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COMPARING ALTERNATIVE DEFINITIONS OF BOOLEAN-VALUED FUZZY SETS

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Two definitions of fuzzy sets with Boolean-valued membership functions, introduced by Drossos and Markakis and called by them external and internal Boolean fuzzy sets, are compared with a third, classical definition descending more directly from the original Zadeh's and Goguen's ideas. Under some rather general conditions, internal and classical Boolean fuzzy sets are proved to be equivalent in the sense that there exists a one-to-one mapping to each other conserving the set theoretic operations. On the other side, the space of external Boolean fuzzy sets is richer, so that such a mapping exists only in some rather special cases.

1. THREE DEFINITIONS OF BOOLEAN-VALUED FUZZY SETS

The two basic notions considered and combined together throughout this paper will be that of *fuzzy set* and that of *Boolean algebra*. Let us refer to [4] as far as the notion of fuzzy sets, their properties and basic results are concerned, let us refer to [5] for Boolean algebras. Fuzzy sets, in their classical setting with numerical real-valued membership functions, were conceived by Zadeh in 1965 [6] with the aim to develop a mathematical tool for uncertainty quantification and processing, alternative to that one represented by the classical probability theory. Hence, a fuzzy subset E of a nonempty basic space or universe A was defined by and identified with a function μ_E defined on A and taking its values in the unit interval $\langle 0, 1 \rangle$ of real numbers. For a number of reasons, as soon as in 1967 Goguen presented the idea of fuzzy sets with non-numerical membership functions, cf. [3] for more details and motivation. Then, in 1985, [1] Drossos and Markakis argued in favour of taking profit of Boolean algebras when defining fuzzy sets, however, both the definitions of fuzzy sets, suggested by the same authors in [2], differ from the definitions resulting from direct applications of Goguen's ideas. So, the aim of this paper will be to compare the two definitions of Boolean-valued fuzzy sets from [2] with the third, classical or Goguen-like one.

Let $\mathbb{B} = \langle B, \vee, \wedge, \neg, \mathbf{0}_{\mathbb{B}}, \mathbf{1}_{\mathbb{B}} \rangle$ be a Boolean algebra over a nonempty support set B, hence, for each $e, f \in B$, $e \vee f$ is the supremum and $e \wedge f$ the infimum of e, f, $\neg e$ is the complement of e, $\mathbf{0}_B$ is the zero and $\mathbf{1}_B$ the unit (or: the minimal and the maximal) element of $\mathbb{B}$. The partial ordering $\leq$ on B will be defined in the usual way, i. e., for $e, f \in B$, $e \leq f$ holds iff $e \vee f = f$, or, what is the same, iff $e \wedge f = e$, here $=$ is the identity (relation) on B. There are numerous settings of the set of axioms which the operations and the distinguished elements of a Boolean algebra are to obey and we shall not repeat them here referring, e. g., to axioms $A_1 - A_5$ in [5].

In what follows, we shall always suppose that:

i) The Boolean algebra $\mathbb{B}$ is *complete*, hence, for each $\emptyset \neq C \subset B$ there exist $f, g \in B$ such that

   (a) $e \leq f$ for each $e \in C$ and, if $e \leq f_1$ holds for some $f_1 \in B$ and each $e \in C$, then $f \leq f_1$; such an f is denoted by $\bigvee_{e \in C} e$ and called *supremum of* C.

   (b) $g \leq e$ for each $e \in C$ and, if $g_1 \leq e$ holds for some $g_1 \in B$ and each $e \in C$, then $g_1 \leq g$; such a g is denoted by $\bigwedge_{e \in C} e$ and called *infimum of* C.

(ii) The operations of supremum and infnimum are conventionally extended to the case when C is empty, setting $\bigwedge_{e \in \emptyset} e = \mathbf{1}_B$, $\bigvee_{e \in \emptyset} e = \mathbf{0}_B$.

(iii) The Boolean algebra $\mathbb{B}$ is *nonempty* and *nontrivial*, i. e. $B \neq \emptyset$ and $\mathbf{0}_B \neq \mathbf{1}_B$.

As the most simple example of a Boolean algebra satisfying all the demands above let us mention the system of all subsets of an at least two-element universe with respect to the common set-theoretical operations of union, joint (intersection) and complement, with the empty set and the universe playing the role of the distinguished elements. In what follows, the Boolean algebra $\mathbb{B}$ will be taken as fixed, so that it will not be always explicitly introduced as a free parameter of the notions and constructions presented below.

Definition 1.1. Let $e \in B$, let $C \subset B$, i. e., $C \in \mathcal{P}(B) = \{E : E \subset B\}$. The set C is called a *decomposition of (the element)* e if, for each $f_1, f_2 \in C$, $f_1 \wedge f_2 = \mathbf{0}_B$ and if $\bigvee_{f \in C} f = e$. The set of all decompositions of e will be denoted by $\mathrm{Dcp}(e)$ ($\subset \mathcal{P}(B)$).

Let A be a nonempty abstract set, fixed throughout all this paper.

Definition 1.2. $\mathbb{B}$*-fuzzy element of (the set)* A is a mapping x taking A into B and such that $\{x(a)\}_{a \in A}$ is a decomposition of $\mathbf{1}_B$.

The set of all $\mathbb{B}$-fuzzy elements of A will be denoted by $A^{\#}$, or by $A[\mathbb{B}]$, if $\mathbb{B}$ is to be expressed explicitly, so that

$$A^{\#} = \left\{x : x \in B^A,\ \{x(a)\}_{a \in A} \in \mathrm{Dcp}(\mathbf{1}_B)\right\}. \tag{1.1}$$

Definition 1.3. (cf. [2] for (ii), (iii))

(i) *Classical* $\mathbb{B}$*-fuzzy subset of (the set)* A or: classical $\mathbb{B}$-fuzzy set over A is a mapping $\mathcal{X}$ defined on A and taking its values in B. Hence, the set A^* of classical $\mathbb{B}$-fuzzy sets over A is defined setting $A^* = B^A$.

(ii) *Internal* $\mathbb{B}$*-fuzzy subset of (the set)* A or: internal $\mathbb{B}$-fuzzy set over A is a $\mathbb{B}$-fuzzy element of $\mathcal{P}(A)$. Hence, the set of all internal $\mathbb{B}$-fuzzy sets over A is the set $[\mathcal{P}(A)]^\#$ defined, according to (1.1), by

$$[\mathcal{P}(A)]^\# = \left\{X \in B^{\mathcal{P}(A)} : \{X(E)\}_{E \in \mathcal{P}(A)} \; (= \{X(E)\}_{E \subset A}) \in \mathrm{Dcp}\,(\mathbf{1}_B)\right\}. \qquad (1.2)$$

(iii) *External* $\mathbb{B}$*-fuzzy subset of (the set)* A or: external $\mathbb{B}$-fuzzy set over A is a classical (crisp) subset of the set $A^\#$. Hence, the set of all external $\mathbb{B}$-fuzzy sets over A is the power-set $\mathcal{P}(A^\#)$ of the set $A^\#$ of $\mathbb{B}$-fuzzy elements of A.

Fact 1.1. In general, $[\mathcal{P}(A)]^\#$ is a proper subset of $\mathcal{P}(A^\#)$. Hence, each internal $\mathbb{B}$-fuzzy subset of A is also an external one, but, in general, not vice versa.

Proof. Cf. [2] and references introduced there. □

In the rest of this paper we shall investigate some relations between internal and classical $\mathbb{B}$-fuzzy sets over A and between external and classical $\mathbb{B}$-fuzzy sets over A. In the extensional setting, we shall investigate the relations between $[\mathcal{P}(A)]^\#$ and A^*, and between $\mathcal{P}(A^\#)$ and A^*.

2. MUTUAL EMBEDDINGS OF INTERNAL AND CLASSICAL $\mathbb{B}$-FUZZY SETS

Let h_{ic} be a mapping defined on $[\mathcal{P}(A)]^\#$, taking its values in A^* and such that, for each internal $\mathbb{B}$-fuzzy set (over A) $X \in [\mathcal{P}(A)]^\#$, $h_{ic}(X)$, denoted also by $X^{(*)}$, is the classical $\mathbb{B}$-fuzzy set (over A) defined by

$$h_{ic}(X)(a) = X^{(*)}(a) = \bigvee_{E \subset A,\, a \in E} X(E) \qquad (2.1)$$

for all $a \in A$; here *ic* abbreviates "internal to classical". It is evident that $X^{(*)} \in A^*$, moreover, the mapping h_{ic} is one-to-one, as Theorem 2.1 proves.

Theorem 2.1. Let $X_1, X_2 \in [\mathcal{P}(A)]^\#$, let $X_1 \neq X_2$, then $X_1^{(*)} \neq X_2^{(*)}$.

