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ISOTROPY OF STATIONARY RANDOM FIELDS ON LATTICE[1]

ANTONÍN OTÁHAL

The notion of a stationary weakly isotropic random field on an n-lattice (n-dimensional square lattice) is suggested in the paper to constitute a discrete model of stationary isotropic spatial random events. The spectral representation of a stationary weakly isotropic random field is derived and, on the base of that, the hypothesis is tested under which a stationary n-lattice field is supposed to be weakly isotropic.

0. INTRODUCTION

The definition of a stationary random field on an n-lattice is very similar to that of a stationary random field on an n-dimensional Euclidean space; cf. e.g. [4, 9]. It is not the case if we take into consideration the notion of a stationary isotropic random field. On a Euclidean space such a field is naturally defined by aid of the condition saying that the covariance function depends just on the (Euclidean) distance or, equivalently, the covariance function is invariant with respect to the group of all the distance-preserving space transforms; cf. e.g. [8, 9]. Trying to define a stationary isotropic random field on an n-lattice we come to problems – cf. [4], Sec.V. We might, of course, suppose the covariance function to depend just on the distance, coming to what we could call stationary "strongly" isotropic random field. The point is, such fields are rather difficult to describe effectively. That is why we choose an easier way in this paper and define a stationary weakly isotropic random field on an n-lattice by aid of the condition saying that the covariance function is invariant with respect to the group of all the distance-preserving n-lattice transforms. Unlike in the continuous case, these two ways of definition are no more equivalent, the latter being weaker.

In our approach an important role is played by the hyperoctaedral group of

[1] The results presented in the paper are based on the author's dissertation "Spectral Analysis of Stationary Weakly Isotropic Random Fields" (in Czech), ÚTIA ČSAV, Prague 1984.

degree n, whose properties we study in the first part of the paper. The second one is devoted to spectral representation of stationary weakly isotropic random fields. In the third part of the paper, as an example of application of the spectral representation, there is derived a statistical procedure for testing the hypothesis under which a stationary n-lattice random field is supposed to be weakly isotropic.

1. HYPEROCTAEDRAL GROUP

We will introduce the definition of the hyperoctaedral group H_n of degree n, characterize its matrix representations and study functions and measures invariant with respect to H_n. (Concepts of matrix – or linear – group representations are introduced e.g. in [1, 5, 7].)

Let n be a fixed positive integer. Through S_n we denote the symmetric group of degree n, i.e. the group of all permutations on $\{1, \ldots, n\}$. It is well known that $|S_n| = n!$ where, for a set C, the symbol $|C|$ denotes the number of elements of C. $A = (A, *)$ will denote the two-element group $\{0, 1\}$ with $0 * 0 = 1 * 1 = 0$, $0 * 1 = 1 * 0 = 1$ and A^n will denote the nth direct power of A. That is, for $\alpha = (\alpha_1, \ldots \alpha_n) \in A^n$, we define $\alpha * \beta = (\alpha_1 * \beta_1, \ldots, \alpha_n * \beta_n)$. Obviously $|A^n| = 2^n$.

For $\alpha \in A^n$, $\varrho \in S_n$ and $j, k = 1, \ldots, n$ we define $u_{jk}(\alpha) = (-1)^{\alpha_j} \delta_{jk}$, $v_{jk}(\varrho) = \delta_{j\varrho(k)}$ where δ is the Kronecker symbol. Further we define $n \times n$ matrices $\boldsymbol{u}(\alpha) = (u_{jk}(\alpha))$, $\boldsymbol{v}(\varrho) = (v_{jk}(\varrho))$, $\boldsymbol{h}(\alpha, \varrho) = \boldsymbol{u}(\alpha) \, . \, \boldsymbol{v}(\varrho)$.

The *hyperoctaedral group* H_n of degree n is the set $\{\boldsymbol{h}(\alpha, \varrho): \alpha \in A^n, \varrho \in S_n\}$ together with the operation of the usual product of matrices. It is easy to verify that H_n is a group of orthogonal $n \times n$ matrices, number of them being $2^n \, . \, n!$, and an orthogonal $n \times n$ matrix belongs to H_n if and only if all its elements are integers.

Let us denote $U = \{\boldsymbol{u}(\alpha): \alpha \in A^n\}$ and $V = \{\boldsymbol{v}(\varrho): \varrho \in S_n\}$. Then U, V are subgroups of H_n, the subgroup U is normal and the only element of $U \cap V$ is a unit matrix. So H_n is a semidirect product of U, V (cf. [1]) or, equivalently, H_n is isomorphic to a semidirect product of the groups A^n, S_n. Hence one corollary of Clifford's theorem (cf. [1], 1.11) enables us to characterize matrix representations of H_n through those of A^n and S_n.

Let G be a finite group and $\boldsymbol{M}^1, \ldots, \boldsymbol{M}^k$ be irreducible matrix representations of G such that for every irreducible matrix representation $\boldsymbol{M}$ of G there exists $j \in \{1, \ldots, k\}$ for which the representations $\boldsymbol{M}, \boldsymbol{M}^j$ are mutually equivalent. Then we say $\boldsymbol{M}^1, \ldots, \boldsymbol{M}^k$ to constitute a *complete set of irreducible representations* of G.

We say a vector $\lambda = (\lambda_1, \ldots, \lambda_m)$ of positive integers to be a *partition* of n if $\lambda_1 \geqq \ldots \geqq \lambda_m$ and $\lambda_1 + \ldots + \lambda_m = n$. As shown in [5], by means of Young's algorithm it is possible, for every partition λ of n, to define certain irreducible orthogonal representation $\boldsymbol{T}^\lambda$ (whose dimension we denote $r(\lambda)$) of the symmetric group S_n. The set $\{\boldsymbol{T}^\lambda: \lambda \text{ is a partition of } n\}$ is a complete set of irreducible representations of S_n.

Formally we put the set of partitions of the number 0 to be equal to (0), the group S_0 to be equal to S_1 and the representation $\boldsymbol{T}^{(0)}$ to be equal to 1.

