Kybernetika

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DECENTRALIZED DESIGN OF FEEDBACK CONTROL
FOR LARGE-SCALE SYSTEMS

LUBOMÍR BAKULE, JAN LUNZE

Feedback control of large-scale interconnected systems necessitates the decentralization of both the on-line and the design tasks. Whereas the properties of decentralized control systems have been investigated extensively, the decentralization of the design process has been the subject of only a few papers. Decentralization of the design task means that the whole design problem is divided into several 'weakly' interconnected subproblems that are solved independently. In contrast to hierarchical algorithms, no complete coordination is carried out to compensate the interdependencies between these subproblems and their solutions.

This paper surveys and extends the recent results concerning decentralized design of decentralized controllers. First, several concepts for decomposing the overall system into weakly coupled subsystems are described. They form the basis of the different principles for decentralizing the design process. These principles concern the design for hierarchically decomposed systems, sequential design methods, the design on the basis of disjoint or overlapping decomposition of the plant model, the design for singularly perturbed systems, the decentralization of the design on the basis of symmetry properties of the plant, and specific design methods for decentralized PI-controllers.

For each of these methods the main principle is described and summarized in theorems and algorithms. Extensions are surveyed and relevant literature is given. Most of the design ideas are illustrated by means of numerical examples.

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2. INTRODUCTION

As many technological environmental or societal systems have a high complexity, large scale systems became the subject of intensive research in systems and control theory. The complexity of the system leads to severe difficulties that are encountered in the tasks of analysing the system and designing and implementing appropriate control strategies and algorithms. These difficulties arise mainly from the following three reasons:

Dimensionality. The system has a large dimension. That is, it consists of many subsystems and has many inputs and outputs. The model has a high dynamical order etc.

Uncertainty. The behaviour and properties of the overall system cannot be completely described by a mathematical model. Uncertainties occur because complete identification of the overall system is impossible or because some disturbances or control signals are unknown. Further uncertainties result from model aggregation

or simplification, which have been deliberately performed to make the model manageable.

Information structure constraints. The availability of information about the system behaviour and the control is restricted. Generally, no decision maker knows the system completely. In a typical situation, the subsystem authorities have a complete model of their own subsystems, but possess only a limited knowledge about the properties and the control of the subsystems.

For these reasons, the analysis and synthesis tasks cannot be solved economically in a single step as it is possible for similar analysis and design tasks for small scale systems. As no widely accepted definition of the notion of large scale systems is available, many authors take the pragmatic view of considering a system as 'large scale' if it cannot be tackled by 'conventional' methods.

Typical examples of large scale systems include interconnected power systems which consists of many subsystems (synchronous machines, power stations, distribution networks) with strong interactions, ecological systems in which a large number of different entities work in a closed interconnection with each other, water systems which are widely distributed in space, traffic systems with many external signals, or large space structures.

To cope with the aforementioned appearance of complexity several general methodologies have been and are being elaborated. Most of them belong to one of the following three groups:

Decomposition. As the amount of computation required to analyse and design a system grows faster than the size of the system it is beneficial to decompose the whole problem into smaller subproblems, solve these subproblems separately and combine their solutions to the global solution of the original task. The subproblems are not independent and some coordination or modification of the solutions of the subproblems is necessary in order to have regard to the effect of the interactions among the subsystems.

Approximation and Robustness. The dimension of the system equations can be reduced by means of model aggregation. Uncertainties can be taken into consideration by using approximate models and estimating the difference between the model and the real system. In both situations, the solution which has been received by means of the approximate model, must be robust enough so that it provides a reasonable solution of the original problem as well. Robustness is, therefore, one of the main issues in large scale systems theory.

Decentralization. The division of the analysis and synthesis tasks into completely decoupled subproblems aims at the consideration of information structure constraints. Several different decision makers have to solve tasks which, altogether, replace the overall problem. In the narrow sense, decentralization does not permit

information exchange among the decision makers. As the subproblems cannot generally be completely separated from each other this restriction brings about reduction of the quality of the solution. In contrast to decomposition methods which include the coordination of the solutions, decentralization gives up the desire to reach the best possible result. For example, whereas a hierarchical scheme for calculating the optimal control is advantageous because of the smallness of the subproblems but ends up with the best possible solution due to the coordination, a decentralized scheme without any coordination is simpler and quicker, but leads only to a sub-optimal control.

