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Boris M. Schein

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On Some Problems in the Theory of Partial Automata

BORIS SCHEIN

The paper shows how some semigroup-theory methods can be used in automata problems.

The theory of automata is closely connected with the theory of semigroups (or, to be more exact, with the theory of representations of semigroups by transformations). The concept of automaton without outputs is equivalent to the concept of representation of free semigroup by transformations. If we consider everywhere or not everywhere defined, one-valued or many-valued transformations, we obtain full or partial, deterministic or nondeterministic automata. All this was discussed by the author in his lecture at the International symposium on relay circuits and finite automata in Moscow, September 1962 [1].

Many results in the theory of transformation semigroups may be interpreted as results on automata (and vice versa). Unfortunately, the main concepts and results of the theory of transformation semigroups are almost unknown among the specialists in the automata theory. Quite a few recent results on automata turn out to be well known after being translated into semigroup language.

The aim of this paper is to present some results on transformation semigroups as results on automata. This semigroup-theoretic results have been partly published in [4, 5]. The main ideas of these results, the underlying point of view (the so-called "relation algebras") were exposed in [2] and, in a much shorter form, in [3].

The output function of an automaton does not play any rôle in this paper. We consider automata without outputs.

A (finite) automaton is an ordered triple $A = (X, S, \delta)$ where X is a (finite) set of input symbols, S is a (finite) set of inner states and δ is a transition function, i.e. a partial mapping of the set $X \times S$ into S. If δ is everywhere defined, the automaton A is called full. If $\delta(x, s)$ is not defined, it means that the automaton is destroyed when the input is x while the inner state is s. One can consider multi-valued δ. In this case the automaton A, which is in the state s, goes to some state from the set $\delta\langle x, s\rangle$ where

$\delta\langle x, s\rangle$ is the set of images of $(x, s) \in X \times S$ under δ. If $\delta\langle x, s\rangle = \emptyset$ it means that A is destroyed if x is the input while s is its inner state. If δ is multivalued, the automaton is called nondeterministic.

The function δ may be easily extended to a function φ defined on a subset of the set $X^* \times S$ where X^* is the free semigroup with identity e generated by X. We define $\varphi(e, s) = s$, $\varphi(x_1 x_2 \dots x_n, s) = \delta(x_n, \varphi(x_1 \dots x_{n-1}, s))$. Clearly, φ is partial or multivalued if δ is partial or multivalued as well.

The input word α is called applicable to the inner state s of A if A with the inner state s is not destroyed by the input α, i.e. when $\varphi(\alpha, s)$ is defined. If A is nondeterministic, α is applicable to s if $\varphi(\alpha, s) \neq \emptyset$, i.e. if A need not be necessarily destroyed when α is its input while s is the initial state.

Let A be an automaton with a set of inputs X. Let us define binary relations χ_A and ζ_A on X^* by the following conditions:

$(\alpha, \beta) \in \chi_A$ means that whenever α is applicable to some inner state, then β is applicable to this state;

$(\alpha, \beta) \in \zeta_A$ means that if α is applicable to some state $s \in S$, then $\varphi(\alpha, s) = \varphi(\beta, s)$. (For nondeterministic automata $\varphi(\alpha, s) \subset \varphi(\beta, s)$.)

Clearly, χ_A and ζ_A are quasiorder relations (i.e., they are reflexive and transitive). These relations possess a rather simple and natural "automaton meaning": $(\alpha, \beta) \in \chi_A$ means that if A is not destroyed by the input word α, it is not destroyed by the input word β; $(\alpha, \beta) \in \zeta_A$ means that if the input word α does not destroy the automaton, then α and β lead to the same transformation of the inner state. We write $\alpha \sqsubset_A \beta$ and $\alpha \prec_A \beta$ instead of $(\alpha, \beta) \in \chi_A$ and $(\alpha, \beta) \in \zeta_A$ respectively.

Our aim is to find an abstract characterization of these binary relations, that is to find conditions characteristical for $\sqsubset_A$ and $\prec_A$ among all quasiorder relations on X^*.

If ξ is a quasiorder relation then its symmetric part is the following relation ε_ξ: $(\alpha, \beta) \in \varepsilon_\xi$ means that $(\alpha, \beta) \in \xi$ and $(\beta, \alpha) \in \xi$. The index of ξ is the number of different equivalence classes modulo ε_ξ.

Main Theorem. *Let $\sqsubset$ and $\prec$ be two quasiorder relations on a semigroup X^*. There exists a (finite) nondeterministic automaton A with the set of input symbols X such that $\sqsubset = \chi_A$ and $\prec = \zeta_A$ if and only if $\sqsubset$ is left regular, i.e.*

$$\alpha \sqsubset \beta \rightarrow \gamma\alpha \sqsubset \gamma\beta\,, \tag{1}$$

right negative, i.e.

$$\alpha\beta \sqsubset \alpha\,, \tag{2}$$

$\prec$ is stable, i.e.

$$\alpha_1 \prec \beta_1, \alpha_2 \prec \beta_2 \rightarrow \alpha_1\alpha_2 \prec \beta_1\beta_2\,, \tag{3}$$

 $\prec$ *is stronger than* $\vdash$, *i.e.*

$$(4) \qquad \alpha \prec \beta \to \alpha \vdash \beta$$

(*and* $\prec$ *has finite index*).