For $k = 0, \ldots, n$ we define a subgroup $S_{n,k}$ of S_n as the set of exactly those permutations $\varrho \in S_n$ for which $j \in \{1, \ldots, k\}$ implies $\varrho(j) \in \{1, \ldots, k\}$. That is, $\varrho \in S_{n,k}$ if and only if ϱ is a composition of permutations $\varrho_1 \in S_k$ and $\varrho_2 \in S_{n-k}$, ϱ_2 acting on $\{k + 1, \ldots, n\}$. Through $(n\colon k)$ we denote the set of all k-element subsets of the set $\{1, \ldots, n\}$ and to every $B \in (n\colon k)$ we assign a permutation $\tau_B \in S_n$ in the following way. We denote $B_1 = \{1, \ldots, k\} \setminus B$ and $B_2 = B \setminus \{1, \ldots, k\}$. It is $|B_1| = |B_2|$, therefore we may write $B_1 = \{x_1, \ldots, x_r\}$, $B_2 = \{y_1, \ldots, y_r\}$ and define, for every $j = 1, \ldots, n$,

$$\tau_B(j) = \begin{cases} y_\nu & \text{for } j = x_\nu\,, \quad \nu = 1, \ldots, r\,, \\ x_\nu & \text{for } j = y_\nu\,, \quad \nu = 1, \ldots, r\,, \\ j & \text{otherwise}\,. \end{cases}$$

Let $k \in \{0, \ldots, n\}$, λ be a partition of k, μ be a partition of $(n - k)$, $g \in H_n$, $g = \boldsymbol{h}(\alpha, \varrho)$. We define an $(r(\lambda) \,.\, r(\mu)) \times (r(\lambda) \,.\, r(\mu))$ matrix

$$\boldsymbol{P}^{k,\lambda,\mu}(g) = \begin{cases} (-1)^{\sum_{j=1}^{k} \alpha_j}\, \boldsymbol{T}^\lambda(\varrho_1) \otimes \boldsymbol{T}^\mu(\varrho_2) & \text{for} \quad \varrho \in S_{n,k} \\ 0 \quad \text{otherwise} \end{cases}$$

where $\otimes$ denotes the Kronecker product of matrices and $\varrho_1 \in S_k$, $\varrho_2 \in S_{n-k}$ are the permutations which correspond to $\varrho \in S_{n,k}$ in the sense mentioned above.

Further we define a representation $\boldsymbol{M}^{k,\lambda,\mu}$ of H_n for every $g \in H_n$ as

$$\boldsymbol{M}^{k,\lambda,\mu}(g) = (\boldsymbol{P}^{k,\lambda,\mu}(\boldsymbol{v}(\tau_A) \,.\, g \,.\, \boldsymbol{v}(\tau_B)))_{A,B \in (n:k)}\,;$$

the matrix on the right hand side is written in the block form.

1.1. Theorem. Let us denote $A(n) = \{(k, \lambda, \mu)\colon k = 0, \ldots, n,\ \lambda$ is a partition of k, μ is a partition of $(n - k)\}$. Then $\{\boldsymbol{M}^{\boldsymbol{a}};\ \boldsymbol{a} \in A(n)\}$ is a complete set of irreducible representations of the hyperoctaedral group H_n and every representation in this set is orthogonal.

Proof. The statement follows from the application of Clifford's theorem mentioned above – cf. [1], 1.11. □

1.2. Remark. We denote through $d(\boldsymbol{a})$ the dimension of the representation $\boldsymbol{M}^{\boldsymbol{a}}$. From the preceding theorem it follows that, for every $\boldsymbol{a}, \boldsymbol{b} \in A(n)$. every $j, k = 1, \ldots, d(\boldsymbol{a})$ and every $s, t = 1, \ldots, d(\boldsymbol{b})$, the relation

$$\frac{1}{|H_n|} \sum_{g \in H_n} m^{\boldsymbol{a}}_{jk}(g)\, m^{\boldsymbol{b}}_{st}(g) = \frac{1}{d(\boldsymbol{a})}\, \delta_{\boldsymbol{ab}} \delta_{js} \delta_{kt}$$

holds – cf. [7], Chap. V. 20.

Symbols $\mathbb{Z}$, $\mathbb{R}$ will denote the sets of all integers, reals (respectively). A subset M of $\mathbb{R}^n$ we say to be *H_n-invariant* if, for every (column vector) $\boldsymbol{x} \in M$ and every $g \in H_n$, it is $g\boldsymbol{x} \in M$.

Let M be an H_n-invariant subset of $\mathbb{R}^n$. A function f which is defined on M we say to be H_n-*invariant* if it is $f(g\mathbf{x}) = f(\mathbf{x})$ for every $\mathbf{x} \in M$ and every $g \in H_n$. A measure m defined on the sigma-algebra $\mathscr{B}(M)$ of all Borel subsets of M we say to be H_n-*invariant* if, for every $g \in H_n$ and every $A \in \mathscr{B}(M)$, $m(gA) = m(A)$ where $gA = \{g\mathbf{x}: \mathbf{x} \in A\}$.

Let us denote

$$I^n = \{x \in \mathbb{R}^n: -\tfrac{1}{2} \leqq x_j \leqq \tfrac{1}{2} \text{ for all } j = 1, \dots, n\},$$

$$R_n = \{x \in \mathbb{R}^n: 0 \leqq x_1 \leqq \dots \leqq x_n \leqq \tfrac{1}{2}\}.$$

1.3. Lemma. Let P be an H_n-invariant probability measure defined on $\mathscr{B}(I^n)$. Then there exists a unique probability measure Q defined on $\mathscr{B}(R_n)$ such that, for every (Borel) measurable bounded function f on I^n, it holds

$$\int_{I^n} f(\mathbf{x})\,\mathrm{d}P(\mathbf{x}) = \frac{1}{|H_n|} \sum_{g \in H_n} \int_{R_n} f(g\mathbf{r})\,\mathrm{d}Q(\mathbf{r}). \tag{1}$$