Decentralization occurs in connection with the control law, with the design method, and with the implementation. The first leads to decentralized control, i.e. to controllers that consist of several independent control stations without any information exchange. Decentralized design means that whole design task is divided into several subtasks. Although these subproblems are related to each other, they are solved completely independently and the solution is applied with minor or without any modification to the overall system. The third kind of decentralization concerns the implementation of the control law. Here, decentralization means that the different parts of the controller are implemented by different hardware or software components.

These different decentralizations may occur in all possible combinations. We may design a decentralized controller in a decentralized way where each control station is adjusted to the corresponding subsystem without any information about the interaction among the subsystems, and implement the control by means of different, locally distributed control devices. But we may also merely decentralize the control law but design the controller from the overall system point of view and implement it by means of a common microprocessor unit.

The theory of decentralized control has been developing since the mid-70's and several survey papers and monographs are now available on this subject, for example [4], [21], [29], [34], [44], [58], [62], [75], [78], [91], [93], [96], [98], [112]. The majority of the results refer to the decentralization of the control law and extend the design principles of (centralized) multivariable feedback control to structurally constraint feedbacks. For example, the stabilizability by decentralized feedback has been investigated and methods for pole assignment and optimal control elaborated. Contrary to this, the problem of decentralizing (and not merely decomposing) the design task was tackled not earlier than in the end-70's. Whereas the early investigations were primarily stability-oriented, recent methods and results deal with the whole design problem including the suitable shaping of the input-output (I/O)-behaviour of the closed-loop system and restrictions of the a-priori information from which the plant model is derived.

The main concern of this paper is the decentralization of the design. 'Centralized' methods are only surveyed as necessary for the understanding of this text. For

example, hierarchical schemes for calculating optimal decentralized control laws are not considered here as they are design methods which start from the knowledge of a complete model of the overall plant and determine the control law according to global design requirements.

In more detail, the utilisation of structural properties of the plant for the division of the overall design problem into 'weakly coupled' design problems at the sub-system level is the main aim of the paper. We survey the different known methods to achieve conceptual or numerical simplifications of the overall problem and present some new results.

In Chapter 3 the model of interconnected systems are given and principles for decomposing the overall system description are surveyed. These different concepts provide the basis for the different design perspectives, which will be described in the main part of this paper. Chapters 4 and 5 summarise the design problem and general guidelines to the decentralization of the design tasks. The different principles of decentralization of the design process are surveyed in the subsequent chapters. They concern the design for hierarchically decomposed systems, sequential design methods, the design on the basis of disjoint or overlapping decomposition of the plant model, the design for singularly perturbed systems, the decentralization of the design on the basis of symmetry properties of the plant, and specific design methods for decentralized PI-controllers.

In all the chapters, the survey of known principles and possible extensions are combined with new results that have been elaborated by the authors of the present paper. The paper is intended primarily to give a thorough view on the state of the art of decentralized design of feedback controllers. Therefore, different decentralization principles have been put together and are described from a common point of view. However, several new results have been added, in particular in Chapters 6, 8, 11 and 12.

3. DECOMPOSITION OF INTERCONNECTED SYSTEMS

Several decomposition methods have been elaborated in order to simplify the analysis and design tasks for large scale systems, see e.g. [43], [89], [95], [96]. Starting from the description of the overall system and utilising the kind and strength of the interactions among the different parts, these methods aim at a division of the whole model into several sub-models of lower order.

The ultimate goal of decomposition is the reduction of computational complexity of the solution of given analysis and design problems. Therefore, the submodels must be 'weakly' coupled in one or another sense. The question what 'weakly coupled' means can be answered in different ways. Accordingly, different decomposition principles have been established. In this chapter, the decomposition of the overall system into a hierarchy of subsystems, into disjoint or overlapping subsystems with small interactions, and into subsystems with separated time scales will be surveyed.

These concepts provide the basis of the decomposition and decentralization of the design task as they are described in the subsequent chapters.

To get a complete view of the models of interconnected systems, the chapter begins with the development of the overall system model from the subsystem models and the interaction relations. Whereas this way is 'bottom-up', the other subchapters concern the 'top-down' way of dividing the overall system reasonably into different parts. A comparison of both ways makes clear under what condition the structure imposed by decomposition concepts coincides with the structure given by technological reasons.