There exists a (finite) deterministic automaton A *with the set of input symbols* X *and such that* $\vdash \; = \chi_A$ *and* $\prec \; = \zeta_A$ *if and only if conditions* (1)–(4) *are satisfied,* ($\prec$ *has finite index*) *and*

$$(5) \qquad \alpha \prec \gamma, \beta \prec \gamma, \alpha \vdash \beta \to \alpha \prec \beta\,,$$

$$(6) \qquad \gamma \vdash \alpha, \gamma \vdash \beta\delta, \alpha \prec \beta \to \gamma \vdash \alpha\delta\,.$$

Outline of the proof. The necessity of these conditionsis verified straighforwardly. Each of these conditions has a simple "automatic meaning", e.g. (2) means that if an input does not destroy the automaton, then any beginning of this input also does not destroy the automaton.

Sufficiency. Let the conditions (1)–(4) be satisfied. Let us consider an automaton A with the input set X and with inner states $(\bar{\alpha}, \bar{\beta})$ where $\bar{\alpha}$ is the equivalence class modulo $\varepsilon_{\prec}$ containing α; $\bar{\beta}$ is the equivalence class modulo $\varepsilon_{\vdash}$ containing β and $\beta \vdash \alpha$. The condition (4) implies that if $\prec$ has finite index then $\vdash$ also has finite index and the set of inner states of A is finite. Therefore, A is finite if $\prec$ has finite index.

By definition, the input symbol x is applicable to the inner state $(\bar{\alpha}, \bar{\beta})$ if and only if $\beta \vdash \alpha x$. The applicability of x does not depend on the choice of representatives α and β in the classes $\bar{\alpha}$ and $\bar{\beta}$. If x is applicable to $(\bar{\alpha}, \bar{\beta})$ then x sends this inner state to one of the states $(\bar{\gamma}, \bar{\beta})$ where $\gamma \prec \alpha x$. Clearly, A is nondeterministic. The reader may easily verify that $\vdash \; = \chi_A$ and $\prec \; = \zeta_A$, Q.E.D.

Now let the conditions (1)–(6) be satisfied. Let T be the set of ordered pairs $(\bar{\alpha}, \bar{\beta})$ as defined above. Let ε be an equivalence relation on T defined as follows: $(\bar{\alpha}_1, \bar{\beta}_2) \equiv (\bar{\alpha}_2, \bar{\beta}_2) \pmod{\varepsilon}$ means that either $(\bar{\alpha}_1, \bar{\beta}_1) = (\bar{\alpha}_2, \bar{\beta}_2)$ or $\bar{\beta}_1 = \bar{\beta}_2$ and there exists $\gamma \in X^*$ such that $\gamma \prec \alpha_1$, $\gamma \prec \alpha_2$ and $\bar{\gamma} = \bar{\beta}_1$. The reflexivity and the symmetry of ε are self-evident, the transitivity follows from (5). Let S denote the quotient set T/ε. Let us consider an automaton A with input symbols X and inner states S transition function δ of which is defined as follows: $\delta(x, s)$ is defined if an only if $\beta \vdash \alpha x$ for some α and β such that $(\bar{\alpha}, \bar{\beta}) \in s$. It was mentioned above that this condition does not depend on the choice of α and β in the equivalence classes $\bar{\alpha}$ and $\bar{\beta}$. It does not depend on the choice of $(\bar{\alpha}, \bar{\beta})$ in s. In effect, let $(\bar{\alpha}_1, \bar{\beta}_1) \equiv (\bar{\alpha}, \bar{\beta}) \pmod{\varepsilon}$. Then there exists $\gamma \in X^*$ such that $\gamma \prec \alpha$, $\gamma \prec \alpha_1$ and $\bar{\gamma} = \bar{\beta} = \bar{\beta}_1$. But $\beta \vdash \alpha x$, hence, by condition (6), $\beta \vdash \gamma x$. We have $\beta_1 \vdash \beta$ and $\gamma x \prec \alpha_1 x$, therefore $\gamma x \vdash \alpha_1 x$, i.e. $\beta_1 \vdash \alpha_1 x$.

If x is applicable to s, then, by definition, $\delta(x, s) = t$ where $(\bar{\alpha}, \bar{\beta}) \in s$ and $(\overline{\alpha x}, \bar{\beta}) \in t$ for some α and β. It is easy to verify that the function δ is one-valued, i.e. the automaton A is deterministic. If $\prec$ has finite index, then, evidently, A is finite. We omit the straightforward verification of the equalities $\vdash \; = \chi_A$ and $\prec \; = \zeta_A$.

The theorem is proved.