Proof. First, let us prove a more general auxiliary assertion: let $C \subset B$, $C \in \mathrm{Dcp}\,(\mathbf{1}_B)$, let F be a nonempty set, let $\emptyset \neq \mathcal{U}(A) \subset C$ for each $a \in F$, then

$$\bigwedge_{a \in F} \bigvee_{e \in \mathcal{U}(a)} e = \bigvee_{e \in \bigcap_{a \in F} \mathcal{U}(a)} e. \qquad (2.2)$$

Write $\bigcap \mathcal{U}(a)$ instead of $\bigcap_{a \in F} \mathcal{U}(a)$. As $\bigcap \mathcal{U}(a) \subset \mathcal{U}(a)$ for each $a \in F$, we obtain $\bigvee_{e \in \bigcap \mathcal{U}(a)} e \leq \bigvee_{e \in \mathcal{U}(a)} e$ for each $a \in F$, so that $\bigvee_{e \in \bigcap \mathcal{U}(a)} e \leq \bigwedge_{a \in F} \bigvee_{e \in \mathcal{U}(a)} e$. Let $f \in$

$C-\bigcap\mathcal{U}(a)$, then $f \in C-\mathcal{U}(a_f)$ for some $a_f \in F$, hence, $f \wedge \bigvee_{e\in\mathcal{U}(a_f)} e = \bigvee_{e\in\mathcal{U}(a_f)}(f \wedge e) = \mathbf{0}_B$, as $f \neq e$ for each $e \in \mathcal{U}(a_f)$, consequently, $f \wedge e = \mathbf{0}_B$. So, for $f \in C - \bigcap\mathcal{U}(a)$

$$\bigwedge_{a\in F}\left(f \wedge \bigvee_{e\in\mathcal{U}(a)} e\right) = f \wedge \left(\bigwedge_{a\in F}\bigvee_{e\in\mathcal{U}(a)} e\right) = \mathbf{0}_B, \tag{2.3}$$

so that

$$\bigvee_{f\in C-\bigcap\mathcal{U}(a)}\left(f \wedge \bigwedge_{a\in F}\bigvee_{e\in\mathcal{U}(a)} e\right) = \left(\bigvee_{f\in C-\bigcap\mathcal{U}(a)} f\right) \wedge \left(\bigwedge_{a\in F}\bigvee_{e\in\mathcal{U}(a)} e\right) = \mathbf{0}_B. \tag{2.4}$$

Hence,

$$\begin{aligned} \bigwedge_{a\in F}\bigvee_{e\in\mathcal{U}(a)} e = \left(\bigwedge_{a\in F}\bigvee_{e\in\mathcal{U}(a)} e\right) \wedge \mathbf{1}_B = \qquad\qquad (2.5)\\ = \left(\bigwedge_{a\in F}\bigvee_{e\in\mathcal{U}(a)} e\right) \wedge \left(\bigvee_{f\in C-\bigcap\mathcal{U}(a)} f \vee \bigvee_{f\in\bigcap\mathcal{U}(a)} f\right) = \\ = \left(\left(\bigwedge_{a\in F}\bigvee_{e\in\mathcal{U}(a)} e\right) \wedge \left(\bigvee_{f\in C-\bigcap\mathcal{U}(a)} f\right)\right) \vee \left(\left(\bigwedge_{a\in F}\bigvee_{e\in\mathcal{U}(a)} e\right) \wedge \left(\bigvee_{f\in\bigcap\mathcal{U}(a)} f\right)\right) = \\ = \mathbf{0}_B \vee \bigvee_{f\in\bigcap\mathcal{U}(a)} f = \bigvee_{e\in\bigcap\mathcal{U}(a)} e, \end{aligned}$$

as $\bigvee_{e\in\bigcap\mathcal{U}(a)} e \leq \bigwedge_{a\in F}\bigvee_{e\in\mathcal{U}(a)} e$ is nothing else than

$$\left(\bigwedge_{a\in F}\bigvee_{e\in\mathcal{U}(a)} e\right) \wedge \left(\bigvee_{e\in\bigcap\mathcal{U}(a)} e\right) = \bigvee_{e\in\bigcap\mathcal{U}(a)} e, \tag{2.6}$$

hence, (2.2) is proved.

Now, let us prove that, for each $X \in [\mathcal{P}(A)]^{\#}$ and each $F \subset A$,

$$X(F) = \bigwedge_{a\in F} X^{(*)}(a) \wedge \bigwedge_{a\in A-F} \neg X^{(*)}(a). \tag{2.7}$$

Or, supposing that (2.7) holds, and considering X_1, $X_2 \in [\mathcal{P}(A)]^{\#}$ such that $X_1^{(*)}(a) = X_2^{(*)}(a)$ for all $a \in A$, we obtain that $X_1(F) = X_2(F)$ for all $F \subset A$. Hence, if there exists $F \subset A$ such that $X_1(F) \neq X_2(F)$, it must also exist $a \in A$ such that $X_1^{(*)}(a) \neq X_2^{(*)}(a)$.

Due to (2.1), (2.7) yields that

$$X(F) = \bigwedge_{a\in F}\bigvee_{E\subset A,\, a\in E} X(E) \wedge \bigwedge_{a\in A-F} \neg \bigvee_{E\subset A,\, a\in E} X(E). \tag{2.8}$$

As $\bigcap_{a\in F}\{E : E \subset A, a \in E\} = \{E : E \subset A, a \in E, \text{ for all } a \in F\} = \{E : F \subset E\}$, (2.2) yields that

$$\bigwedge_{a\in F} \bigvee_{E\subset A, a\in F} X(E) = \bigvee_{E\supset F} X(E). \tag{2.9}$$

Moreover,

$$\bigcup_{a\in A-F} \{E : E \subset A, a \in E\} = \{E : E \subset A, E \not\subset F\}, \tag{2.10}$$

so that

$$\bigwedge_{a\in A-F} \neg \bigvee_{E\subset A, a\in E} X(E) = \neg \bigvee_{a\in A-F} \bigvee_{E\subset A, a\in E} X(E) = \neg \bigvee_{E\not\subset F} X(E). \tag{2.11}$$

Consequently, (2.7) is equivalent to

$$X(F) = \bigvee_{E\supset F} X(E) \wedge \neg \bigvee_{E\not\subset F} X(E) \tag{2.12}$$

which we are to prove. But,

$$\bigvee_{E\supset F} X(E) \wedge \neg \bigvee_{E\not\subset F} X(E) = \bigvee_{G\supset F} \left[X(G) \wedge \bigwedge_{E\not\subset F} \neg X(E) \right]. \tag{2.13}$$

If $G \supset F$, $G \neq F$, then $G \not\subset F$, hence, $G \in \{E : E \not\subset F\}$, so that

$$X(G) \wedge \bigwedge_{E\not\subset F} \neg X(E) = X(G) \wedge \neg X(G) \wedge \bigwedge_{E\not\subset F, E\neq G} \neg X(E) = \mathbf{0}_B, \tag{2.14}$$

so that

$$\begin{aligned} \bigvee_{E\supset F} X(E) \wedge \neg \bigvee_{E\not\subset F} X(E) &= \bigvee_{G\in\{F\}} \left[X(G) \wedge \bigwedge_{E\not\subset F} \neg X(E) \right] = \\ = X(F) \wedge \bigwedge_{E\not\subset F} \neg X(E) &= \bigwedge_{E\not\subset F} (X(F) \wedge \neg X(E)) = X(F), \end{aligned} \tag{2.15}$$

as $E \not\subset F$ implies $E \neq F$, so that $X(E) \wedge X(F) = \mathbf{0}_B$, hence

$$\begin{aligned} X(F) \wedge \neg X(E) &= (X(F) \wedge \neg X(E)) \vee (X(F) \wedge X(E)) = \\ = X(F) \wedge (\neg X(E) \vee X(E)) &= X(F) \wedge \mathbf{1}_B = X(F) \end{aligned} \tag{2.16}$$

for each $E \not\subset F$. Hence, (2.12), (2.7) and Theorem 1 are proved. □

When trying to define a mapping from classical $\mathbb{B}$-fuzzy sets into the internal ones we shall use relation (2.7). So, let h_{ci} be the mapping defined on A^*, taking its values in the set $B^{\mathcal{P}(A)}$ of all mappings from $\mathcal{P}(A)$ into B, and such that, for each $\mathcal{X} \in A^*$ and each $F \subset A$,

$$h_{ci}(\mathcal{X})(F) = \bigwedge_{a\in F} \mathcal{X}(a) \wedge \bigwedge_{a\in A-F} \neg \mathcal{X}(a). \tag{2.17}$$

We shall write also $\mathcal{X}^{(\#)}$ nd $\mathcal{X}^{(\#)}(F)$ instead of $h_{ci}(\mathcal{X})$ and $h_{ci}(\mathcal{X})(F)$. Here, again, ci abbreviates "classical to internal", but in this time it is not evident that $\mathcal{X}^{(\#)}$ is in $[\mathcal{P}(A)]^{\#}$ so that we have to prove it.