3.1. Subsystem models and overall system model

The overall system consists of N subsystems which are described by the state space model

$$\dot{x}_i = A_i x_i + B_i u_i + E_i s_i, \qquad x_i(0) = x_{i0},$$

(3.1)
$$y_i = C_i x_i + D_i u_i + F_i s_i,$$

$$z_i = C_{zi} x_i + D_{zi} u_i + F_{zi} s_i, \quad i = 1, \ldots, N,$$

where $x_i \in \mathbb{R}^{n_i}$, $u_i \in \mathbb{R}^{m_i}$, $s_i \in \mathbb{R}^{m_{si}}$, $y_i \in \mathbb{R}^{p_i}$, $z_i \in \mathbb{R}^{p_{zi}}$, are the vectors of the subsystem states, control inputs, interconnection inputs, control output, and interconnection
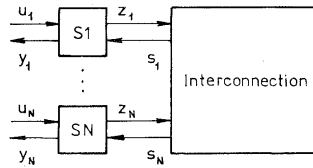

Fig. 1.

output, respectively (Fig. 1). The interactions among the subsystems are, without loss of generality, described by the algebraic relation

(3.2)
$$s = Lz,$$

where $s = (s_1^T, \ldots, s_N^T)^T$ and $z = (z_1^T, \ldots, z_N^T)^T$. If the whole matrix L is partitioned according to the partitions of s and z,

(3.3)
$$L = \begin{pmatrix} L_{11} & \ldots & L_{1N} \\ \vdots & & \vdots \\ L_{N1} & \ldots & L_{NN} \end{pmatrix}$$

the $(m_{si} \times p_{zj})$ block L_{ij} reflects the direct coupling between subsystem j and subsystem i.

As the overall system (3.1), (3.2) is linear, an alternative description can be given

in the form

$$(3.4) \qquad \dot{x} = Ax + Bu , \quad x(0) = x_0 ,$$

$$y = Cx + Du$$

with $u = \left(u_1^T, \ldots, u_N^T\right)^T$ and $y = \left(y_1^T, \ldots, y_N^T\right)^T$. The relation between constant matrices A, B, C, D and A_i, B_i, C_i, $\ldots$ are obtained in the following way. Eq. (3.1) for $i = 1, \ldots$ $\ldots, N$ yields

$$(3.5) \qquad \dot{x} = \mathrm{diag}\, A_i x + \mathrm{diag}\, B_i u + \mathrm{diag}\, E_i s ,$$

$$y = \mathrm{diag}\, C_i x + \mathrm{diag}\, D_i u + \mathrm{diag}\, F_i s$$

with $x = \left(x_1^T, \ldots, x_N^T\right)^T$.

From (3.1) and (3.2) we have

$$(3.6) \qquad s = Lz = L\mathrm{diag}\, C_{zi} x + L\mathrm{diag}\, D_{zi} u + L\mathrm{diag}\, F_{zi} s , \quad \text{and}$$

$$s = \left(I - L\mathrm{diag}\, F_{zi}\right)^{-1} L\mathrm{diag}\, C_{zi} x + \left(I - L\mathrm{diag}\, F_{zi}\right)^{-1} L\mathrm{diag}\, D_{zi} u .$$

Eqs. (3.5) and (3.6) lead to (3.4) with

$$(3.7) \qquad A = \mathrm{diag}\, A_i + \mathrm{diag}\, E_i \left(I - L\mathrm{diag}\, F_{zi}\right)^{-1} L\mathrm{diag}\, C_{zi} ,$$

$$B = \mathrm{diag}\, B_i + \mathrm{diag}\, E_i \left(I - L\mathrm{diag}\, F_{zi}\right)^{-1} L\mathrm{diag}\, D_{zi} ,$$

$$C = \mathrm{diag}\, C_i + \mathrm{diag}\, F_i \left(I - L\mathrm{diag}\, F_{zi}\right)^{-1} L\mathrm{diag}\, C_{zi} ,$$

$$D = \mathrm{diag}\, D_i + \mathrm{diag}\, F_i \left(I - L\mathrm{diag}\, F_{zi}\right)^{-1} L\mathrm{diag}\, D_{zi} .$$

In Eq. (3.5) it has been assumed that the subsystem states are disjoint, i.e. the overall system state includes all the subsystem states

$$(3.8) \qquad x = \left(x_1^T, \ldots, x_N^T\right)^T .$$

While this is true for many real systems there are examples for which two or more subsystems have the same state variables. A typical example is an interconnected power system in which for long time horizon investigations a common frequency of all the subsystems is assumed to exist (cf. Chapter 8). This frequency occurs in each subsystem model (3.1) as state variable, but it should appear in the overall system model (3.4) only once.

