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Ladislav Nebeský

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# An Outerplanar Test of Linguistic Projectivity

LADISLAV NEBESKÝ

In the present paper, a mathematical model (called here an $L$-tree) of the dependency structure of the sentence is considered. From the linguistic point of view the most important $L$-trees are the projective ones. For any $L$-tree $L$ we define a graph $G$ such that $G$ uniquely determines $L$ (Theorem 1) and that $L$ is projective if and only if $G$ is outerplanar (Theorem 2). The outerplanar test of projectivity of $L$-trees given by Theorem 2 is relative to the planar test of projectivity of $L$-trees given in [5].

In [5] we defined an $L$-tree as an quadruple $L = (V_0, E_0, r, \leqq_L)$ such that $(V_0, E_0)$ is a tree, $r$ is one of the vertices of $V_0$ and $\leqq_L$ is a complete ordering of $V_0$. We said that an $L$-tree $L$ is projective if for every vertices $u$, $v$ and $w$ such that $uw$ is an edge of $E_0$ and that either $u <_L v <_L w$ or $w <_L v <_L u$ it holds, that if $u$ lies on the path from $r$ to $w$, then $u$ also lies on the path from $r$ to $v$ (notice that in the present paper we use a rather different graphical terminology and notation than in [5].

The concept of $L$-trees is an apparatus useful for modelling the sentence structure in dependency syntax; the most important $L$-trees are the projective ones. For position of the concept of projectivity in algebraic linguistics, see Marcus [3], Chapter VI (our concept of $L$-trees corresponds to Marcus' concept of simple strings, but Marcus studied projectivity more generally, not only for simple strings). For another mathematical discussion of projectivity of $L$-trees, see, for example, [4], Chapter IV. For linguistic questions of projectivity or non-projectivity of sentence structures, see, for example, Novák [7] and Uhlířová [8].

In the present paper, for any $L$-tree $L$ we shall construct a certain graph $G$ and prove that $L$ is projective if and only if $G$ is outerplanar. Outerplanar graphs represent a simple class of planar graphs. A graph $G$ is outerplanar if it can be embedded in the plane such that all the vertices of $G$ lie on the exterior region. Chartrand and Harary [2] proved that a graph is outerplanar if and only if it contains no subgraph homeomorphic from the complete graph $K_4$ or the complete bipartite graph $K_{2,3}$. A graph $H$ is homeomorphic from a graph $H_0$ if $H$ is isomorphic either to $H_0$ or to a graph which can be obtained from $H_0$ by a suitable insertion of vertices of degree 2 into the edges of $H_0$ (the concept „homeomorphic from" is different from the concept „homeomorphic with"; see [1] and [2]).

Now, we shall define the main concept of the present paper:

**Definition.** Let $L = (V_0, E_0, r, \leqq_L)$ be an $L$-tree such that $V_0 = \{v_1, \ldots, v_n\}$, $n \geqq 1$, and $v_1 <_L \ldots <_L v_n$. We say that a graph $G = (V, E)$ is a graphical expansion of $L$ if there is a set $W = \{w_0, \ldots, w_{n+1}\}$ disjoint with $V_0$ and such that $V = V_0 \cup W$ and

$$E = E_0 \cup \{rw_{n+1}\} \cup \{w_0v_1, v_1w_1, \ldots, w_{n-1}v_n, v_nw_n, w_nw_{n+1}\}.$$

Obviously, any two graphical expansion of an $L$-tree $L$ are isomorphic. A close connection between $L$-trees and their graphical expansions is given in the following theorem:

**Theorem 1.** *Let $G$ be a graphical expansion of an $L$-tree $L$. Then $G$ is a graphical expansion of the only $L$-tree.*

Proof. We can assume that $L$ and $G$ are the same as in the definition. For every $u \in V$ it holds that $u \in V_0$ if and only if $u$ has degree at least 3 in $G$. Similarly, for every $uv \in E$ it holds that $uv \in E_0$ if and only if both $u$ and $v$ are in $V_0$. There is exactly one vertex of degree 1 in $G$; it is $w_0$. Further, we have $w_0v_1 \in E$. For any $i$, $1 \leqq i < n$, there is exactly one vertex $w \in W$ and exactly one vertex $v \in V_0$ such that $v \neq v_i$ and $v_iw$, $wv \in E$; obviously $w = w_i$ and $v = v_{i+1}$. There are exactly two vertices $w'$, $w'' \in W - \{w_0, \ldots, w_{n-1}\}$; obviously, $w'w'' \in E$. If $v_nw'$, $v_nw'' \in E$, then $r = v_n$. Otherwise, there is $j$, $1 \leqq j < n$, such that either $v_jw'$, $v_nw'' \in E$, or $v_jw''$, $v_nw' \in E$; then $r = v_j$. This means that $G$ uniquely determines $L$. Hence the theorem.