Proof. According to [2] every H_n-invariant probability measure is represented by means of H_n-ergodic measures. In order to complete the proof it is sufficient to take into account the one-to-one correspondence between H_n-ergodic measures and points of R_n. In fact, if we denote $M_\mathbf{r} = \{g\mathbf{r}: g \in H_n\}$ for every $\mathbf{r} \in R_n$, the corresponding H_n-ergodic measure $m_\mathbf{r}$ is for every $A \in \mathscr{B}(I^n)$ defined as

$$m_\mathbf{r}(A) = |M_\mathbf{r} \cap A| / |M_\mathbf{r}|. \qquad \square$$

Let f be a complex function defined on $\mathbb{Z}^n$. We say f to be *positive semidefinite* if, for every positive integer m, every m-tuple of complex numbers $c_1, \dots, c_m$ and every m-tuple $\mathbf{K}^1, \dots, \mathbf{K}^m \in \mathbb{Z}^n$ the inequality

$$\sum_{j=1}^{m} \sum_{k=1}^{m} c_j \,.\, c_k^* \,.\, f(\mathbf{K}^j - \mathbf{K}^k) \geqq 0$$

holds (asterisk denotes the complex conjugate).

1.4. Lemma. Let f be an H_n-invariant positive semidefinite function defined on $\mathbb{Z}^n$. There exists a unique H_n-invariant finite measure F on I^n such that, for every $\mathbf{K} \in \mathbb{Z}^n$, it is

$$f(\mathbf{K}) = \int_{I^n} \mathrm{e}^{2\pi\mathrm{i}\langle \mathbf{K}, \mathbf{t}\rangle}\,\mathrm{d}F(\mathbf{t}) \tag{2}$$

where $\langle .\,, .\rangle$ denotes the usual scalar product, $\langle \mathbf{x}, \mathbf{y}\rangle = \sum x_j y_j$.

Proof. Let us denote $J^n = \{\mathbf{t} \in \mathbb{R}^n: -\tfrac{1}{2} < t_j \leqq \tfrac{1}{2} \text{ for all } j = 1, \dots, n\}$. According to a generalized version of the Bochner theorem (cf. [7]) there exists a unique measure G on J^n such that

$$f(\mathbf{K}) = \int_{J^n} \mathrm{e}^{2\pi\mathrm{i}\langle \mathbf{K}, \mathbf{t}\rangle}\,\mathrm{d}G(\mathbf{t})$$

holds for every $\mathbf{K} \in Z^n$. We define a measure Q on I^n as $Q(A) = G(A \cap J^n)$ for every $A \in \mathscr{B}(I^n)$ and, for every $g \in H_n$, a measure Q_g as $Q_g(A) = Q(gA)$. Then $F = (1/|H_n|) \,.\, \sum_{g \in H_n} Q_g$ is an H_n-invariant measure for which (2) holds.

In order to prove the uniqueness of F we define a function mod: $I^n \to J^n$; for $\mathbf{x} \in I^n$ we define $\mathbf{y} = \mathrm{mod}(\mathbf{x})$ as

$$y_j = \begin{cases} \frac{1}{2} & \text{for } x_j = -\frac{1}{2}, \\ x_j & \text{otherwise}, \end{cases}$$

$j = 1, \ldots, n$. From this definition it follows that, for every H_n-invariant set $A \in \mathscr{B}(I^n)$, the equality $A = \mathrm{mod}^{-1}(\mathrm{mod}(A))$ holds. Let F^1, F^2 be H_n-invariant measures on I^n for which (2) is true. As e^z is a $2\pi i$-periodic function of a complex variable z, it is, for every $\mathbf{K} \in \mathbb{Z}^n$ and $j = 1, 2$,

$$f(\mathbf{K}) = \int_{J^n} e^{2\pi i \langle \mathbf{K}, \mathbf{t} \rangle} \, d(F^j \, \mathrm{mod}^{-1})(\mathbf{t})$$

where we have denoted $(F^j \, \mathrm{mod}^{-1})(A) = F^j(\mathrm{mod}^{-1}(A))$ for every $A \in \mathscr{B}(I^n)$. It is $F^1 \, \mathrm{mod}^{-1} = F^2 \, \mathrm{mod}^{-1}$ because the n-tuple Fourier coefficients of these two measures are the same (cf. [6], Thm. VII. 1.7). If it were $F^1 \neq F^2$, Theorems 1 and 3 in [6] would imply the existence of an H_n-invariant set $A \in \mathscr{B}(I^n)$ for which $F^1(A) \neq F^2(A)$. Equivalently, it would hold $(F^1 \, \mathrm{mod}^{-1})(\mathrm{mod}(A)) = F^1(A) \neq F^2(A) = (F^2 \, \mathrm{mod}^{-1})(\mathrm{mod}(A))$; that is, the contradiction $F^1 \, \mathrm{mod}^{-1} \neq F^2 \, \mathrm{mod}^{-1}$ would follow. □

2. SPECTRAL REPRESENTATION OF STATIONARY WEAKLY ISOTROPIC RANDOM FIELDS

We define a *random field* $\mathbf{X}$ on $\mathbb{Z}^n$ as a system of (generally complex) random variables, $\mathbf{X} = (X(\mathbf{K}): \mathbf{K} \in \mathbb{Z}^n)$ – i.e. the index set is an n-lattice $\mathbb{Z}^n$ – such that every random variable from $\mathbf{X}$ has zero mean and finite variance.

A random field $\mathbf{X}$ on $\mathbb{Z}^n$ is *stationary* if there exists a *covariance function* $\mathbf{B}$ defined on $\mathbb{Z}^n$ such that it is $\mathsf{E}\{X(\mathbf{K}) \, . \, X(\mathbf{L})^*\} = \mathbf{B}(\mathbf{K} - \mathbf{L})$ for every $\mathbf{K}, \mathbf{L} \in \mathbb{Z}^n$. A stationary random field $\mathbf{X}$ on $\mathbb{Z}^n$ is *weakly isotropic* if, for every $g \in H_n$ and every $\mathbf{K} \in \mathbb{Z}^n$, $\mathbf{B}(\mathbf{K}) = \mathbf{B}(g\mathbf{K})$ holds. From the definition of the hyperoctaedral group H_n it is easy to see that the condition

for every distance-preserving transformation T on $\mathbb{Z}^n$ and every $\mathbf{K}, \mathbf{L} \in \mathbb{Z}^n$ it is $\mathsf{E}\{X(\mathbf{K}) \, X(\mathbf{L})^*\} = \mathsf{E}\{X(T(\mathbf{K})) \, X(T(\mathbf{L}))^*\}$

is a necessary and sufficient one for the random field $\mathbf{X}$ to be weakly isotropic.