Theorem 3.1. $[37]$. The model (3.4) exists if and only if $\left(I - L\mathrm{diag}\, F_{zi}\right)$ is invertible.

Eq. (3.7) shows in which way the subsystem parameters combine to the overall system parameters and makes it possible to derive the assumptions that must be made to end up with the specific forms of the model which will be used in the succeeding chapters. These models can and will be used to simplify the notation or even to make a design method under consideration reasonable. The overall system model (3.4) is often pragmatically decomposed into the form

$$(3.9) \qquad \dot{x}_i = A_{ii} x_i + \sum_j A_{ij} x_j + B_i u_i ,$$

$$y_i = C_i x_i ,$$

where A_{ii} is considered as system matrix of the subsystem i (cf. A_i in Eq. (3.1)). Eq. (3.7) makes clear that $A_{ii} = A_i$ holds only if $F_{zi} = 0$ or if L and F_{zi} have special forms. Furthermore, B is block diagonal only if, for example, $D_{zi} = 0$ is assumed, and $D = 0$ holds if $D_i = 0$ and $D_{zi} = 0$.

3.2. Decomposition into a hierarchy of subsystems

Most of the difficulties in the analysis and design of interconnected systems are raised by the complete interdependence of the subsystems. Conceptual simplifications can be obtained if some subsystems have only a one-way effect on some others. In such a situation the interconnection matrix L in Eq. (3.3) is lower block triangular or can be made so be reordering the subsystems.

An example should illustrate this way of decomposition. Consider an overall system (3.1), (3.2) with 6 subsystems and

$$L = \begin{bmatrix} . & . & . & L_{14} & . & . \\ . & . & . & . & L_{25} & . \\ L_{31} & . & . & L_{34} & . & . \\ L_{41} & . & . & . & . & L_{46} \\ . & L_{52} & L_{53} & . & . & . \\ L_{61} & . & . & . & . & . \end{bmatrix},$$

where all empty places denote zero matrices. The interconnection structure of the overall system can be made clear by means of a directed graph in which each vertex symbolizes a subsystem. An edge is drawn from subsystem i to subsystem j if $L_{ij} \neq 0$ (Fig. 2).
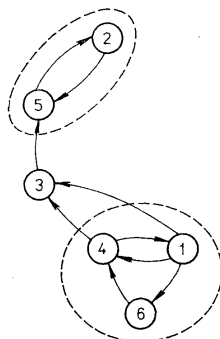

Fig. 2.

Definition 3.1. The subsystems i and j are called *strongly coupled* if there exist a path from vertex i to vertex j and a path from vertex j to vertex i.

Obviously, only the subsystems 1, 4 and 6, or 5 and 2, respectively, are strongly coupled with each other. Hence, the overall system can be decomposed into 3 sub-systems which are internally strongly coupled but have only one-directional inter-actions among them. After changing the order of the subsystems, the matrix $\bar{L}$ has the lower block triangular form.

$$\bar{L} = \begin{matrix} & \\ 1 \\ 6 \\ 4 \\ 3 \\ 5 \\ 2 \end{matrix} \begin{matrix} 1 & 6 & 4 & 3 & 5 & 2 \\ \left[\begin{array}{ccc|cc|c} . & . & L_{14} & . & . & . \\ L_{61} & . & . & . & . & . \\ L_{41} & L_{46} & . & . & . & . \\ \hline L_{31} & . & L_{34} & . & . & . \\ . & . & L_{54} & . & . & L_{52} \\ . & . & . & . & L_{25} & . \end{array} \right] \end{matrix}.$$

The compound consists of subsystems 1, 4, 6 influencing subsystem 3 but not vice versa, and subsystem 3 influencing subsystems 5 and 2. They are 'hierarchically ordered' by permuting rows and columns of the original matrix into desirable lower triangular form. This decomposition can be carried out by graph-theoretic methods.

Theorem 3.2 [107]. The decomposition of the overall system into strongly coupled subsystems is given by the equivalence relation on the set V of subsystems (vertices of the system graph). V is decomposed into disjoint subsets V_i

$$V = \bigcup V_i ,$$
$$V_i \cap V_j = \emptyset , \quad i \neq j ,$$

where all elements of V_i are strongly coupled with each other. In a special case of such systems, all the subsystems are coupled in a chain connection. That is

(3.10) $s_{i+1} = z_i .$
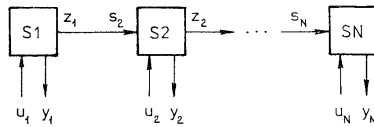
This case is illustrated in Fig. 3.