The complete Boolean algebra $\mathbb{B}$ is called *completely set-isomorphic*, if there exists a set S and a mapping H defined on B, taking its values in the power-set $\mathcal{P}(S)$ over S and such that, for each $e \in B$ and each $C \subset B$,

$$H(\neg e) = S - H(e), \qquad H\left(\bigvee_{e \in C} e\right) = \bigcup_{e \in C} H(e). \tag{2.18}$$

Consequently, also

$$H(\mathbf{0}_B) = \emptyset, \qquad H(\mathbf{1}_B) = S, \qquad H\left(\bigwedge_{e \in C} e\right) = \bigcap_{e \in C} H(e) \tag{2.19}$$

hold, as can be easily proved. When considering only finite operations, i. e., finite sets C in (2.18) and (2.19), each Boolean algebra is finitely set-isomorphic, due to the well-known Stone representation theorem (cf., e. g., [5], § 8). For infinite operations the existence of an isomorphism between a complete Boolean algebra and the field of subsets of a set is a nontrivial property of the Boolean algebra in question, as the two facts introduced below demonstrate.

An element $\mathbf{0}_B \neq e \in B$ is called an *atom* of the Boolean algebra $\mathbb{B}$, if for each $f \in B$ such that $f \leq e$, either $f = \mathbf{0}_B$ or $f = e$. Boolean algebra $\mathbb{B}$ is called *atomic*, if for each $\mathbf{0}_B \neq f \in B$ there exists an atom $e \in B$ such that $e \leq f$. Complete Boolean algebra $\mathbb{B}$ is called *completely distributive*, if for each $\{e_{ts}\}_{t \in T, s \in S} \subset B$,

$$\bigcap_{t \in T} \bigcup_{s \in S} e_{ts} = \bigcup_{\varphi \in S^T} \bigcap_{t \in T} e_{t,\varphi(t)}, \tag{2.20}$$

or, what is the same due to the fact that de Morgan rules are valid also for infinite operations, if for each $\{e_{ts}\}_{t \in T, s \in S} \subset B$,

$$\bigcup_{t \in T} \bigcap_{s \in S} e_{ts} = \bigcap_{\varphi \in S^T} \bigcup_{t \in T} e_{t,\varphi(t)}. \tag{2.21}$$

Fact 2.1. A complete Boolean algebra is completely set-isomorphic iff it is atomic (cf. assertion 25.1, [5], § 25).

Fact 2.2. A complete Boolean algebra is completely set-isomorphic, iff it is completely distributive (cf. assertion 25.2, [5], § 25).

Theorem 2.2. Let $\mathbb{B}$ be completely set-isomorphic, then $\mathcal{X}^{(\#)} \in [\mathcal{P}(A)]^{\#}$ for each $\mathcal{X} \in A^*$.

Proof. We have to prove that $\{\mathcal{X}^{(\#)}(F)\}_{F\subset A} \in \mathrm{Dcp}(\mathbf{1}_B)$. Let $E, F \subset A$, $E \neq F$, then (2.17) yields that

$$\begin{aligned}\mathcal{X}^{(\#)}(E) \wedge \mathcal{X}^{(\#)}(F) &= \bigwedge_{a\in E} \mathcal{X}(a) \wedge \bigwedge_{a\in A-E} \neg\mathcal{X}(a) \wedge \bigwedge_{a\in F} \mathcal{X}(a) \\ \wedge \bigwedge_{a\in A-F} \neg\mathcal{X}(a) &= \bigwedge_{a\in E\cup F} \mathcal{X}(a) \wedge \bigwedge_{a\in A-(E\cap F)} \neg\mathcal{X}(a) \leq \\ &\leq \mathcal{X}(a_0) \wedge \neg\mathcal{X}(a_0) = \mathbf{0}_B,\end{aligned} \tag{2.22}$$

where $a_0 \in (E \cup F) - (E \cap F)$; $E \neq F$ implies that $E \cup F \neq E \cap F$, hence, such an a_0 exists.

Let S be a set and H a complete isomorphic mapping of B into $\mathcal{P}(S)$. So,

$$\begin{aligned}H\left(\bigvee_{F\subset A}\left(\bigwedge_{a\in F}\mathcal{X}(a) \wedge \bigwedge_{a\in A-F}\neg\mathcal{X}(a)\right)\right) &= \\ = \bigcup_{F\subset A}\left(\bigcap_{a\in F} H(\mathcal{X}(a)) \cap \bigcap_{a\in A-F}(S - H(\mathcal{X}(a)))\right).\end{aligned} \tag{2.23}$$

Let $s_0 \in S$, let $F(s_0) = \{a \in A : s_0 \in H(\mathcal{X}(a))\}$, then $s_0 \in \bigcap_{a\in F(s_0)} H(\mathcal{X}(a))$, but also $s_0 \notin H(\mathcal{X}(a))$ for each $a \in A - F(s_0)$, so that $s_0 \in \bigcap_{a\in A-F(s_0)}(S - H(\mathcal{X}(a)))$. So, for each $s_0 \in S$ there exists $F \subset A$ such that $s_0 \in H\left(\mathcal{X}^{(\#)}(F)\right)$. Consequently,

$$\begin{aligned}\bigcup_{F\subset A} H\left(\mathcal{X}^{(\#)}(F)\right) = H\left(\bigvee_{F\subset A}\mathcal{X}^{(\#)}(F)\right) &= \\ = \bigcup_{F\subset A}\left(\bigcap_{a\in F} H(\mathcal{X}(a)) \cap \bigcap_{a\in A-F}(S - H(\mathcal{X}(a)))\right) &= S.\end{aligned} \tag{2.24}$$

However, $\mathbf{1}_B$ is the only element of B which is mapped onto S by H. Hence, it follows $\bigvee_{F\subset A}\mathcal{X}^{(\#)}(F) = \mathbf{1}_B$, so that $\{\mathcal{X}^{(\#)}(F)\}_{F\subset A} \in \mathrm{Dcp}(\mathbf{1}_B)$. The assertion is proved. □

Theorem 2.3. Let $\mathbb{B}$ be completely set-isomorphic, let $\mathcal{X}_1, \mathcal{X}_2 \in A^*$, let $\mathcal{X}_1 \neq \mathcal{X}_2$, then $\mathcal{X}_1^{(\#)} \neq \mathcal{X}_2^{(\#)}$.

Proof. Let us prove that, for each $\mathcal{X} \in A^*$ and each $a_0 \in A$,

$$\mathcal{X}(a_0) = \bigvee_{F\ni a_0}\mathcal{X}^{(\#)}(F) = \bigvee_{F\ni a_0}\left(\bigwedge_{a\in F}\mathcal{X}(a) \wedge \bigwedge_{a\in A-F}\neg\mathcal{X}(a)\right). \tag{2.25}$$

Supposing that (2.25) holds and that $\mathcal{X}_1^{(\#)}(F) = \mathcal{X}_2^{(\#)}(F)$ for each $F \subset A$, then $\mathcal{X}_1(a_0) = \mathcal{X}_2(a_0)$ for each $a_0 \in A$, hence, $\mathcal{X}_1(a) \neq \mathcal{X}_2(a)$ for some $a \in A$ implies that $\mathcal{X}_1^{(\#)}(F) \neq \mathcal{X}_2^{(\#)}(F)$ for some $F \subset A$.