For every $\mathbf{a} \in A(n)$, $j, k = 1, \ldots, d(\mathbf{a})$, $\mathbf{K} \in \mathbb{Z}^n$ and $\mathbf{x} \in \mathbb{R}^n$ we define

$$e^{\mathbf{a}}_{jk}(\mathbf{K}, \mathbf{x}) = \frac{1}{|H_n|} \sum_{g \in H_n} e^{2\pi i \langle \mathbf{K}, g\mathbf{x} \rangle} \, m^{\mathbf{a}}_{jk}(g)$$

(cf. 1.1 and 1.2). The functions $(e^{\mathbf{a}}_{jk}: \mathbf{a} \in A(n), j, k = 1, \ldots, d(\mathbf{a}))$ we shall call *generalized goniometric functions* with regard to that for $n = 1$ these functions are $\cos Kx$, $i \, . \sin Kx$ and some properties of functions $(e^{\mathbf{a}}_{jk})$ are similar to those of goniometric functions.

2.1. Lemma. For every $\mathbf{K} \in \mathbb{Z}^n$, $\mathbf{x} \in \mathbb{R}^n$, $g \in H_n$, $\mathbf{a} = (p, \lambda, \mu) \in A(n)$ and $j, k = 1, \dots$ $\dots, d(\mathbf{a})$ the following assertions are true:

(i) $\mathrm{e}^{2\pi\mathrm{i}\langle \mathbf{K}, g\mathbf{x}\rangle} = \sum_{\mathbf{a}\in A(n)} \mathrm{d}(\mathbf{a}) \sum_{j=1}^{d(\mathbf{a})} \sum_{k=1}^{d(\mathbf{a})} e^{\mathbf{a}}_{jk}(\mathbf{K}, \mathbf{x})\, m^{\mathbf{a}}_{jk}(g)$,

(ii) $e^{\mathbf{a}}_{jk}(g\mathbf{K}, \mathbf{x}) = \sum_{s=1}^{d(\mathbf{a})} m^{\mathbf{a}}_{js}(g)\, e^{\mathbf{a}}_{sk}(\mathbf{K}, \mathbf{x})$,

(iii) $e^{\mathbf{a}}_{jk}(\mathbf{K}, g\mathbf{x}) = \sum_{s=1}^{d(\mathbf{a})} m^{\mathbf{a}}_{ks}(g)\, e^{\mathbf{a}}_{js}(\mathbf{K}, \mathbf{x})$,

(iv) $\sum_{\mathbf{a}\in A(n)} d(\mathbf{a}) \sum_{s=1}^{d(\mathbf{a})} \sum_{t=1}^{d(\mathbf{a})} |e^{\mathbf{a}}_{st}(\mathbf{K}, \mathbf{x})|^2 = 1$,

(v) $e^{(p,\lambda,\mu)}_{jk}(\mathbf{K}, \mathbf{x})$ is real for p even and purely imaginary for p odd.

Proof. The assertions (i)–(iv) follow from the definitions of generalized goniometric functions and of the representations $M^{\mathbf{a}}$ (cf. 1.2).

Let us prove the assertion (v). We denote through $\mathbf{E}$ the $n \times n$ unit matrix. It is $(-\mathbf{E}) \in H_n$ and from the definition of $M^{\mathbf{a}}$ the relation

$$m^{\mathbf{a}}_{jk}(-\mathbf{E}) = \begin{cases} \delta_{jk} & \text{for } p \text{ even} \\ -\delta_{jk} & \text{for } p \text{ odd} \end{cases}$$

follows. Obviously $e^{\mathbf{a}}_{jk}(\mathbf{K}, \mathbf{x})^* = e^{\mathbf{a}}_{jk}(-\mathbf{K}, \mathbf{x})$; so putting $g = -\mathbf{E}$ in (ii) we obtain

$$e^{\mathbf{a}}_{jk}(\mathbf{K}, \mathbf{x})^* = \begin{cases} e^{\mathbf{a}}_{jk}(\mathbf{K}, \mathbf{x}) & \text{for } p \text{ even} \\ -e^{\mathbf{a}}_{jk}(\mathbf{K}, \mathbf{x}) & \text{for } p \text{ odd.} \end{cases}$$

□

The generalized goniometric function $e^{(0,(0),(n))}$, which corresponds to the unit representation $M^{(0,(0),(n))}$ of H_n, we shall denote shortly e_1. Taking into account the relation $2\cos x = e^{\mathrm{i}x} + e^{-\mathrm{i}x}$ we see that

$$e_1(\mathbf{K}, \mathbf{x}) = \frac{1}{n!} \sum_{\varrho\in S_n} \prod_{j=1}^{n} \cos\left(2\pi K_j x_{\varrho(j)}\right)$$

holds for every $\mathbf{K} \in \mathbb{Z}^n$, $\mathbf{x} \in \mathbb{R}^n$.

2.2. Theorem. Let $\mathbf{B}$ be a covariance function of a stationary weakly isotropic random field on $\mathbb{Z}^n$. Then there exists exactly one finite measure S on R_n such that it is

$$\mathbf{B}(\mathbf{K}) = \int_{R_n} e_1(\mathbf{K}, \mathbf{r})\, \mathrm{d}S(\mathbf{r}) \tag{3}$$

for every $\mathbf{K} \in \mathbb{Z}^n$.

Proof follows immediately from 1.3 and 1.4 because a covariance function is positive semidefinite. □

The relation (3) is called the *spectral representation* of the covariance function $\mathbf{B}$ and the measure S in (3) we call the *spectral measure* of $\mathbf{B}$.