Fig. 3. Chain connected subsystems.

3.3. Decomposition into disjoint subsystems

In what follows, we assume that no simplifications due to the absence of inter-actions are possible as in the preceding chapter and consider the problem of de-composition of a strongly coupled overall system (3.4). Note that 'strongly' coupled in the sense of Definition 3.1 refers to the presence of interactions among all parts

of the system whereas in this and the next chapters 'weak' and 'strong' interactions have regard to the magnitude of present interconnection lines.

Disjoint decomposition means that the overall system (3.4) is divided into subsystems

$$(3.11) \qquad \dot{x}_i = A_{ii}x_i + \sum_{i \neq j} A_{ij}x_j + B_i u ,$$

$$y_i = C_i x + D_i u , \quad i = 1, \dots, N ,$$

where $x = (x_1^{\mathrm{T}}, \dots, x_N^{\mathrm{T}})^{\mathrm{T}}$, $A = (A_{ij})$, $B = (B_1^{\mathrm{T}}, \dots, B_N^{\mathrm{T}})^{\mathrm{T}}$, $C = (C_1^{\mathrm{T}}, \dots, C_N^{\mathrm{T}})^{\mathrm{T}}$, $D = (D_1^{\mathrm{T}}, \dots, D_N^{\mathrm{T}})^{\mathrm{T}}$. The isolated subsystem, i.e. Eq. (3.11) for $x_j = 0$, has the form

$$(3.12) \qquad \dot{x}_i = A_{ii}x_i + B_i u ,$$

$$y_i = C_i x_i + D_i u .$$

So far, no simplifications have been gained because y_i depends on $u_1, \dots, u_N$. The decomposition must be performed in such a way that the dependencies of y_i on u_j for $i \neq j$ are zero or small. This can only be done appropriately if the internal structure of the given system is utilised (cf. remarks following Eq. (3.9)).

The system (3.4) is input-decentralized if

$$(3.13) \qquad \dot{x}_i = A_{ii}x_i + \sum_{i \neq j} A_{ij}x_j + B_i u_i ,$$

$$y_i = C_i x$$

holds, i.e. the input u_i excites directly only the subsystem state x_i. It is output-decentralized, if it has the form

$$(3.14) \qquad \dot{x}_i = A_{ii}x_i + \sum_{i \neq j} A_{ij}x_j + B_i u ,$$

$$y_i = C_i x_i .$$

The motivation for decentralization is the following: An input-output decentralized system can be designed using local feedbacks for independent subsystems and the influence of interconnections on global system performance can be tested. This idea is based on the assumption that the interactions are weak, i.e. they do not strongly influence the system performance. If for given decomposition (3.12) the decentralized structure (3.13) or (3.14) is not reached, a supplementary procedure called the decentralization has to be used. We describe the procedure for the input decentralization of the system

$$(3.15) \qquad \dot{x} = Ax + Bu ,$$

$$y = x .$$

Eq. (3.15) is Eq. (3.12) for $C_i = I$ [94]. Suppose that all free subsystems are controllable. The procedure follows the idea that at each scalar equation of subsystem one or none control acts. The isolated subsystems are transformed into Luenberger's canonical form via

$$\tilde{x}_i = Q_i^{-1} x_i$$

with

$$Q_i = \begin{pmatrix} b_1^i & A_{ii}b_1^i & \dots & A_{ii}^{n_{ij}-1}b_1^i & b_2^i & \dots & A_{ii}^{p_{ir}-1}b_r^i \end{pmatrix}$$

so that

(3.16)
$$\dot{\tilde{x}}_i = \tilde{A}_{ii}\tilde{x}_i + \sum_{i \neq j} \tilde{A}_{ij}\tilde{x}_j + \tilde{B}_i u ,$$

where

$$\tilde{A}_{ii} = Q_i^{-1}A_{ii}Q_i , \quad \tilde{A}_{kj} = Q_k^{-1}A_{kj}Q_j , \quad \tilde{B}_i = Q_i^{-1}B_i = \text{diag}\left(b_1^i, \dots, b_r^i\right) .$$

If n_{ij} denotes the number of rows in b_j^i then $\sum_j n_{ij} = n_i$. To obtain $\tilde{B}_i$ from B_i each diagonal block of B_i is changed to one (if $n_{ij} = 1$) or by a column with the same dimension, where the upper left element is one, the others are zero.

