$	$\hat{g}(x)$	0·00	0·00	0·00	0·14	0·98	3·06	5·96	6·08	2·72	0·86	0·20	0·00	0·00	0·00
$N = 80$	$g(x)$	0·00	0·00	0·00	0·05	0·60	2·96	6·39	6·39	2·96	0·60	0·05	0·00	0·00	0·00
$m = 40$, $n = 40$	$\hat{g}(x)$	0·00	0·00	0·00	0·06	0·72	3·30	5·94	6·08	3·08	0·74	0·08	0·00	0·00	0·00
$m = 30$, $n = 50$	$\hat{g}(x)$	0·00	0·00	0·00	0·06	0·62	2·70	6·26	6·42	3·04	0·80	0·10	0·00	0·00	0·00
$m = 20$, $n = 60$	$\hat{g}(x)$	0·00	0·00	0·00	0·10	0·74	2·94	6·28	6·34	2·82	0·72	0·06	0·00	0·00	0·00
$N = 90$	$g(x)$	0·00	0·00	0·00	0·02	0·47	2·81	6·70	6·70	2·81	0·47	0·02	0·00	0·00	0·00
$m = 40$, $n = 50$	$\hat{g}(x)$	0·00	0·00	0·00	0·02	0·64	2·68	6·80	6·66	2·62	0·56	0·02	0·00	0·00	0·00
$m = 30$, $n = 60$	$\hat{g}(x)$	0·00	0·00	0·00	0·00	0·50	3·00	6·58	6·80	2·72	0·36	0·04	0·00	0·00	0·00
$N = 100$	$g(x)$	0·00	0·00	0·00	0·01	0·36	2·64	6·99	6·99	2·64	0·36	0·01	0·00	0·00	0·00
$m = 50$, $n = 50$	$\hat{g}(x)$	0·00	0·00	0·00	0·02	0·56	2·80	6·64	6·80	2·70	0·46	0·02	0·00	0·00	0·00
$m = 40$, $n = 60$	$\hat{g}(x)$	0·00	0·00	0·00	0·02	0·46	2·84	6·68	6·54	3·00	0·42	0·04	0·00	0·00	0·00
$N = 110$	$g(x)$	0·00	0·00	0·00	0·01	0·27	2·48	7·24	7·24	2·48	0·27	0·01	0·00	0·00	0·00
$m = 50$, $n = 60$	$\hat{g}(x)$	0·00	0·00	0·00	0·00	0·24	2·70	7·06	7·30	2·46	0·22	0·02	0·00	0·00	0·00
$N = 120$	$g(x)$	0·00	0·00	0·00	0·00	0·21	2·31	7·48	7·48	2·31	0·21	0·00	0·00	0·00	0·00
$m = 60$, $n = 60$	$\hat{g}(x)$	0·00	0·00	0·00	0·02	0·28	2·08	7·54	7·48	2·44	0·14	0·02	0·00	0·00	0·00

Table 4. Median statistic.