Take $a_0 \in A$ arbitrarily and set $A_0 = A - \{a_0\}$. Define a $\mathbb{B}$-fuzzy subset $\mathcal{X}_0$ of A_0 (classical), setting $\mathcal{X}_0(a) = \mathcal{X}(a)$ for each $a \in A_0$, so that $\mathcal{X}_0 \in A_0^* = B^{A_0}$. Set, for each $E \subset A_0$,

$$\mathcal{X}_0^{(\#)}(E) = \bigwedge_{a\in E} \mathcal{X}_0(a) \wedge \bigwedge_{a\in A_0-E} \neg\mathcal{X}_0(a). \tag{2.26}$$

Applying Theorem 2.2 to A_0 and $\mathcal{X}_0$ we obtain that

$$\bigvee_{E\subset A_0} \mathcal{X}_0^{(\#)}(E) = \bigvee_{E\subset A_0} \left(\bigwedge_{a\in E} \mathcal{X}_0(a) \wedge \bigwedge_{a\in A_0-E} \neg\mathcal{X}_0(a)\right) = \mathbf{1}_B. \tag{2.27}$$

But,

$$\bigvee_{F\subset A,\, F\ni a_0} \mathcal{X}^{(\#)}(F) = \bigvee_{F\subset A,\, F\ni a_0} \left(\bigwedge_{a\in F} \mathcal{X}(a) \wedge \bigwedge_{a\in A-F} \neg\mathcal{X}(a)\right) = \tag{2.28}$$

$$= \bigvee_{E\subset A_0} \mathcal{X}^{(\#)}(E\cup\{a_0\}) = \bigvee_{E\subset A_0} \left(\left(\bigwedge_{a\in E} \mathcal{X}(a)\right) \wedge \mathcal{X}(a_0) \wedge\right.$$

$$\left.\wedge \bigwedge_{a\in(A-E)-\{a_0\}} \neg\mathcal{X}(a)\right).$$

As $(A-E)-\{a_0\} = A_0 - E$ for each $E \subset A_0$ and $\mathcal{X}(a) = \mathcal{X}_0(a)$ for each $a \in E \subset A_0$ and each $a \in A_0 - E$, (2.28) yields that

$$\bigvee_{F\subset A,\, F\ni a_0} \mathcal{X}^{(\#)}(F) = \mathcal{X}(a_0) \wedge \bigvee_{E\subset A_0} \left(\bigwedge_{a\in E} \mathcal{X}_0(a) \wedge\right. \tag{2.29}$$

$$\left.\wedge \bigwedge_{a\in A_0-E} \neg\mathcal{X}_0(a)\right) = \mathcal{X}(a_0) \wedge \mathbf{1}_B = \mathcal{X}(a_0)$$

due to (2.27), so that (2.25) and Theorem 2.3 are proved. □

3. SET-THEORETIC OPERATIONS OVER CLASSICAL AND INTERNAL $\mathbb{B}$-FUZZY SETS

Going on in our effort to compare classical and internal $\mathbb{B}$-fuzzy sets, let us introduce the set-theoretic operations of joint (intersection), union and complement for such sets. We shall begin with the classical $\mathbb{B}$-valued fuzzy sets, where the situation is more simple.

Definition 3.1. Let $\mathcal{X} \in A^*$ be a classical $\mathbb{B}$-fuzzy set over A, then its *complement* is denoted by $\mathcal{X}^c$ and defined by $\mathcal{X}^c(a) = \neg\mathcal{X}(a)$ for each $a \in A$. Let $C \subset A^*$ be a nonempty set of classical $\mathbb{B}$-fuzzy sets over A, then the *joint* or *intersection* of the $\mathbb{B}$-fuzzy sets from C is denoted by $\bigcap_{\mathcal{X}\in C} \mathcal{X}$ and defined by $\left(\bigcap_{\mathcal{X}\in C} \mathcal{X}\right)(a) = \bigwedge_{\mathcal{X}\in C} \mathcal{X}(a)$ for each $a \in A$. The *union* of the $\mathbb{B}$-fuzzy sets from C is denoted by $\bigcup_{\mathcal{X}\in C}\mathcal{X}$ and defined by $\left(\bigcup_{\mathcal{X}\in C}\mathcal{X}\right)(a) = \bigvee_{\mathcal{X}\in C}\mathcal{X}(a)$ for each $a \in A$.

As the Boolean algebra $\mathbb{B}$ is complete, $\mathcal{X}^c$, $\bigcap_{\mathcal{X}\in C}\mathcal{X}$, and $\bigcup_{\mathcal{X}\in C}\mathcal{X}$ are obviously in A^*, also the validity of de Morgan rules for these operations immediately follows from the validity of these rules in $\mathbb{B}$. If $C = \{\mathcal{X}_1, \mathcal{X}_2\}$, we shall write $\mathcal{X}_1 \cap \mathcal{X}_2$ and $\mathcal{X}_1 \cup \mathcal{X}_2$ instead of $\bigcap_{\mathcal{X}\in\{\mathcal{X}_1,\mathcal{X}_2\}}\mathcal{X}$ and $\bigcup_{\mathcal{X}\in\{\mathcal{X}_1,\mathcal{X}_2\}}\mathcal{X}$.

In the case of internal $\mathbb{B}$-fuzzy sets the corresponding definitions are more complicated in the sense that the membership of the resulting objects in the space $[\mathcal{P}(A)]^\#$ of internal $\mathbb{B}$-fuzzy sets is neither necessary nor evident.

Definition 3.2. Let $X \in [\mathcal{P}(A)]^\#$ be an internal $\mathbb{B}$-fuzzy set over A, then its *complement* will be denoted by X^c and defined by $X^c(E) = X(A - E)$ for each $E \subset A$. Let T be a nonempty parametric set, let $\{X_t\}_{t\in T} \subset [\mathcal{P}(A)]^\#$, i.e., each X_t, $t \in T$, is an internal $\mathbb{B}$-fuzzy set over A, then the *joint* or *intersection* of the $\mathbb{B}$-fuzzy sets X_t is denoted by $\cap_{t\in T}X_t$ and defined, for each $E \subset A$, by

$$\left(\bigcap_{t\in T} X_t\right)(E) = \bigvee_{\{F_t\}_{t\in T}\subset\mathcal{P}(A),\ \bigcap_{t\in T}F_t = E}\ \bigwedge_{t\in T} X_t(F_t). \tag{3.1}$$

The *union* of the $\mathbb{B}$-fuzzy sets X_t is denoted by $\bigcup_{t\in T}X_t$ and defined, for each $E \subset A$, by

$$\left(\bigcup_{t\in T} X_t\right)(E) = \left(\bigcap_{t\in T} X_t^c\right)^c (E). \tag{3.2}$$

If $T = \{1, 2\}$, we shall write $X_1 \cap X_2$ and $X_1 \cup X_2$ instead of $\bigcap_{t\in\{1,2\}} X_t$ and $\bigcup_{t\in\{1,2\}} X_t$, as can be easily seen,

$$(X_1 \cap X_2)(E) = \bigvee_{F,G\subset A,\ F\cap G=E} (X_1(F) \wedge X_2(G)). \tag{3.3}$$

Theorem 3.1. For each X, X_1, $X_2 \in [\mathcal{P}(A)]^\#$, X^c, $X_1 \cap X_2$, and $X_1 \cup X_2$ are also in $[\mathcal{P}(A)]^\#$. If the Boolean algebra $\mathbb{B}$ is completely distributive, then $\bigcap_{t\in T} X_t$ and $\bigcup_{t\in T} X_t$ are in $[\mathcal{P}(A)]^\#$ for each nonempty parametric set T.

Proof. If E, $F \subset A$, $E \neq F$, then $A - E \neq A - F$, so that $X^c(E) \wedge X^c(F) = X(A - E) \wedge X(A - F) = \mathbf{0}_B$, moreover, $\bigvee_{E\subset A} X^c(E) = \bigvee_{E\subset A} X(A - E) = \bigvee_{E\subset A} X(E) = \mathbf{1}_B$, so that $\{X^c(E)\}_{E\subset A} \in \mathrm{Dcp}(\mathbf{1}_B)$, i.e., $X^c \in [\mathcal{P}(A)]^\#$.

Let X_1, $X_2 \in [\mathcal{P}(A)]^\#$, let E_1, $E_2 \subset A$, $E_1 \neq E_2$, then

$$\begin{aligned} &(X_1 \cap X_2)(E_1) \wedge (X_1 \cap X_2)(E_2) = \\ =& \bigvee_{F_1,G_1\subset A,\ F_1\cap G_1=E_1} (X_1(F_1) \wedge X_2(G_1)) \wedge \end{aligned} \tag{3.4}$$

$$\wedge \bigvee_{F_2,G_2\subset A,\, F_2\cap G_2=E_2} (X_1(F_2)\wedge X_2(G_2)) =$$

$$= \bigvee_{F_1,G_1\subset A,\, F_1\cap G_1=E_1} \bigvee_{F_2,G_2\subset A,\, F_2\cap G_2=E_2} (X_1(F_1)\wedge X_2(G_1)\wedge X_1(F_2)\wedge X_2(G_2)).$$

But, if $E_1 \neq E_2$, $F_1\cap G_1 = E_1$, and $F_2\cap G_2 = E_2$, then either $F_1\neq F_2$, or $G_1\neq G_2$, so that either $X_1(F_1)\wedge X_1(F_2) = \mathbf{0}_B$, or $X_2(G_1)\wedge X_2(G_2) = \mathbf{0}_B$, consequently, $X_1(F_1)\wedge X_2(G_1)\wedge X_1(F_2)\wedge X_2(G_2) = \mathbf{0}_B$ in every case. Hence,

$$(X_1\cap X_2)(E_1)\wedge (X_1\cap X_2)(E_2) = \mathbf{0}_B. \tag{3.5}$$

Moreover,

$$\bigvee_{E\subset A}(X_1\cap X_2)(E) = \bigvee_{E\subset A}\left(\bigvee_{F,G\subset A,\, F\cap G=E}(X_1(F)\wedge X_2(G))\right) = \tag{3.6}$$

$$= \bigvee_{F,G\subset A}(X_1(F)\wedge X_2(G)) = \bigvee_{F\subset A}\bigvee_{G\subset A}(X_1(F)\wedge X_2(G)) =$$

$$= \bigvee_{F\subset A}\left(X_1(F)\wedge \bigvee_{G\subset A} X_2(G)\right) = \bigvee_{F\subset A}(X_1(F)\wedge \mathbf{1}_B) =$$

$$= \bigvee_{F\subset A} X_1(F) = \mathbf{1}_B,$$

hence, $\{(X_1\cap X_2)(E)\}_{E\subset A}\in \mathrm{Dcp}\,(\mathbf{1}_B)$, so that $X_1\cap X_2\in[\mathcal{P}(A)]^{\#}$.