Example. Consider a system

$$\dot{x}_1 = A_{11}x_1 + A_{12}x_2 + \begin{pmatrix} 1 & 1 \\ 0 & 4 \end{pmatrix} u ,$$

$$\dot{x}_2 = A_{22}x_2 + A_{21}x_1 + \begin{pmatrix} 2 & 2 \\ 1 & 0 \\ 2 & 3 \end{pmatrix} u ,$$

where $A_{11}, A_{12}, A_{22}, A_{21}$ are $2 \times 2, 2 \times 2, 3 \times 3, 3 \times 2$ matrices so that the pairs $\left(A_{11}, B_1\right), \left(A_{22}, B_2\right)$ are controllable, then

$$B_1 = \begin{pmatrix} 1 & 1 \\ 0 & 4 \end{pmatrix} , \qquad B_2 = \begin{pmatrix} 2 & 2 \\ 1 & 0 \\ 2 & 3 \end{pmatrix} ,$$

$$\tilde{B}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} , \qquad \tilde{B}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} .$$

3.4. Decomposition into overlapping subsystems

Decomposition into disjoint subsystems means that the whole system is completely divided into different parts. As these parts should be weakly coupled, such a decomposition may not be successful in case of large-magnitude interactions. This is the motivation for overlapping decomposition, where the resulting subsystems have same part in common. These subsystems may be weakly coupled although disjoint subsystems are not.

A systematic way to overlapping decomposition begins with an expansion of the original system (3.4). The expanded system is decomposed into disjoint subsystems in the same way as described in Chapter 3.3. The question whether the original and the decomposed systems have the same trajectory is answered by the inclusion principle.

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Consider a pair of systems $\mathcal{S}, \bar{\mathcal{S}}$

$$(3.17) \qquad \mathcal{S}: \quad \dot{x} = Ax + Bu , \quad x(0) = x_0 ,$$
$$y = Cx ,$$

$$(3.18) \qquad \bar{\mathcal{S}}: \quad \dot{\bar{x}} = \bar{A}\bar{x} + \bar{B}u , \quad \bar{x}(0) = \bar{x}_0 ,$$
$$y = \bar{C}\bar{x} ,$$

where $x \in \mathbb{R}^n$, $\bar{x} \in \mathbb{R}^{\bar{n}}$, $n \leqq \bar{n}$. $x(t; x_0, u)$ and $\bar{x}(t; \bar{x}_0, u)$ denote the unique solutions of Eqs. (3.17) and (3.18) for the initial time $t = 0$, the initial state x_0, and fixed control u for $t \in \langle 0, +\infty \rangle$.

Definition 3.2. [40]. A system $\bar{\mathcal{S}}$ *includes* a system $\mathcal{S}$ if there exists an ordered pair of matrices (T, T^+) such that $TT^+ = I$, and for any $x_0 \in X$ of $\mathcal{S}$ and any fixed input $u(t)$ holds

$$(3.19) \qquad x(t; x_0, u) = T\bar{x}(t; T^+\bar{x}_0, u) ,$$
$$y[x(t)] = y[\bar{x}(t)] \quad \text{for all } t ,$$

where T^+ is a generalized inverse of T. Therefore, the states $x, \bar{x}$ are related by a linear transformation

$$(3.20) \qquad \bar{x} = T^+ x .$$

The systems $\mathcal{S}, \bar{\mathcal{S}}$ satisfying the equations (3.17), (3.18) are called a *contraction* or *expansion*, respectively.

Under which conditions are expansions generated for a given $\mathcal{S}$? For a given transformation matrix T, the matrices $\bar{A}, \bar{B}, \bar{C}$ are expressed as

$$(3.21) \qquad \bar{A} = TAT^+ + M , \quad \bar{B} = TB + N ,$$
$$\bar{C} = CT^+ + L,$$

where M is a complementary matrix of appropriate dimension. For $\bar{\mathcal{S}}$ to be an expansion of $\mathcal{S}$ a proper choice of M is required. This is specified by the following theorem.

Theorem 3.3. [39], [40], [95]. The system $\bar{\mathcal{S}}$ of (3.18) is an expansion of the system (3.17) if and only if

$$(3.22) \qquad T^+ M^i T \quad = 0 ,$$
$$T^+ M^{i-1} N = 0 ,$$
$$L M^{i-1} T \quad = 0 ,$$
$$L M^{i-1} N \quad = 0 \quad \text{for all} \quad i = 1, \ldots, \bar{n} .$$

A direct consequence of Definition 3.2 for $u = 0$ and Theorem 3.3 is the following theorem [39] [40] [95].