An outerplanar test of projectivity of $L$-trees is given in the following theorem:

**Theorem 2.** *Let $L$ be an $L$-tree and $G$ be a graphical expansion of $L$. A necessary and sufficient condition for $L$ to be projective is that $G$ be outerplanar.*

Proof. We assume that $L$ and $G$ are the same as in the definition.

Necessity: Let $L$ be projective. If $1 \leqq i \leqq n$, then by $d_i$ we denote the distance between $r$ and $v_i$ in $(V_0, E_0)$. For every vertex $v$ in $V$ we denote the points $P_v$ and $Q_v$ in the cartesian plane as follows:

$$P_{v_i} = (i - 1, -d_i), \quad \text{for} \quad 1 \leqq i \leqq n;$$
$$P_{w_0} = (-1/2, -d_1);$$
$$P_{w_j} = (j - (1/2), -\max(d_j, d_{j+1})), \quad \text{for} \quad 1 \leqq j \leqq n - 1;$$
$$P_{w_n} = (n - (1/2), -d_n);$$

$$P_{w_{n+1}} = (n, 1)\,;$$
$$\text{if } P_v = (x, y)\,, \quad \text{then} \quad Q_v = (x, -n)\,, \quad \text{for every} \quad v \in V\,.$$

If $P$ and $P'$ are points then by $PP'$ we denote the straight-line segment which connects $P$ and $P'$. Denote $S_0 = \{P_u P_v \mid uv \in E_0\}$, $S = \{P_u P_v \mid uv \in E\}$, $T_0 = \{P_u Q_u \mid u \in \in V_0\}$ and $T = \{P_u Q_u \mid u \in V\}$. As $L$ is projective then no two straight-line segments in $S_0 \cup T_0$ cross; cf. [3], pp. 237–240. The set $S$ gives an embedding of $G$ in the plane. It is easy to see that no two straight-line segments in $S \cup T$ cross. This means that $G$ is outerplanar.

Sufficiency: Let $L$ be not projective. Then, there are $u$, $v$ and $w$ in $V_0$ such that (i) $uw$ is in $E_0$, (ii) $u$ lies on the path from $r$ to $w$, (iii) $u$ does not lie on the path from $r$ to $v$, and (iv) either $u <_L v <_L w$ or $w <_L v <_L u$. It is obvious that $u \neq r \neq w$. Without loss of generality we assume that $u <_L v <_L w$.

Let either $r <_L u$ or $w <_L r$. Then there is an edge $st$ in $E_0$ such that either $t <_L u <_L s <_L w$ or $u <_L s <_L w <_L t$. Without loss of generality we assume that $u <_L s <_L w <_L t$. There are $i$, $j$ such that $1 < i < j - 1 < n$ and $s = v_i$, $t = v_j$. It is evident that $G$ contains a subgraph which includes the vertices $u$, $w_{i-1}$, $s$, $w_i$, $w$, $w_{j-1}$, $t$ and which is homeomorphic from $K_{2,3}$.

Let $u <_L r <_L w$. There is $k$ such that $1 \leqq k \leqq n$ and $r = v_k$. It is evident that $G$ contains a subgraph which includes the vertices $u$, $w_{k-1}$, $r$, $w_k$, $w$, $w_n$, $w_{n+1}$ and which is homeomorphic from $K_{2,3}$. Thus $G$ is not outerplanar which completes the proof.

The test of projectivity of $L$-trees given by Theorem 2 is relative to the planar test of projectivity of $L$-trees given in [5] (cf. also [6]).

Notice that there is an $L$-tree with a non-planar graphical expansion; for example an $L$-tree $(V_0, E_0, r, \leqq_L)$ with $V = \{v_1, \ldots, v_6\}$, $E_0 = \{v_3v_6, v_6v_1, v_1v_4, v_4v_2, v_2v_5\}$, $r = v_1$, $v_1 <_L \ldots <_L v_6$.

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## REFERENCES


[1] M. Behzad and G. Chartrand: Introduction to the Theory of Graphs. Allyn and Bacon, Inc., Boston 1971.

[2] G. Chartrand and F. Harary: Planar permutation graphs. Annales de l'Institut Henri Poincaré *3* (1967), Section B, 433–438.

[3] S. Marcus: Algebraic Linguistics; Analytical Models. Academic Press, New York 1967.

[4] L. Nebeský: Algebraic Properties of Trees. Karlova universita, Praha 1969.

[5] L. Nebeský: A planar test of linguistic projectivity. Kybernetika *8* (1972), 351–354.

[6] L. Nebeský: Projectivity in linguistics and planarity in graph theory. Prague Studies in Mathematical Linguistics 5 (submitted).

[7] P. Novák: Postscript, [4], 83–95.

[8] L. Uhlířová: On the non-projective constructions in Czech. Prague Studies in Mathematical Linguistics 3, Academia, Praha 1972, pp. 171–181.


*RNDr. Ladislav Nebeský, CSc., filosofická fakulta Karlovy university (Faculty of Philosophy, Charles University), nám. Krasnoarmějců 2, 116 38 Praha 1. Czechoslovakia.*