Let $(D, \mathcal{D}, \xi)$ be a finite measure space. Further let there be given, for every $j =$

$= 1, \ldots, p$ and every $C \in \mathscr{D}$, a random variable $Z_j(C)$ with zero mean and finite variance and suppose, for every $j, k = 1, \ldots, p$ and every $C_1, C_2 \in \mathscr{D}$, the relation $\mathsf{E}\{Z_j(C_1) \, . \, Z_k(C_2)^*\} = \delta_{jk} \, . \, \xi(C_1 \cap C_2)$ to hold. Then we say that a *finite orthogonal system* $\boldsymbol{Z} = (Z_1, \ldots, Z_p)$ *of random measures* on D consistent with the measure ξ is given.

If there is given a random measure Z cosnistent with the measure ξ (i.e. if it is $p = 1$ in the situation considered above) then it is possible to define, for any function f which is square integrable with respect to ξ, a stochastic integral $\int_D f \, dZ$ – the integral is understood in the square mean sense, for details cf. e.g. [3], Chap. IV. 5.

2.3. Theorem. Let $\boldsymbol{X}$ be a stationary weakly isotropic random field on $\mathbb{Z}^n$ with the spectral measure S on R_n. Then there exists a finite orthogonal system $\boldsymbol{Z} = (Z^{\boldsymbol{a}}_{jk}: \boldsymbol{a} \in A(n), j, k = 1, \ldots, d(\boldsymbol{a}))$ of random measures on R^n consistent with S such that, for every $\boldsymbol{K} \in \mathbb{Z}^n$, it holds

$$X(\boldsymbol{K}) = \sum_{\boldsymbol{a} \in A(n)} (d(\boldsymbol{a}))^{1/2} \sum_{j=1}^{d(\boldsymbol{a})} \sum_{k=1}^{d(\boldsymbol{a})} \int_{R_n} e^{\boldsymbol{a}}_{jk}(\boldsymbol{K}, \boldsymbol{r}) \, dZ^{\boldsymbol{a}}_{jk}(\boldsymbol{r}) \, . \tag{4}$$

Proof. Let $\boldsymbol{K}, \boldsymbol{L} \in \mathbb{Z}^n$. With regard to 2.2 we may write

$$\mathsf{E}\{X(\boldsymbol{K}) \, . \, X(\boldsymbol{L})^*\} = \boldsymbol{B}(\boldsymbol{K} - \boldsymbol{L}) = \frac{1}{|H_n|} \sum_{g \in H_n} \int_{R_n} e^{2\pi i \langle \boldsymbol{K} - \boldsymbol{L}, g\boldsymbol{r} \rangle} \, dS(\boldsymbol{r}) \, .$$

Substituting for $e^{2\pi i \langle \boldsymbol{K}, g\boldsymbol{r} \rangle}$ and $e^{2\pi i \langle \boldsymbol{L}, g\boldsymbol{r} \rangle}$ according to 2.1. (i) and taking into account Remark 2.1, we obtain

$$\mathsf{E}\{X(\boldsymbol{K}) \, . \, X(\boldsymbol{L})^*\} = \sum_{\boldsymbol{a} \in A(n)} d(\boldsymbol{a}) \sum_{j=1}^{d(\boldsymbol{a})} \sum_{k=1}^{d(\boldsymbol{a})} \int_{R_n} e^{\boldsymbol{a}}_{jk}(\boldsymbol{K}, \boldsymbol{r}) \, e^{\boldsymbol{a}}_{jk}(\boldsymbol{L}, \boldsymbol{r})^* \, dS(\boldsymbol{r})$$

and the assertion of the theorem follows from the Karhunen theorem (cf. e.g. [3], Chap. IV. 5). In fact, let us denote γ the "counting" measure on $\Gamma = \{(\boldsymbol{a}, j, k): \boldsymbol{a} \in A(n), j, k = 1, \ldots, d(\boldsymbol{a})\}$. Then, by the Karhunen theorem referred to, a random measure ζ on $\Gamma \times R_n$ corresponds to the product measure $\gamma \times S$ on $\Gamma \times R_n$ such that $Z^{\boldsymbol{a}}_{jk}(C) = \zeta((\boldsymbol{a}, j, k) \times C)$. □

The relation (4) we call the *spectral representation of the stationary weakly isotropic random field* $\boldsymbol{X}$.

3. TEST OF WEAK ISOTROPY

The random measures which occur in the spectral representation (4) are mutually orthogonal, i.e. uncorrelated. This property characterizes stationary weakly isotropic random fields in the class of stationary random fields and will be used to derive a test of weak isotropy for a stationary random field on the base of values observed in some finite region of indices.

For the sake of simplicity, we suppose the stationary random field in question to be Gaussian and to have an exponential decay of covariances.

A stationary randon field $\mathbf{X}$ on $\mathbb{Z}^n$ we say to be *Gaussian* if all random variables in $\mathbf{X}$ are real and, for every positive integer m and every m-tuple of indices $\mathbf{K}^1, \ldots, \mathbf{K}^m$, the random vector $(X(\mathbf{K}^1), \ldots, X(\mathbf{K}^m))$ is Gaussian with zero mean and variance matrix $(\mathbf{B}(\mathbf{K}^j - \mathbf{K}^k))_{j,k=1}$ where $\mathbf{B}$ is the covariance function of $\mathbf{K}$.

A function b defined on $\mathbb{Z}^n$ fulfils the condition of *exponential decay* if

(5) there exist $M > 0, \varepsilon > 0$ such that, for every $\mathbf{K} \in \mathbb{Z}^n$,

$$b(\mathbf{K}) \leqq M \,.\, \exp\{-\varepsilon \,.\, \sum_{j=1}^{n} |K_j|\} \quad \text{holds}\,.$$

3.1. Lemma. Let b be a nonvanishing complex function on $\mathbb{Z}^n$ which fulfils the condition of exponential decay. We define a function φ on $J^n = \{\mathbf{x} \in \mathbb{R}^n: -\frac{1}{2} < x_j \leqq \leqq \frac{1}{2}$ for all $j = 1, \ldots, n\}$ as

$$\varphi(\mathbf{x}) = \sum_{\mathbf{K}} b(\mathbf{K}) \,.\, e^{2\pi i \langle \mathbf{K}, \mathbf{x} \rangle}$$

where $\mathbf{K}$ runs over $\mathbb{Z}^n$. Then the n-dimensional Lebesgue measure $\mathbf{L}_n$ of the set $N = \{\mathbf{x} \in J^n: \varphi(\mathbf{x}) = 0\}$ is zero.