Theorem 3.4. [40]. If $\bar{\mathcal{S}}, \mathcal{S}$ is an expansion or contradiction respectively, then the asymptotic stability in the large of $\bar{\mathcal{S}}$ implies the asymptotic stability in the large

of $\mathscr{S}$. For the eigenvalues it holds

$$\Pi(\bar{A}) = \Pi(A)\,\Pi(M)\,,$$

where Π denotes the characteristic polynomial.

Example. Consider a system $\mathscr{S}$ in the form

$$(3.23) \qquad\qquad\qquad \mathscr{S}: \quad \dot{x} = Ax\,,$$

where $x = (x_1^{\mathrm{T}}, x_2^{\mathrm{T}}, x_3^{\mathrm{T}})^T$, $A = (A_{ij})\,(i,j = 1, 2, 3)$. Decompose the system (3.23) into two overlapping components $\bar{x}_1 = (x_1^{\mathrm{T}}, x_2^{\mathrm{T}})_f$, $\bar{x}_2 = (x_2^{\mathrm{T}}, x_3^{\mathrm{T}})^T$, $\bar{x} = (\bar{x}_1^{\mathrm{T}}, \bar{x}_2^{\mathrm{T}})^T$. Eq. (3.21) with

$$T = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix}, \qquad T^+ = \begin{pmatrix} I_1 & 0 & 0 & 0 \\ 0 & 0{\cdot}5I_2 & 0{\cdot}5I_2 & 0 \\ 0 & 0 & 0 & I_3 \end{pmatrix}$$

yields

$$A = \begin{pmatrix} A_{11} & A_{12} & 0 & A_{13} \\ A_{21} & A_{22} & 0 & A_{23} \\ A_{21} & 0 & A_{22} & A_{23} \\ A_{31} & 0 & A_{32} & A_{33} \end{pmatrix}, \qquad M = \begin{pmatrix} 0 & A_{12} & -A_{12} & 0 \\ 0 & A_{22} & -A_{22} & 0 \\ 0 & -A_{22} & A_{22} & 0 \\ 0 & -A_{32} & A_{32} & 0 \end{pmatrix}$$

$$A = (\bar{A}_{ij})\,, \quad i,j = 1,2$$

$$\bar{A}_{11} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \qquad \bar{A}_{22} = \begin{pmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{pmatrix}.$$

3.5. Time-scale decomposition

So far, we have described decomposition methods whose aim was to divide the system into different parts with small-magnitude interconnections. The following method is motivated by temporal considerations: If the overall system consists of a fast and slow subsystem, then the fast subsystem will arrive at its final state before the slow subsystem had begun the important part of its motion. Hence, the fast subsystem will behave as a static system from the slow subsystem point of view whereas the slow subsystem seems to be quiescent if it is considered in the time horizon of the fast part. Both have a weak influence on the other. Sandell et al. [91] classify the terms of weak or strong coupling according to the occurrence of a perturbation term in the right-hand side or in the left-hand side of a differential equation following the notion of a nonsingular or singular perturbation, respectively. We recognize these terms in accordance with Definition 3.1.

For the formalisation of time-scale decomposition consider the system (3.4) in the form

$$(3.24) \qquad\qquad \dot{x} = A_{11}x + A_{12}z + B_1u\,, \quad x(t_0) = x_0\,,$$
$$\varepsilon\dot{z} = A_{21}x + A_{22}z + B_2u\,, \quad z(t_0) = z_0\,,$$
$$y = C_1x + C_2z\,,$$

where $x \in \mathbb{R}^{n_s}$, $z \in \mathbb{R}^{n_f}$ are state vectors. ε is a small parameter $0 < \varepsilon \ll 1$. If we use $\varepsilon = 0$, the fast subsystem is considered as infinitely fast and approximated by an algebraic equation. The degenerated system has the form

$$\begin{aligned}
\dot{\hat{x}} &= A_{11}\hat{x} + A_{12}\hat{z} + B_1 u , \\
0 &= A_{21}\hat{x} + A_{22}\hat{z} + B_2 u , \\
\hat{y} &= C_1 \hat{x} + C_2 \hat{z} .
\end{aligned}$$
(3.25)