Let $\{X_t\}_{t\in T}\subset[\mathcal{P}(A)]^{\#}$, let $E_1,\, E_2\subset A,\; E_1\neq E_2$, then

$$\left(\bigcap_{t\in T}X_t\right)(E_1)\wedge\left(\bigcap_{t\in T}X_t\right)(E_2) = \tag{3.7}$$

$$= \bigvee_{\{F_t\}_{t\in T}\subset\mathcal{P}(A),\,\bigcap_{t\in T}F_t=E_1} g(\{F_t\}_{t\in T})\wedge$$

$$\wedge \bigvee_{\{G_t\}_{t\in T}\subset\mathcal{P}(A),\,\bigcap_{t\in T}G_t=E_2} h(\{G_t\}_{t\in T})$$

where

$$g(\{F_t\}_{t\in T}) = \bigwedge_{t\in T}X_t(F_t), \tag{3.8}$$

$$h(\{G_t\}_{t\in T}) = \bigwedge_{t\in T}X_t(G_t).$$

So,

$$\left(\bigcap_{t\in T}X_t\right)(E_1)\wedge\left(\bigcap_{t\in T}X_t\right)(E_2) = \tag{3.9}$$

$$= \bigvee_{\{F_t\}_{t\in T}\subset\mathcal{P}(A),\bigcap_{t\in T}F_t=E_1} \bigvee_{\{G_t\}_{t\in T}\subset\mathcal{P}(A),\bigcap_{t\in T}G_t=E_2} (g(\{F_t\}_{t\in T}) \wedge h(\{G_t\}_{t\in T})) =$$

$$= \bigvee_{\{F_t\}}\bigvee_{\{G_t\}}\left(\bigwedge_{t\in T}(X_t(F_t)\wedge X_t(G_t))\right) = \mathbf{0}_B,$$

as if $\bigcap_{t\in T} F_t = E_1$, $\bigcap_{t\in T} G_t = E_2$, and $E_1 \neq E_2$, then there exists $t_0 \in T$ such that $F_{t_0} \neq G_{t_0}$, so $X_{t_0}(F_{t_0}) \wedge X_{t_0}(G_{t_0}) = \mathbf{0}_B$, consequently, $\bigwedge_{t\in T} X_t(F_t) \wedge X_t(G_t) = \mathbf{0}_B$.

Finally,

$$\bigvee_{E\subset A}\left(\bigcap_{t\in T}X_t\right)(E) = \tag{3.10}$$

$$= \bigvee_{E\subset A}\bigvee_{\{F_t\}_{t\in T}\subset\mathcal{P}(A),\bigcap_{t\in T}F_t=E}\left(\bigwedge_{t\in T}X_t(F_t)\right) =$$

$$= \bigvee_{\{F_t\}_{t\in T}\subset\mathcal{P}(A)}\bigwedge_{t\in T}X_t(F_t) = \bigvee_{\varphi\in\mathcal{P}(A)^T}\bigwedge_{t\in T}X_t(\varphi_t).$$

If $\mathbb{B}$ is completely distributive, then for each nonempty parametric sets T, S, and each $\{e_{ts}\}_{t\in T, s\in S} \subset B$,

$$\bigvee_{\varphi\in S^T}\bigwedge_{t\in T}e_{t,\varphi(t)} = \bigwedge_{t\in T}\bigvee_{s\in S}e_{ts}. \tag{3.11}$$

Setting $S = \mathcal{P}(A)$ and $e_{tE} = X_t(E)$ for each $t \in T$, $E \subset A$, i.e., $E \in S$, we obtain from (3.10), that

$$\bigvee_{E\subset A}\left(\bigcap_{t\in T}X_t\right)(E) = \bigwedge_{t\in T}\bigvee_{E\subset A}X_t(E) = \mathbf{1}_B, \tag{3.12}$$

as $\bigvee_{E\subset A} X_t(E) = \mathbf{1}_B$ for each $t \in T$. Hence,

$$\left\{\left(\bigcap_{t\in T}X_t\right)(E)\right\}_{E\subset A} \in \mathrm{Dcp}\,(\mathbf{1}_B), \tag{3.13}$$

so that $\bigcap_{t\in T} X_t \in [\mathcal{P}(A)]^{\#}$. The same assertion for $\bigcup_{t\in T} X_t$ follows immediately from (3.2). □

Let us recall that external $\mathbb{B}$-fuzzy sets over A are defined as classical (crisp) sets of fuzzy elements of A, i.e., as elements of $\mathcal{P}(A^{\#})$, so that the set-theoretic operations over external sets are defined in the usual way.

4. COMPATIBILITY OF SET-THEORETIC OPERATIONS AND EMBEDDINGS FOR CLASSICAL AND INTERNAL B-FUZZY SETS

In this chapter we shall prove that, roughly speaking, the set-theoretic operations commute with the mappings h_{ic} and h_{ci} defined and investigated above. In other words, we shall prove that these mappings define homomorphic embeddings between classical and internal $\mathbb{B}$-fuzzy sets over the same basic set A. A more formal description of what this commutativity means is given in the conditions of the assertions introduced and proved below.

Theorem 4.1. Let $X, Y \in [\mathcal{P}(A)]^{\#}$, then

$$(X^c)^{(*)} = \left(X^{(*)}\right)^c, \qquad (X \cup Y)^{(*)} = X^{(*)} \cup Y^{(*)}, \qquad (X \cap Y)^{(*)} = X^{(*)} \cap Y^{(*)}. \tag{4.1}$$

If the Boolean algebra $\mathbb{B}$ is completely distributive and if $X_t \in [\mathcal{P}(A)]^{\#}$ for each $t \in T \neq \emptyset$, then

$$\left(\bigcup_{t \in T} X_t\right)^{(*)} = \bigcup_{t \in T} X_t^{(*)}, \qquad \left(\bigcap_{t \in T} X_t\right)^{(*)} = \bigcap_{t \in T} X_t^{(*)}. \tag{4.2}$$

Proof. Let $X \in [\mathcal{P}(A)]^{\#}$, let $a \in A$, then

$$\begin{aligned} (X^c)^{(*)}(a) &= \bigvee_{E \ni a} X^c(E) = \bigvee_{E \ni a} X(A - E) = \bigvee_{E \not\ni a} X(E) = \\ &= \neg \bigvee_{E \ni a} X(E) = \neg X^{(*)}(a) = \left(X^{(*)}\right)^c(a), \end{aligned} \tag{4.3}$$

as

$$\left(\bigvee_{E \ni a} X(E)\right) \vee \left(\bigvee_{E \not\ni a} X(E)\right) = \mathbf{1}_B, \tag{4.4}$$

$$\left(\bigvee_{E \ni a} X(E)\right) \wedge \left(\bigvee_{E \not\ni a} X(E)\right) = \mathbf{0}_B, \tag{4.5}$$

so that $(X^c)^{(*)} = \left(X^{(*)}\right)^c$.

Let $X, Y \in [\mathcal{P}(A)]^{\#}$, let $a \in A$, then

$$(X \cap Y)^{(*)}(a) = \bigvee_{E \ni a} (X \cap Y)(E) = \tag{4.6}$$

$$\begin{aligned} &= \bigvee_{E \ni a} \left(\bigvee_{F, G \subset A,\, F \cap G = E} (X(F) \wedge Y(G)) \right) = \\ &= \bigvee_{F, G \subset A,\, F \cap G \ni a} (X(F) \wedge Y(G)) = \bigvee_{F \ni a,\, G \ni a} (X(F) \wedge Y(G)) = \\ &= \bigvee_{F \ni a} \bigvee_{G \ni a} (X(F) \wedge Y(G)) = \bigvee_{F \ni a} \left(X(F) \wedge \bigvee_{G \ni a} Y(G) \right) = \\ &= \left(\bigvee_{F \ni a} X(F)\right) \wedge \left(\bigvee_{G \ni a} Y(G)\right) = X^{(*)}(a) \wedge Y^{(*)}(a) = \left(X^{(*)} \cap Y^{(*)}\right)(a), \end{aligned}$$

so that $(X \cap Y)^{(*)} = X^{(*)} \cap Y^{(*)}$. The assertion for $(X \cup Y)^{(*)}$ follows immediately from de Morgan rules.