Proof. It is sufficient to prove

(6) from $\mathbf{L}_n(N) > 0$ it follows $b \equiv 0$, i.e. $b(\mathbf{K}) = 0$ for all $\mathbf{K} \in \mathbb{Z}^n$.

The proof of (6) we shall carry out by induction with respect to n.

1) For $n = 1$ it follows from (5) that the complex function $\bar{\varphi}$ of the complex variable z which is defined

$$\bar{\varphi}(z) = \sum_{k=-\infty}^{\infty} b(k) \,.\, z^k$$

is holomorphic in the domain $U = \{z: e^{-\varepsilon} < |z| < e^{\varepsilon}\}$. If the $\mathbf{L}_n$ measure of $N = = \{x \in J: \bar{\varphi}(e^{2\pi i x}) = 0\}$ is positive, then the set $\bar{N} = \{2\pi i x: x \in N\}$ has a cluster point in U which implies $\bar{\varphi} \equiv 0$ on U.

2) Let $n \geqq 2$ and (6) hold for $n - 1$. $\mathbf{L}_n(N) > 0$ implies $\mathbf{L}_{n-1}(M) > 0$ where $M = \{\mathbf{t} \in J^{n-1}: \mathbf{L}_1(N_{\mathbf{t}}) > 0\}$, $N_{\mathbf{t}} = \{y \in J, \varphi(\mathbf{t}, y) = 0\}$. For $k \in \mathbb{Z}$, $\mathbf{t} \in J^{n-1}$ we denote

$$c(k, \mathbf{t}) = \sum_{\mathbf{K}} b(\mathbf{K}, k) \,.\, e^{2\pi i \langle \mathbf{K}, \mathbf{t} \rangle}$$

where $\mathbf{K}$ runs over $\mathbb{Z}^{n-1}$. From (5)

$$|c(k, \mathbf{t})| \leqq M \left(\frac{1 + e^{-\varepsilon}}{1 - e^{-\varepsilon}} \right)^{n-1} e^{-\varepsilon k}$$

follows. As $\varphi(\mathbf{t}, y) = \sum_{k=-\infty}^{\infty} e^{2\pi i k y} \,.\, c(k, \mathbf{t})$, it is (cf. 1)) $c(k, \mathbf{t}) = 0$ for every $k \in \mathbb{Z}$ and the induction assumption implies $b \equiv 0$. □

Let us denote, for every $\mathbf{x} \in J^n$,

(7) $$f(\mathbf{x}) = \sum_{\mathbf{K}} \mathbf{B}(\mathbf{K}) \,.\, e^{-2\pi i \langle \mathbf{K}, \mathbf{x} \rangle}$$

where $\boldsymbol{K}$ runs over $\mathbb{Z}^n$. If $\boldsymbol{B}$ fulfils the condition of exponential decay (5), the last sum converges and f is bounded and continuous. The function f is called *spectral density* of the stationary random field $\boldsymbol{X}$, whose covariance function is $\boldsymbol{B}$, because (7) implies, for every $\boldsymbol{K} \in \mathbb{Z}^n$.

$$\boldsymbol{B}(\boldsymbol{K}) = \int_{J^n} e^{2\pi i\langle \boldsymbol{K}, \boldsymbol{x}\rangle} f(\boldsymbol{x})\, d\boldsymbol{x}\,.$$

The measure whose Radon-Nikodym derivative with respect to n-dimensional Lebesgue measure is f is denoted F.

From the Karhunen theorem it follows that there exists a random measure Z on J^n consistent with F such that, for every $\boldsymbol{K} \in \mathbb{Z}^n$, it is

$$X(\boldsymbol{K}) = \int_{J^n} e^{2\pi i\langle \boldsymbol{K}, \boldsymbol{x}\rangle}\, dZ(\boldsymbol{x})\,. \tag{8}$$

3.2. Lemma. Let Z be the random measure which corresponds, by means of (8), to a stationary field $\boldsymbol{X}$ on $\mathbb{Z}^n$ with exponential decay of covariances. Then, for every Borel subset C of J^n,

$$Z(C) = \sum_{\boldsymbol{K}} X(\boldsymbol{K}) \int_C e^{-2\pi i\langle \boldsymbol{K}, \boldsymbol{x}\rangle}\, d\boldsymbol{x} \tag{9}$$

holds where $\boldsymbol{K}$ runs over $\mathbb{Z}^n$ and the sum is understood in the square mean sense.

Proof. For $\boldsymbol{y} \in J^n$ we define

$$1_C(\boldsymbol{y}) = \begin{cases} 1 & \text{for } \boldsymbol{y} \in C\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

As $\int_C e^{-2\pi i\langle \boldsymbol{K}, \boldsymbol{x}\rangle}\, dx$, $\boldsymbol{K} \in \mathbb{Z}^n$, are n-tuple Fourier coefficients of 1_C, the relation

$$\sum_{\boldsymbol{K}} e^{2\pi i\langle \boldsymbol{K}, \cdot\rangle} \int_C e^{-2\pi i\langle \boldsymbol{K}, \boldsymbol{x}\rangle}\, d\boldsymbol{x} = 1_C(\cdot)$$

holds; the convergence is understood in the square mean sense with respect to the n-dimensional Lebesgue measure. The sum converges in the square mean also with respect to F because the spectral density f is bounded. Therefore (9) follows from substitution for $X(\boldsymbol{K})$ from (8) to (9) with respect to that, for a random measure W consistent with a measure ξ, $\operatorname{l.i.m.}_{m\to\infty} \int \varphi_m\, dW = \int \varphi\, dW$ is equivalent to $\operatorname{l.i.m.}_{m\to\infty} \varphi_m = \varphi$ with respect to ξ (cf. [3], Chap. IV. 5). □

Let $\mathscr{C} = (C_1, \ldots, C_Q)$ be a partition of the set $R_n = \{\boldsymbol{x} \in \mathbb{R}^n : 0 \leqq x_1 \leqq \ldots \leqq x_n \leqq \leqq \frac{1}{2}\}$. We say $\mathscr{C}$ to be a *regular partition* if every $C \in \mathscr{C}$ is a Borel subset of R_n and the n-dimensional Lebesgue measure of C is positive.