If a stable quasiorder relation ξ on X has finite index, then, by the well-known result of S. C. Kleene, the equivalence classes modulo ε_ξ are regular events over the alphabeth X. Let $\alpha \in X^*$. Then the set of all β such that $(\alpha, \beta) \in \xi$ is a regular event as well. It would be interesting to consider these events and their interconnection with the automaton when $\xi = \zeta_A$ or $\xi = \chi_A$ (in the latter case these events are also regular).

Corollary 1. *Let $\ulcorner$ be a quasiorder relation on a semigroup X^*. There exists (finite) automaton A with the set of input symbols X and such that $\chi_A = \ulcorner$ if and only if $\ulcorner$ is left regular and right negative quasiorder relation (of finite index). These conditions are necessary and sufficient for both deterministic and non-deterministic cases.*

Proof. The necessity follows from the main theorem. Now let $\ulcorner$ satisfy the conditions (1) and (2). Let us consider the identical order relation $\prec$ (i.e., $\alpha \prec \beta$ means that $\alpha = \beta$). Clearly, $\ulcorner$ and $\prec$ satisfy the conditions (1)–(6), hence, by the main theorem, there exists a deterministic automaton A such that $\ulcorner = \chi_A$ and $\prec = \zeta_A$. This automaton is infinite.

Now let $\ulcorner$ be of finite index and S be the set of inner states of A. Let us define an equivalence relation η on S by the following condition: $s \equiv t(\text{mod } \eta)$ means that for every $\alpha \in X^*$ α is applicable to s if and only if it is applicable to t. This relation η defines the state-homomorphism of A onto some automaton A/η with the input set X and the set of states S/η. Using the definition of η, one can easily deduce that $\ulcorner = \chi_{A/\eta}$.

Let $\alpha_1, \ldots, \alpha_n$ be representatives of all equivalence classes modulo $\varepsilon_\ulcorner$. Then $s \equiv t$ $(\text{mod } \eta)$ if and only if α_i is applicable to s exactly when α_i is applicable to t for $i = 1, \ldots, n$. It is evident now that η has finite index, i.e., S/η is finite. Hence, A is a finite automaton.

Corollary 2. *Let $\prec$ be a quasiorder relation on a semigroup X. There exists a (finite) nondeterministic automaton A with the set X of input symbols and such that $\zeta_A = \prec$ if and only if $\prec$ is stable (and of finite index). There exists a (finite) deterministic automaton A with the set of input symbols X and such that $\zeta_A = \prec$ if and only if $\prec$ is stable and weakly steady, i.e.*

$$\alpha \prec \beta\gamma,\ \alpha \prec \delta\eta,\ \beta \prec \delta \rightarrow \alpha \prec \beta\eta \tag{7}$$

and is a quasiorder relation (of finite index).

Proof. The necessity of these conditions is verifiable straightforwardly. If $\prec$ is stable (and of finite index), then let us define $\alpha \ulcorner \beta$ for all $\alpha, \beta \in X^*$. Clearly $\prec$ and $\ulcorner$ satisfy the conditions (1)–(4) and, by the main theorem, there exists a nondeterministic (finite) automaton A such that $\zeta_A = \prec$.

Now let $\prec$ be stable and weakly steady. We shall construct a deterministic automaton A with the input set X and with the set of inner states consisting of all non-empty subsets of X^* saturated for $\prec$ (a subset $H \subset X^*$ is called saturated for $\prec$ if $\alpha \in H$, $\alpha \prec \beta \rightarrow \beta \in H$). Clearly, if $\prec$ has finite index, then the set of all saturated subsets is finite and A turns out to be a finite automaton. The input x is applicable to the inner state H if and only if H contains a word the last letter of which is x. In this case $\delta(x, H) = H_1$ where $\alpha \in H_1 \leftrightarrow \alpha\, x \in H$. It is easy to verify that H_1 is an inner state and that $\zeta_A = \prec$.

We have considered several semigroup-theoretical problems discussed in such a way that they appear to be automata problems. The choice of these (and not some other) problems was purely by chance. Our aim was to show some possible applications of the theory of representations of semigroups by transformations to the automata theory and to draw attention of specialists in the automata field to possibilities of the semigroup theory.

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O některých problémech v teorii částečných automatů

Boris Schein

Automatem se rozumí trojice (X, S, δ), kde X jsou vstupy, S stavy a δ přechodová funkce definovaná ne nutně všude (proto částečné automaty); přitom se připouští deterministický i nedeterministický případ. Definují se dvě kvasiuspořádání $\sqsubset$ a $\prec$ na X^* a řeší se následující problém: Jsou-li na volné pologrupě X^* definovány dvě binární relace kvasiuspořádání, pak se mají určit nutné a postačující podmínky pro to, aby existoval automat v uvedeném slova smyslu, který předepsané binární relace určuje jako své relace $\sqsubset$ a $\prec$, a to jak v případě deterministickém tak i nedeterministickém.

Cílem článku je ukázat, jak se dají použít metody teorie pologrup na problematiku z teorie automatů.

Boris Schein, Mihailovskaia Str. 2—111, Saratov, U.S.S.R.