In the general case, when $X_t \in [\mathcal{P}(A)]^{\#}$ for each $t \in T \neq \emptyset$, we obtain, for each $a \in A$, that

$$\left(\bigcap_{t\in T} X_t\right)^{(*)}(a) = \bigvee_{E\ni a}\left(\bigcap_{t\in T} X_t\right)(E) = \tag{4.7}$$
$$= \bigvee_{E\ni a}\ \bigvee_{\{F_t\}_{t\in T}\subset\mathcal{P}(A),\,\bigcap_{t\in T}F_t=E}\ \bigwedge_{t\in T} X_t(F_t) =$$
$$= \bigvee_{\{F_t\}_{t\in T}\subset\mathcal{P}(A),\,\bigcap_{t\in T}F_t\ni a}\ \bigwedge_{t\in T} X_t(F_t) =$$
$$= \bigvee_{\varphi\in S^T}\bigwedge_{t\in T} X_t(F_t),$$

where $S = \{E : a \in E \subset A\} \subset \mathcal{P}(A)$. If $\mathbb{B}$ is completely distributive, the last expression in (4.7) can be written as

$$\bigwedge_{t\in T}\bigvee_{E\in S} X_t(E) = \bigwedge_{t\in T}\ \bigvee_{E\ni a,\,E\subset A} X_t(E) = \tag{4.8}$$
$$= \bigwedge_{t\in T} X_t^{(*)}(a) = \left(\bigcap_{t\in T} X_t^{(*)}\right)(a),$$

so that $\left(\bigcap_{t\in T} X_t\right)^{(*)} = \bigcap_{t\in T} X_t^{(*)}$. The dual assertion for $\left(\bigcup_{t\in T} X_t\right)^{(*)}$ again follows immediately from de Morgan rules. The assertion is proved. □

Theorem 4.2. Let the Boolean algebra $\mathbb{B}$ be completely set-isomorphic, let $\mathcal{X} \in A^*$, let $\mathcal{X}_t \in A^*$ for each $t \in T \neq \emptyset$, then

$$(\mathcal{X}^c)^{(\#)} = (\mathcal{X}^{(\#)})^c, \qquad \left(\bigcup_{t\in T}\mathcal{X}_t\right)^{(\#)} = \bigcup_{t\in T}\mathcal{X}_t^{(\#)}, \tag{4.9}$$
$$\left(\bigcap_{t\in T}\mathcal{X}_t\right)^{(\#)} = \bigcap_{t\in T}\mathcal{X}_t^{(\#)}.$$

Proof. Let $\mathcal{X} \in A^*$, let $E \subset A$, then

$$(\mathcal{X}^c)^{(\#)}(E) = \bigwedge_{a\in E}\mathcal{X}^c(a) \wedge \bigwedge_{a\in A-E}\neg\mathcal{X}^c(a) = \tag{4.10}$$
$$= \bigwedge_{a\in E}\neg\mathcal{X}(a) \wedge \bigwedge_{a\in A-E}\mathcal{X}(a) = \mathcal{X}^{(\#)}(A-E) = \left(\mathcal{X}^{(\#)}\right)^c(E),$$

so that $(\mathcal{X}^c)^{(\#)} = \left(\mathcal{X}^{(\#)}\right)^c$.

Let $\mathcal{X}_t \in A^*$ for each $t \in T \neq \emptyset$, let $E \subset A$, then we obtain

$$\left(\bigcap_{t\in T} \mathcal{X}_t\right)^{(\#)}(E) = \bigwedge_{a\in E}\left(\bigcap_{t\in T}\mathcal{X}_t\right)(a) \wedge \bigwedge_{a\in A-E} \neg\left(\bigcap_{t\in T}\mathcal{X}_t\right)(a) = \tag{4.11}$$
$$= \bigwedge_{a\in E}\bigwedge_{t\in T}\mathcal{X}_t(a) \wedge \bigwedge_{a\in A-E}\neg\left(\bigwedge_{t\in T}\mathcal{X}_t(a)\right).$$

From the other side we obtain that

$$\left(\bigcap_{t\in T}\mathcal{X}_t^{(\#)}\right)(E) = \bigvee_{\{F_t\}_{t\in T}\subset\mathcal{P}(A),\bigcap_{t\in T}F_t=E}\left(\bigwedge_{t\in T}\mathcal{X}_t^{(\#)}(F_t)\right) = \tag{4.12}$$
$$= \bigvee_{\{F_t\}_{t\in T}\subset\mathcal{P}(A),\bigcap_{t\in T}F_t=E}\ \bigwedge_{t\in T}\left[\bigwedge_{a\in F_t}\mathcal{X}_t(a)\wedge\bigwedge_{a\in A-F_t}\neg\mathcal{X}_t(a)\right] =$$
$$= \bigvee_{\{F_t^0\}_{t\in T}\subset\mathcal{P}(A-E),\bigcap_{t\in T}F_t^0=\emptyset}\ \bigwedge_{t\in T}\left[\bigwedge_{a\in E\cup F_t^0}\mathcal{X}_t(a)\wedge\bigwedge_{a\in(A-E)-F_t^0}\neg\mathcal{X}_t(a)\right] =$$
$$= \bigwedge_{a\in E}\bigwedge_{t\in T}\mathcal{X}_t(a)\wedge$$
$$\wedge \bigvee_{\{F_t^0\}_{t\in T}\subset\mathcal{P}(A-E),\bigcap_{t\in T}F_t^0=\emptyset}\ \bigwedge_{t\in T}\left(\bigwedge_{a\in F_t^0}\mathcal{X}_t(a)\wedge\bigwedge_{a\in(A-E)-F_t^0}\neg\mathcal{X}_t(a)\right),$$

so that the only we have to prove is that

$$\bigwedge_{a\in A-E}\neg\left(\bigwedge_{t\in T}\mathcal{X}_t(a)\right) = \bigwedge_{a\in A-E}\left(\bigvee_{t\in T}\neg\mathcal{X}_t(a)\right) = \tag{4.13}$$
$$= \bigvee_{\{F_t^0\}_{t\in T}\subset\mathcal{P}(A-E),\bigcap_{t\in T}F_t^0=\emptyset}\ \bigwedge_{t\in T}\left(\bigwedge_{a\in F_t^0}\mathcal{X}_t(a)\wedge\bigwedge_{a\in(A-E)-F_t^0}\neg\mathcal{X}_t(a)\right).$$

Supposing that $\mathbb{B}$ is completely set-isomorphic we may also suppose, without any loss of generality and in order to simplify our notation, that $\mathcal{X}_t(a)$ is a subset of a basic space S for each $t \in T$, $a \in A$, and that (4.13) converts into

$$H_1 = \bigcap_{a\in A-E}\bigcup_{t\in T}(S-\mathcal{X}_t(a)) = \tag{4.14}$$
$$= \bigcup_{\{F_t^0\}_{t\in T}\subset\mathcal{P}(A-E),\bigcap_{t\in T}F_t=\emptyset}\ \bigcap_{t\in T}\left(\bigcap_{a\in F_t^0}\mathcal{X}_t(a)\cap\bigcap_{a\in(A-E)-F_t^0}(S-\mathcal{X}_t(a))\right)$$
$$= H_2,$$

where H_1 and H_2 denote abbreviately the corresponding sets.

Let $s \in S$, $s \in H_1$, set, for each $t \in T$,

$$\alpha_t(s) = \{a \in A, s \in \mathcal{X}_t(a)\}, \qquad F_t^0(s) = \alpha_t(s) - E. \tag{4.15}$$

Obviously,

$$\bigcap_{t\in T} \alpha_t(s) = \left\{a \in A : s \in \bigcap_{t\in T} \mathcal{X}_t(a)\right\} = \tag{4.16}$$

$$= \left\{a \in A : s \in S - \bigcup_{t\in T}(S - \mathcal{X}_t(a))\right\} \subset E,$$

as for each $a \in A - E$, $s \in H_1$ yields that $s \in \bigcup_{t\in T}(S - \mathcal{X}_t(a))$. Hence, $\bigcap_{t\in T} F_t^0 = \bigcap_{t\in T}(\alpha_t(s) - E) = \emptyset$, so that $\{F_t^0(s)\}_{t\in T}$ is one of the sequences over which the union operation in H_2 is taken. Moreover, $F_t^0(s) \subset \alpha_t(s)$, so that $s \in \mathcal{X}_t(a)$ for each $a \in F_t^0(s)$ and each $t \in T$, hence, $s \in \bigcap_{a\in F_t^0} \mathcal{X}_t(a)$ for each $t \in T$. Due to the definition of $F_t^0(s)$, $(A-E)-\alpha_t(s) = (A-E)-F_t^0(s)$, so that $s \in S-\mathcal{X}_t(a)$ for each $a \in (A-E)-F_t^0(s)$ and each $t \in T$, hence, $s \in \bigcap_{a\in(A-E)-F_t^0}(S-\mathcal{X}_t(a))$ for each $t \in T$. Consequently, $s \in H_2$, so that $H_1 \subset H_2$.