For a regular partition $\mathscr{C}$ of R_n, we say a stationary random field $\boldsymbol{X}$ with a spectral density f to be $\mathscr{C}$-isotropic if, for every $C \in \mathscr{C}$ and every $g \in H_n$, it is

$$\int_C f(\boldsymbol{x})\, d\boldsymbol{x} = \int_{gC} f(\boldsymbol{x})\, d\boldsymbol{x}\,. \tag{10}$$

For a fine regular partition $\mathscr{C}$, i.e. a partition which consists of a "large number of small sets", the notion of $\mathscr{C}$-isotropy is near to that of weak isotropy because the weak isotropy is equivalent to holding of (10) for every Borel subset C of R_n.

So we are going to derive a test of $\mathscr{C}$-isotropy instead of that of weak isotropy.

For this purpose we define, for $\boldsymbol{a} \in A'(n)$, $j, k = 1, \ldots, d(\boldsymbol{a})$, $\boldsymbol{K} \in \mathbb{Z}^n$ and $\boldsymbol{x} \in \mathbb{R}^n$, the function

$$(11) \qquad \varepsilon^{\boldsymbol{a}}_{jk}(\boldsymbol{K}, \boldsymbol{x}) = \begin{cases} e^{\boldsymbol{a}}_{jk}(\boldsymbol{K}, \boldsymbol{x}) & \text{for } e^{\boldsymbol{a}}_{jk}(\boldsymbol{K}, \boldsymbol{x}) \text{ real}, \\ -\mathrm{i} \,.\, e^{\boldsymbol{a}}_{jk}(\boldsymbol{K}, \boldsymbol{x}) & \text{otherwise}. \end{cases}$$

According to 2.1. (v), the function $\varepsilon^{\boldsymbol{a}}_{jk}$ is real. Further we define, for the given Gaussian stationary random field $\boldsymbol{X}$ on $\mathbb{Z}^n$ with exponential decay of covariances,

$$(12) \qquad \zeta^{\boldsymbol{a}}_{jk}(C) = (d(\boldsymbol{a}))^{1/2} \sum_{\boldsymbol{K}} X(\boldsymbol{K}) \int_C \varepsilon^{\boldsymbol{a}}_{jk}(\boldsymbol{K}, \boldsymbol{x}) \, \mathrm{d}\boldsymbol{x}$$

where $\boldsymbol{K}$ runs over $\mathbb{Z}^n$ and C is a Borel subset of R_n. The last sum converges in the square mean sense with regard to Lemma 3.2. As the field $\boldsymbol{X}$ is supposed to be Gaussian, the random vector $\zeta(C) = (\zeta^{\boldsymbol{a}}_{jk}(C): \boldsymbol{a} \in A(n), j, k = 1, \ldots, d(\boldsymbol{a}))$ is Gaussian.

3.3. Lemma. Let $\mathscr{C} = (C_1, \ldots, C_Q)$ be a regular partition of R_n and $\zeta(C)$, $C \in \mathscr{C}$, be the random vectors defined above. Then

(i) all the random vectors $\zeta(C_1), \ldots, \zeta(C_Q)$ are mutually (stochastically) independent,

(ii) the field $\boldsymbol{X}$ is $\mathscr{C}$-isotropic if and only if, for every $C \in \mathscr{C}$, all the components of the random vector $\zeta(C)$ are mutually independent,

(iii) if the field $\boldsymbol{X}$ is $\mathscr{C}$-isotropic, the variance of the random variable $\zeta^{\boldsymbol{a}}_{jk}(C)$ is positive for every $C \in \mathscr{C}$, $\boldsymbol{a} \in A(n)$, $j, k = 1, \ldots, d(\boldsymbol{a})$.

Proof. The relations (11), (12) and the definition of generalized goniometric functions yield

$$(13) \qquad \zeta^{\boldsymbol{a}}_{jk}(C) = \gamma(\boldsymbol{a}) \frac{(d(\boldsymbol{a}))^{1/2}}{|H_n|} \sum_{g \in H_n} m^{\boldsymbol{a}}_{jk}(g) \,.\, Z(gC)$$

where

$$\gamma(\boldsymbol{a}) = \begin{cases} 1 & \text{for } \boldsymbol{a} = (p, \lambda, \mu), \quad p \text{ even} \\ \mathrm{i} & \text{for } \boldsymbol{a} = (p, \lambda, \mu), \quad p \text{ odd} \end{cases}$$

and Z corresponds to $\boldsymbol{X}$ by means of (8).

If $C, D \in \mathscr{C}$, $C \neq D$, then obviously gC, hD are disjoint for all $g, h \in H_n$ because for $r, s \in R_n$, $r \neq s$, the sets $\{gr: g \in H_n\}$ and $\{gs: g \in H_n\}$ are disjoint (cf. the proof of Lemma 1.3); from that (i) follows. In fact, values of Z for disjoint sets are uncorrelated and the joint distribution of $\zeta(C)$, $C \in \mathscr{C}$, is Gaussian.