$m = 10$	x	−1·00	−0·80	−0·60	−0·40	−0·20	0	0·20	0·40	0·60	0·80	1·00		
$n = 10$	$g(x)$	0·00	0·00	0·09	0·83	2·41	3·34	2·41	0·83	0·09	0·00	0·00		
$N = 20$	$\hat{g}(x)$	0·00	0·00	0·12	0·81	2·24	3·50	2·41	0·83	0·09	0·00	0·00		
$m = 10$	x	−0·71	−0·57	−0·42	−0·28	−0·14	0	0·14	0·28	0·42	0·57	0·71		
$n = 20$	$g(x)$	0·00	0·03	0·32	1·42	3·22	4·18	3·22	1·42	0·32	0·03	0·00		
$N = 30$	$\hat{g}(x)$	0·00	0·04	0·48	1·33	3·44	4·20	3·13	1·27	0·23	0·03	0·00		
$m = 20$	x	−0·50	−0·40	−0·30	−0·20	−0·10	0	0·10	0·20	0·30	0·40	0·50		
$n = 20$	$g(x)$	0·03	0·21	0·89	2·34	4·08	4·89	4·08	2·34	0·89	0·21	0·03		
$N = 40$	$\hat{g}(x)$	0·02	0·22	0·88	2·62	4·30	4·94	3·58	2·40	0·80	0·22	0·02		
$m = 10$	x	−0·58	−0·46	−0·35	−0·23	−0·12	0	0·12	0·23	0·35	0·46	0·58		
$n = 30$	$g(x)$	0·00	0·07	0·49	1·82	3·84	4·89	3·84	1·82	0·49	0·07	0·00		
$N = 40$	$\hat{g}(x)$	0·00	0·14	0·36	1·51	4·07	5·09	3·85	1·84	0·43	0·04	0·00		
$m = 10$	x	−0·50	−0·40	−0·30	−0·20	−0·10	0	0·10	0·20	0·30	0·40	0·50		
$n = 40$	$g(x)$	0·01	0·10	0·63	2·15	4·36	5·50	4·36	2·15	0·63	0·10	0·01		
$N = 50$	$\hat{g}(x)$	0·00	0·12	0·78	2·30	3·84	6·02	4·28	1·90	0·66	0·10	0·00		
$m = 20$	x	−0·41	−0·33	−0·24	−0·16	−0·08	0	0·08	0·16	0·24	0·33	0·41		
$n = 30$	$g(x)$	0·08	0·41	1·33	2·95	4·72	5·50	4·72	2·95	1·33	0·41	0·08		
$N = 50$	$\hat{g}(x)$	0·05	0·07	1·69	2·74	4·61	5·81	4·85	2·99	1·64	0·02	0·02		
$m = 30$	x	−0·40	−0·33	−0·27	−0·20	−0·13	−0·07	0	0·07	0·13	0·20	0·27	0·33	0·40
$n = 30$	$g(x)$	0·05	0·22	0·77	1·93	3·66	5·34	6·05	5·34	3·66	1·93	0·77	0·22	0·05
$N = 60$	$\hat{g}(x)$	0·00	0·09	0·60	1·89	3·75	5·79	6·12	4·86	3·87	2·10	0·72	0·15	0·06
$m = 20$	x	−0·42	−0·35	−0·28	−0·21	−0·14	−0·07	0	0·07	0·14	0·21	0·28	0·35	0·42
$n = 40$	$g(x)$	0·03	0·14	0·59	1·67	3·44	5·26	6·05	5·26	3·44	1·67	0·59	0·14	0·03
$N = 60$	$\hat{g}(x)$	0·03	0·14	0·42	1·67	2·94	5·40	5·74	5·18	3·79	2·04	0·68	0·20	0·00
$m = 10$	x	−0·54	−0·45	−0·36	−0·27	−0·18	−0·09	0	0·09	0·18	0·27	0·36	0·45	0·54
$n = 50$	$g(x)$	0·00	0·01	0·13	0·75	2·43	4·83	6·05	4·83	2·43	0·75	0·13	0·01	0·00
$N = 60$	$\hat{g}(x)$	0·00	0·00	0·11	0·63	1·95	4·94	6·28	4·92	2·59	0·74	0·18	0·00	0·00
$m = 30$	x	−0·35	−0·29	−0·23	−0·17	−0·12	−0·06	0	0·06	0·12	0·17	0·23	0·29	0·35
$n = 40$	$g(x)$	0·10	0·37	1·07	2·40	4·21	5·87	6·56	5·87	4·21	2·40	1·07	0·37	0·10
$N = 70$	$\hat{g}(x)$	0·04	0·62	1·14	1·97	4·64	5·33	7·21	6·37	4·12	1·97	0·62	0·52	0·07