In order to prove the inverse inclusion, suppose that $s \in S - H_1$. Then there exists $a_0 \in A - E$ such that $s \in S - \bigcup_{t\in T}(S - \mathcal{X}_t(a_0))$. To arrive at a contradiction suppose, moreover, that $s \in H_2$. As $a_0 \in A - E$ and $\bigcap_{t\in T} F_t^0 = \emptyset$, it follows that for each $\{F_t^0\}_{t\in T} \subset \mathcal{P}(A - E)$, $\bigcap_{t\in T} F_t^0 = \emptyset$ such $t_0 \in T$ must exist that a_0 is in $(A - E) - F_{t_0}^0$. Consequently, $s \in S - \mathcal{X}_{t_0}(a_0)$, hence, $s \in \bigcup_{t\in T}(S - \mathcal{X}_t(a))$. But this conclusion contradicts our assumption that $s \in S - H_1$, so that $s \in S - H_1$ implies $s \in S - H_2$. Consequently, $H_1 = H_2$, so that (4.14) and (4.13) are proved and we may conclude that $\left(\bigcap_{t\in T} \mathcal{X}_t\right)^{(\#)} = \bigcap_{t\in T} \mathcal{X}_t^{(\#)}$. The dual result for $\left(\bigcup_{t\in T} \mathcal{X}_t\right)^{(\#)}$ immediately follows from relation (3.2) which defines this internal $\mathbb{B}$-fuzzy set over A through the joint and complement operations in the way preserving de Morgan rules. The assertion is proved. □

Theorem 4.3. For each $X \in [\mathcal{P}(A)]^\#$, $\left(X^{(*)}\right)^{(\#)} = X$. If $\mathbb{B}$ is completely set-isomorphic, then for each $\mathcal{X} \in A^*$, $\left(\mathcal{X}^{(\#)}\right)^{(*)} = \mathcal{X}$.

Proof. Let $X \in [\mathcal{P}(A)]^\#$, let $F \subset A$, then

$$\left(X^{(*)}\right)^{(\#)}(F) = \bigwedge_{a\in F} X^{(*)}(a) \wedge \bigwedge_{a\in A-F} \neg X^{(*)}(a) = \tag{4.17}$$

$$= \bigwedge_{a\in F} \bigvee_{E\subset A,\, a\in E} X(E) \wedge \bigwedge_{a\in A-F} \neg \bigvee_{E\subset A,\, a\in E} X(E) = X(F)$$

due to (2.7), so that $\left(X^{(*)}\right)^{(\#)} = X$. If $\mathbb{B}$ is completely set-isomorphic, then for each $a \in A$,

$$\left(\mathcal{X}^{(\#)}\right)^{(*)}(a) = \bigvee_{F\subset A,\, a\in F} \mathcal{X}^{(\#)}(F) = \mathcal{X}(a) \tag{4.18}$$

due to (2.25), so that $\left(\mathcal{X}^{(\#)}\right)^{(*)} = \mathcal{X}$. The assertion is proved. □

5. EXTERNAL $\mathbb{B}$-FUZZY SETS

Let us recall that external $\mathbb{B}$-fuzzy sets are defined as classical crisp sets of $\mathbb{B}$-fuzzy elements of the set A, so that the set of all external $\mathbb{B}$-fuzzy sets is identical with the power-set $\mathcal{P}(A^{\#})$ over $A^{\#}$. Set-theoretic operations over external $\mathbb{B}$-fuzzy sets are, hence, defined in the usual way.

As already mentioned above (Fact 1.1), the space of external $\mathbb{B}$-fuzzy sets is richer than that of internal $\mathbb{B}$-fuzzy sets or, due to the results obtained above, than that of classical $\mathbb{B}$-fuzzy sets. So, it is not too reasonable to expect that a simple mapping from $\mathcal{P}(A^{\#})$ into A^* or into $[\mathcal{P}(A)]^{\#}$ would conserve all the properties of external $\mathbb{B}$-fuzzy sets and would enable to identify unambiguously the inverse image from $\mathcal{P}(A^{\#})$, given its image in A^* or in $[\mathcal{P}(A)]^{\#}$. To illustrate the situation, let us consider the following straighforward generalizations of the mappings h_{ic} and h_{ci} to external $\mathbb{B}$-fuzzy sets. Let $\mathbb{X} \in \mathcal{P}(A^{\#})$, i.e., let $\mathbb{X} \subset A^{\#}$, set

$$\begin{aligned} \mathbb{X}^{(*)}(a) &= \bigvee_{x \in \mathbb{X}} x(a), \qquad (5.1)\\ \mathbb{X}^{(\#)}(E) &= \bigwedge_{a \in E} \mathbb{X}^{(*)}(a) \wedge \bigwedge_{a \in A-E} \neg \mathbb{X}^{(*)}(a) = \\ &= \bigwedge_{a \in E} \bigvee_{x \in \mathbb{X}} x(a) \wedge \bigwedge_{a \in A-E} \neg \bigvee_{x \in \mathbb{X}} x(a) \end{aligned}$$

for each $a \in A$ and $E \subset A$. It is obvious that $\mathbb{X}^{(*)} \in A^* = B^A$, and it follows from what we have proved above, that $\mathbb{X}^{(\#)} \in [\mathcal{P}(A)]^{\#}$. A simple example proves that, contrary to the case of internal $\mathbb{B}$-fuzzy sets, the mapping defined by (5.1) is not one-to-one, so that there exist $\mathbb{X}, \mathbb{Y} \in \mathcal{P}(A^{\#})$ such that $\mathbb{X} \neq \mathbb{Y}$, but $\mathbb{X}^{(*)} = \mathbb{Y}^{(*)}$ and, consequently, also $\mathbb{X}^{(\#)} = Y^{(\#)}$. Take $A = \{a, b\}$, $a \neq b$, $B = \{\mathbf{0}_B, e, \neg e, \mathbf{1}_B\}$ such that $\mathbf{0}_B \neq e$, $\neg e \neq \mathbf{1}_B$, take $\mathbb{X} = \{x_1, x_2\}$, $\mathbb{Y} = \{y_1, y_2\}$, where

$$\begin{aligned} &x_1(a) = e, \quad x_1(b) = \neg e, \quad x_2(a) = \neg e, \quad x_2(b) = e, \qquad (5.2)\\ &y_1(a) = \mathbf{0}_B, \quad y_1(b) = \mathbf{1}_B, \quad y_2(a) = \mathbf{1}_B, \quad y_2(b) = \mathbf{0}_B, \end{aligned}$$

so that, evidently, $\mathbb{X} \neq \mathbb{Y}$. However,

$$\begin{aligned} \mathbb{X}^{(*)}(a) &= x_1(a) \vee x_2(a) = e \vee (\neg e) = \mathbf{1}_B = \qquad (5.3)\\ &= \mathbf{0}_B \vee \mathbf{1}_B = y_1(a) \vee y_2(a) = \mathbb{Y}^{(*)}(a), \\ \mathbb{X}^{(*)}(b) &= x_1(b) \vee x_2(b) = (\neg e) \vee e = \mathbf{1}_B = \\ &= \mathbf{1}_B \vee \mathbf{0}_B = y_1(b) \vee y_2(b) = \mathbb{Y}^{(*)}(b), , \end{aligned}$$

so that $\mathbb{X}^{(*)} = \mathbb{Y}^{(*)}$.