Using (13) we for every $C \in \mathscr{C}$ calculate

$$(14) \qquad \mathsf{E}\{\zeta^{\boldsymbol{a}}_{jk}(C) \, \zeta^{\boldsymbol{b}}_{st}(C)\} = \gamma(\boldsymbol{a}) \, \gamma(\boldsymbol{b})^* \frac{(d(\boldsymbol{a}) \, d(\boldsymbol{L}))^{1/2}}{|H_n|^2} \sum_{g \in H_n} m^{\boldsymbol{a}}_{jk}(g) \, m^{\boldsymbol{b}}_{st}(g) \, F(gC) \,.$$

In case $\boldsymbol{X}$ is $\mathscr{C}$-isotropic, it is $F(gC) = F(C)$ for every $g \in H_n$, and the last relation can be simplified according to 1.2 as

$$\mathsf{E}\{\zeta_{jk}^{a}(C)\,\zeta_{st}^{b}(C)\} = \begin{cases} \dfrac{F(C)}{|H_n|} & \text{for} \quad \boldsymbol{a} = \boldsymbol{b},\, j = s,\, k = t\,, \\ 0 & \text{otherwise.} \end{cases}$$

That is, under the $\mathscr{C}$-isotropy the components of the vector $\zeta(C)$ are, for every $C \in \mathscr{C}$, mutually independent and they have positive variances because $\mathscr{C}$ is a regular partition which, according to Lemma 3.1, implies $F(C) > 0$ for every $C \in \mathscr{C}$.

On the other hand, let, for a set $C \in \mathscr{C}$, the components of $\zeta(C)$ be mutually independent. According to [7], Chap. V. 22 we may write, for every $g \in H_n$,

$$(15) \qquad F(gC) = \sum_{\boldsymbol{w} \in A(n)} d(\boldsymbol{w}) \sum_{u=1}^{d(\boldsymbol{w})} \sum_{v=1}^{d(\boldsymbol{w})} F_{uv}^{\boldsymbol{w}}(C)\,.\, m_{uv}^{\boldsymbol{w}}(g)$$

where the coefficients $F_{uv}^{\boldsymbol{w}}(C)$ are uniquely determined. Let us substitute, according to (15), on the right hand side of (14) and put $\boldsymbol{a} = (0, (0), (n))$ there. It is $d(\boldsymbol{a}) = 1$, $m^{\boldsymbol{a}}(g) = 1$ for all $g \in H_n$, and $\gamma(\boldsymbol{a}) = 1$. So, for every $\boldsymbol{b} \in A(n)$ and every $s, t = 1, \dots$ $\dots, d(\boldsymbol{b})$,

$$\mathsf{E}\{\zeta^{a}(C)\,\zeta_{st}^{b}(C)\} = \gamma(\boldsymbol{b})^* \frac{(d(\boldsymbol{b}))^{1/2}}{|H_n|^2} \sum_{g \in H_n} m_{st}^{\boldsymbol{b}}(g) \sum_{\boldsymbol{w}} d(\boldsymbol{w}) \sum_{u} \sum_{v} m_{uv}^{\boldsymbol{w}}(g)\, F_{uv}^{\boldsymbol{w}}(gC)\,.$$

The right hand side of the last relation equals, with respect to 1.2, to $\gamma(\boldsymbol{b})^* \,.\, (d(\boldsymbol{b}))^{1/2}\,.$ $.\, F_{st}^{\boldsymbol{b}}(C)/|H_n|$ and the left hand one is supposed to be zero for $\boldsymbol{b} \neq \boldsymbol{a}$. Therefore it is $F_{st}^{\boldsymbol{b}}(C) = 0$ for $\boldsymbol{b} \neq \boldsymbol{a}$ and (15) yields $F(C) = F(gC)$ for every $g \in H_n$. □

The last lemma transforms the problem of testing $\mathscr{C}$-isotropy onto that of testing mutual independence of the components of the vectors $\zeta(C)$, $C \in \mathscr{C}$. As variances of these components are unknown, we use a nonparametric chi-square goodness of fit test.

For a real x we define

$$\Phi(x) = \begin{cases} 0 & \text{for} \quad x \leq 0\,, \\ 1 & \text{for} \quad x > 0\,. \end{cases}$$

Let $\boldsymbol{Y} = (Y_1, \dots, Y_m)$ be a Gaussian random vector with zero mean. We denote $\Phi(\boldsymbol{Y}) = (\Phi(Y_1), \dots, \Phi(Y_m))$. It is easy to prove that $\Phi(\boldsymbol{Y})$ is uniformly distributed on $\{0, 1\}^m$ if and only if the components of $\boldsymbol{Y}$ are mutually independent and their variances are positive.

Now we are able to formulate a test of the hypothesis

(H) stationary random field $\boldsymbol{X}$ is weakly isotropic

on the base of values $X(\boldsymbol{K})$ observed in a finite (sufficiently large) region $\mathscr{K}$. More exactly, we assume the field $\boldsymbol{X}$ to be Gaussian stationary with exponential decay of covariances and we are testing, for some fine regular partition $\mathscr{C} = (C_1, \dots, C_Q)$ of R_n, $\mathscr{C}$-isotropy instead of weak isotropy.

Let us denote $h = |H_n| = 2^n \,.\, n!$, $f = 2^h - 1$ and, for $C \in \mathscr{C}$,

$$\xi^a_{jk}(C, \mathscr{K}) = (d(\boldsymbol{a}))^{1/2} \sum_{\boldsymbol{K}} X(\boldsymbol{K}) \int_C \varepsilon^a_{jk}(\boldsymbol{K}, \boldsymbol{r}) \, d\boldsymbol{r}$$

where $\boldsymbol{K}$ runs over $\mathscr{K}$. For every $\alpha \in \{0, 1\}^h$ we define $T_\alpha(\mathscr{C}, \mathscr{K})$ to be the number of those $C \in \mathscr{C}$ for which $\Phi(\xi(C, \mathscr{K})) = \alpha$ where $\xi = (\xi^a_{jk})$. Then the statistic

$$T(\mathscr{C}, \mathscr{K}) = \frac{f+1}{Q} \sum_\alpha \left(T_\alpha(\mathscr{C}, \mathscr{K}) - \frac{Q}{f+1} \right)^2$$

where α runs over $\{0, 1\}^h$ has asymptotically (for $\mathscr{K} \to \mathbb{Z}^n$, $Q \to \infty$) the distribution χ^2_f.

That is, the hypothesis (H) is rejected on $p\,\%$ level of significance if the value of $T(\mathscr{C}, \mathscr{K})$ exceeds the $p\,\%$ critical value of χ^2_f distribution.

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