$m = 10$	x	−0·49	−0·41	−0·33	−0·24	−0·16	−0·08	0	0·08	0·16	0·24	0·33	0·41	0·49
$n = 60$	$g(x)$	0·00	0·04	0·16	0·85	2·69	5·26	6·56	5·26	2·69	0·85	0·16	0·04	0·00
$N = 70$	$\hat{g}(x)$	0·00	0·00	0·20	0·76	3·33	4·85	6·66	4·87	2·82	0·93	0·05	0·00	0·00
$m = 20$	x	−0·38	−0·32	−0·25	−0·19	−0·13	−0·06	0	0·06	0·13	0·19	0·25	0·32	0·38
$n = 50$	$g(x)$	0·04	0·20	0·74	1·95	3·85	5·74	6·56	5·74	3·85	1·95	0·74	0·20	0·04
$N = 70$	$\hat{g}(x)$	0·03	0·22	0·79	1·87	3·64	5·72	7·65	5·34	3·92	1·71	0·51	0·22	0·03
$m = 40$	x	−0·35	−0·30	−0·25	−0·20	−0·15	−0·10	−0·05	0	0·05	0·10	0·15	0·20	0·25	0·30	0·35
$n = 40$	$g(x)$	0·05	0·20	0·60	1·49	2·96	4·79	6·39	7·02	6·39	4·79	2·96	1·49	0·60	0·20	0·05
$N = 80$	$\hat{g}(x)$	0·04	0·20	0·48	1·36	2·96	4·92	6·08	6·92	6·04	5·36	3·04	1·72	0·68	0·20	0·04
$m = 30$	x	−0·36	−0·31	−0·26	−0·21	−0·15	−0·10	−0·05	0	0·05	0·10	0·15	0·21	0·26	0·31	0·36
$n = 50$	$g(x)$	0·04	0·15	0·51	1·34	2·79	4·67	6·35	7·02	6·35	4·67	2·79	1·34	0·51	0·15	0·04
$N = 80$	$\hat{g}(x)$	0·00	0·08	0·23	1·24	2·33	4·57	7·20	7·05	7·17	3·95	2·83	1·32	0·46	0·19	0·04
$m = 20$	x	−0·40	−0·35	−0·29	−0·23	−0·17	−0·12	−0·06	0	0·06	0·12	0·17	0·23	0·29	0·35	0·40
$n = 60$	$g(x)$	0·01	0·05	0·26	0·88	2·21	4·22	6·19	7·02	6·19	4·22	2·21	0·88	0·25	0·05	0·01
$N = 80$	$\hat{g}(x)$	0·00	0·03	0·24	0·87	1·45	4·33	5·96	7·52	6·37	4·47	2·25	0·76	0·31	0·10	0·00
$m = 40$	x	−0·31	−0·27	−0·22	−0·18	−0·13	−0·09	−0·04	0	0·04	0·09	0·13	0·18	0·22	0·27	0·31
$n = 50$	$g(x)$	0·09	0·30	0·82	1·84	3·24	5·28	6·85	7·46	6·85	5·28	3·42	1·84	0·82	0·30	0·09
$N = 90$	$\hat{g}(x)$	0·18	0·22	1·03	1·65	2·91	5·63	7·83	6·84	6·40	5·95	2·77	1·79	0·94	0·27	0·22
$m = 30$	x	−0·33	−0·28	−0·24	−0·19	−0·14	−0·09	−0·05	0	0·05	0·09	0·14	0·19	0·24	0·28	0·33
$n = 60$	$g(x)$	0·05	0·21	0·64	1·57	3·14	5·08	6·78	7·46	6·78	5·08	3·13	1·57	0·64	0·21	0·05
$N = 90$	$\hat{g}(x)$	0·13	0·13	0·55	1·48	2·88	5·18	6·28	7·68	7·72	5·13	3·10	1·44	0·51	0·08	0·04
$m = 50$	x	−0·28	−0·24	−0·20	−0·16	−0·12	−0·08	−0·04	0	0·04	0·08	0·12	0·16	0·20	0·24	0·28
$N = 50$	$g(x)$	0·16	0·46	1·11	2·27	3·93	5·79	7·30	7·88	7·30	5·79	3·93	2·27	1·11	0·46	0·16
$N = 100$	$\hat{g}(x)$	0·10	0·45	1·25	2·55	3·95	5·70	7·80	7·05	7·55	5·80	4·10	2·10	1·10	0·30	0·15
$m = 40$	x	−0·29	−0·24	−0·20	−0·16	−0·12	−0·08	−0·04	0	0·04	0·08	0·12	0·16	0·20	0·24	0·29
$n = 60$	$g(x)$	0·13	0·40	1·02	2·15	3·81	5·71	7·27	7·88	7·27	5·71	3·81	2·15	1·02	0·40	0·13
$N = 100$	$\hat{g}(x)$	0·13	0·20	1·13	1·91	3·81	5·71	7·30	7·20	7·94	5·14	4·26	2·40	1·27	0·49	0·13
$m = 50$	x	−0·26	−0·22	−0·18	−0·15	−0·11	−0·07	−0·04	0	0·04	0·07	0·11	0·15	0·18	0·22	0·26
$n = 60$	$g(x)$	0·23	0·61	1·37	2·64	4·36	6·23	7·71	8·27	7·71	6·23	4·36	2·64	1·37	0·61	0·23
$N = 100$	$\hat{g}(x)$	0·11	1·04	1·48	3·18	4·27	6·13	6·74	8·22	7·45	6·08	4·60	3·01	1·10	0·93	0·33
$m = 60$	x	−0·23	−0·20	−0·17	−0·13	−0·10	−0·07	−0·03	0	0·03	0·07	0·10	0·13	0·17	0·20	0·23
$n = 60$	$g(x)$	0·34	0·81	1·69	3·06	4·83	6·68	8·11	8·65	8·11	6·68	4·83	3·06	1·69	0·81	0·34
$N = 120$	$\hat{g}(x)$	0·34	0·66	1·69	2·58	4·44	6·18	8·58	8·82	7·56	7·32	4·80	3·66	1·80	0·84	0·36

recommendable for practical purposes. Popularity of the Student's t-test and easy accessibility of rich tables of t-distribution makes T_R-test worth of attention. This is confirmed also by the fact that in the last version of SPSS package of statistical programs presented during the conference COMPSTAT 84 t-transformation was quoted between new and recommendable procedures.

APPENDIX – Durstenfeld's algorithm

```
procedure SHUFFLE (a, n, random).
      value N; integer N; real procedure RANDOM;
      integer array A;
begin
      integer I, J; real B
      for I := N step — 1 until 2 do
         begin J := entier (I RANDOM + 1);
               B := A(I); A(I) := A(J); A(J) := B;
         end loop i
end SHUFFLE
```

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