To compare the extents of the set $\mathcal{P}(A^*)$ of external $\mathbb{B}$-fuzzy sets and of the set A^* of classical $\mathbb{B}$-fuzzy sets ($A^* = B^A$), the following simple computation concerning their cardinalities may be worth introducing explicitly. Let $\mathbb{B} = \langle \mathcal{P}(B_0), \cap, \cup, {}^c, \emptyset, B_0 \rangle$ be the

Boolean algebra of all subsets of a fixed nonempty set B_0 with respect to the common set-theoretic operations, let A be a nonempty set, then

$$\begin{aligned}\operatorname{card}(A^*) &= \operatorname{card}(\mathcal{P}(B_0)^A) = (\operatorname{card}\mathcal{P}(B_0))^{\operatorname{card}(A)} = \\ &= \left(2^{\operatorname{card}(B_0)}\right)^{\operatorname{card}(A)} = 2^{\operatorname{card}(B_0)\operatorname{card}(A)}.\end{aligned} \tag{5.4}$$

At the same time,

$$\operatorname{card}(\mathcal{P}(A^{\#})) = 2^{\operatorname{card}(A^{\#})}. \tag{5.5}$$

In order to obtain a lower estimate for $\operatorname{card}(A^{\#})$, consider the following special $\mathbb{B}$-fuzzy elements of A. Let $C \subset B_0$, $\emptyset \neq C \neq B_0$, let $a_1, a_2 \in A$, let $x(C, a_1, a_2): A \to \mathcal{P}(B_0)$ be defined as follows:

$$\begin{aligned}&x(C, a_1, a_2)(a_1) = C, \\ &x(C, a_1, a_2)(a_2) = C^c = B_0 - C, \\ &x(C, a_1, a_2)(a) = \emptyset\end{aligned} \tag{5.6}$$

for each $a \in A$, $a \neq a_1$, $a \neq a_2$, supposing that $a_1 \neq a_2$. If $a_1 = a_2$, then

$$\begin{aligned}&x(C, a_1, a_2)(a_1) = B_0, \\ &x(C, a_1, a_2)(a) = \emptyset\end{aligned} \tag{5.7}$$

for each $a \in A$, $a \neq a_1$. Obviously, for each triple $\langle C, a_1, a_2\rangle \in (\mathcal{P}(B_0) - \{\emptyset, B_0\}) \times A \times A$, $\{x(C, a_1, a_2)(a)\}_{a \in A} \in \operatorname{Dcp}(\mathbf{1}_B)$, hence, $x(C, a_1, a_2) \in A^{\#}$. Moreover, if $\langle C', a_1', a_2'\rangle \neq \langle C, a_1, a_2\rangle$, then $x(C', a_1', a_2') \neq x(C, a_1, a_2)$, consequently,

$$\begin{aligned}\operatorname{card}(A^{\#}) &\geq \operatorname{card}((\mathcal{P}(B_0) - \{\emptyset, B_0\}) \times A \times A) = \\ &= \operatorname{card}(\mathcal{P}(B_0) - \{\emptyset, B_0\})\,(\operatorname{card}(A))^2 = \\ &= \left(2^{\operatorname{card}(B_0)}\right)(\operatorname{card}(A))^2.\end{aligned} \tag{5.8}$$

Hence, if $\operatorname{card}(A) \geq 2$, $\operatorname{card}(B_0) \geq 2$, then

$$\operatorname{card}(A^{\#}) \geq (\operatorname{card} B_0)(\operatorname{card} A). \tag{5.9}$$

If, moreover, both the sets B_0 and A are finite, then the inequality in (5.9) is strict (>), so that

$$\begin{aligned}\operatorname{card}(\mathcal{P}(A^{\#})) &= 2^{\operatorname{card}(A^{\#})} \geq 2^{(2^{\operatorname{card}(B_0)}-2)(\operatorname{card}(A))^2} \\ &> 2^{\operatorname{card}(B_0)\operatorname{card}(A)} = \operatorname{card}(A^*).\end{aligned} \tag{5.10}$$

Hence, in this case no one-to-one mapping between $\mathcal{P}(A^{\#})$ and A^* exists. The same situation occurs when both A, B_0 are infinite, and $\operatorname{card}(B_0) \geq \operatorname{card}(A)$, as in this case

$$\begin{aligned}\left(2^{\operatorname{card}(B_0)} - 2\right)(\operatorname{card}(A))^2 &= 2^{\operatorname{card}(B_0)} > \operatorname{card}(B_0) = \\ &= \operatorname{card}(B_0)\operatorname{card}(A_0).\end{aligned} \tag{5.11}$$

On the other side, if A is infinite, if $\mathrm{card}(A) \geq 2^{\mathrm{card}(B_0)} = \mathrm{card}(\mathcal{P}(B_0))$ and if $\mathrm{card}(B_0) \geq 2$, then

$$(2^{\mathrm{card}(B_0)} - 2)\ (\mathrm{card}\, A))^2 = \mathrm{card}(A) = \mathrm{card}(A)\,\mathrm{card}(B) \tag{5.12}$$

and our example fails. Namely, if B_0 is finite and A is infinite, $\mathrm{card}(\mathcal{P}(A^{\#})) = \mathrm{card}(A^*)$, as the following reasoning demonstrates.

Let B_0 be a finite set, let $\mathrm{card}(B_0) = b$, let A be an infinite set. Let $\mathcal{K}_n$, $n = 1, 2, \ldots$, be the set of all decompositions of B_0 into n different and nonempty subsets, Let $K_n = \mathrm{card}(\mathcal{K}_n)$. Each decomposition of B_0 can contain at most b different nonempty subsets (the decomposition into singletons), so that $K_n = 0$ for each $n > b$. Obviously, $K_n \leq 2^b - 1 = \mathrm{card}(\mathcal{P}(B_0) - \{\emptyset\})$ for each n. Let $\langle D_1, D_2, \ldots, D_n\rangle$ be a decomposition from $\mathcal{K}_n$, hence, $\emptyset \neq D_i$, $D_i \cap D_j = \emptyset$ for each $1 \leq i \neq j \leq n$, $\bigcup_{i=1}^{n} D_i = B_0$, let $\langle a_1, a_2, \ldots, a_n\rangle$ be an n-tuple of mutually different elements from A. Then, each permutation π of $\langle 1, 2, \ldots, n\rangle$, ascribing to a_i the subset $D_{\pi(i)} \subset B_0$, defines a fuzzy element of A with respect to the Boolean algebra over $\mathcal{P}(B_0)$, hence, each such π defines an element from $A^{\#}$. Obviously, different permutations yield different elements from $A^{\#}$, so that the pair $(\langle a_1, \ldots, a_n\rangle, \langle D_1, \ldots, D_n\rangle)$ of n-tuples defines $n!$ such elements. If $\langle a_1, \ldots, a_n\rangle \neq \langle a'_1, \ldots, a'_n\rangle$ or $\langle D_1, \ldots, D_n\rangle \neq \langle D'_1, \ldots, d'_n\rangle$, then the sets of elements of $A^{\#}$, generated by $(\langle a_1, \ldots, a_n\rangle, \langle D_1, \ldots, D_n\rangle)$ and by $(\langle a'_1, \ldots, a'_n\rangle, \langle D'_1, \ldots, D'_n\rangle)$ are disjoint, so that

$$\begin{aligned} \mathrm{card}(A^{\#}) &= \sum_{n=1}^{\infty} (\{\langle a_1, \ldots, a_n\rangle \in A^n,\ a_i \text{ different}\}) \cdot \\ &\qquad \cdot (\mathrm{card}(\mathcal{K}_n))\ n! \leq \\ &\leq \sum_{n=1}^{b} \mathrm{card}\ (\{\langle a_1, \ldots, a_n\rangle \in A^n\})\ 2^b\, n! \leq \\ &\leq b(\mathrm{card}(A))^b\, b!\, 2^b = \mathrm{card}(A) = \mathrm{card}(A)\,\mathrm{card}(B_0), \end{aligned} \tag{5.13}$$

as $\mathrm{card}(A) \geq \aleph_0$, $\mathrm{card}(B_0) < \aleph_0$. The inverse inequality $\mathrm{card}(A^{\#}) \geq \mathrm{card}(A)$ follows immediately from (5.9), as A is infinite and B_0 finite. Consequently, we obtain that

$$\begin{aligned} \mathrm{card}(\mathcal{P}(A^{\#})) &= 2^{\mathrm{card}(A^{\#})} = 2^{\mathrm{card}(A)\,\mathrm{card}(B_0)} = \\ &= \mathrm{card}(A^*), \end{aligned} \tag{5.14}$$

so that there exists a one-to-one mapping between external and classical $\mathbb{B}$-fuzzy sets.

The immediately emerging questions are as follows:

(1) to find an explicit way how to encode external fuzzy sets by the classical ones in the cases when it is possible, as our proof of such a possibility is of purely non-constructive way;

(2) To investigate, in which sense, degree, and for which purposes, classical or internal $\mathbb{B}$-fuzzy sets can approximate the external ones under the condition that a one-to-one

encoding is either impossible, or if it is not effectively achievable, or if it is hard to proceed for some computational or other reasons. However, let us postpone a more detailed investigation of both these problems till another occasion.

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