Supposing A_{22} to be nonsingular we obtain

$$\hat{z} = -A_{22}^{-1}A_{21}\hat{x} - A_{22}^{-1}B_2 u$$

and the system (3.24) has the form

$$\begin{aligned}
\dot{\hat{x}} &= \hat{A}\hat{x} + \hat{B}u , \\
\hat{y} &= \hat{C}\tilde{x} + \hat{D}u ,
\end{aligned}$$
(3.26)

where

$$\begin{aligned}
\hat{A} &= A_{11} - A_{12}A_{22}^{-1}A_{21} , \quad \hat{B} = B_1 - A_{12}A_{22}^{-1}B_2 , \\
\hat{C} &= C_1 - C_2 A_{22}^{-1}A_{21} , \quad \hat{D} = -C_2 A_{22}^{-1}B_2 .
\end{aligned}$$
(3.27)

Theorem 3.5. [112].

(1) If all the eigenvalues of the matrices $\hat{A}$, A_{22} have negative real parts, then there exists some $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ the equilibrium $x = 0$, $z = 0$ of the system (3.24) is asymptotically stable.

(2) If $\varepsilon \to 0$, then n_s roots of the characteristic equation of the system (3.24) converge to the roots of the characteristic equation of the system (3.25) and the solution of Eq. (3.24) approaches the solution of Eq. (3.25). n_f roots of the characteristic equation of (3.24) becomes infinite.

(3) The system (3.24) is asymptotically stable, if and only if the degenerated system (3.25) and the boundary layer system

$$\dot{\hat{z}}_\tau = A_{21}\hat{x} + A_{22}\hat{z} , \quad \tau = t/\varepsilon$$
(3.28)

are stable.

The generalization of the time scale decomposition to more than two temporally weak interacting parts can be made. A three-time scale system is given by

$$\begin{aligned}
\dot{x} &= A_{11}x + A_{12}z_1 + A_{13}z_2 , \\
\varepsilon_1 \dot{z}_1 &= A_{21}x + A_{22}z_1 + A_{23}z_2 , \\
\varepsilon_2 \dot{z}_2 &= A_{31}x + A_{32}z_1 + A_{33}z_2 ,
\end{aligned}$$
(3.29)

where x, z_1, z_2 are states of appropriate dimensions, and $\varepsilon_2/\varepsilon_1 \ll 1$. Three degenerate subsystems are obtained

$$\begin{aligned}
\mathscr{S}1: \quad \dot{x} &= A_{11}x + A_{12}z_1 , \\
0 &= A_{21}x + A_{22}z_1 , \\
\mathscr{S}2: \quad \varepsilon_1 \dot{z}_1 &= A_{22}z_1 ,
\end{aligned}$$
(3.30)

$$\mathscr{S}3: \quad 0 = A_{11}x + A_{12}z_1 + A_{13}z_2\,,$$
$$0 = A_{21}x + A_{22}z_1 + A_{23}z_2\,,$$
$$\varepsilon_2\dot{z}_2 = A_{31}x + A_{32}z_1 + A_{33}z_2\,.$$

Desoer and Shensa [26] proved the result that if the subsystems $\mathscr{S}1$, $\mathscr{S}2$, $\mathscr{S}3$ are stable, then there exist $\varepsilon_{10} > 0$ and $\varepsilon_{20} > 0$ such that for all $\varepsilon_1 \in (0, \varepsilon_{10})$, $\varepsilon_2 \in (\varepsilon_{20}, +\infty)$ the system (3.29) is asymptotically stable. Another form of generalization of the two time scale decomposition concerns systems that consist of a slow 'core' system and several subsystems

$$(3.31) \qquad \dot{x} = A_{11}x + \sum_j A_{1j}z_j + \sum_j B_{1j}u_j\,,$$
$$\varepsilon_i\dot{z}_i = A_{i1}x + A_{ii}z_i + \sum_j \varepsilon_{ij}A_{ij}z_j + B_{ii}u_i \quad \text{for} \quad i = 1, \ldots, N\,.$$

x and z_i are state vectors, ε_i are different small parameters, ε_{ij} provide weak coupling. The slow subsystem is, in general, directly excited by the inputs u_j of the fast subsystems [44]–[48].

4. THE DESIGN PROBLEM

The design problem can be formulated as follows:

Find a decentralized controller

$$(4.1) \qquad \dot{x}_{ri} = A_{ri}x_{ri} + B_{ri}u_i + E_{ri}v_i\,,$$
$$u_i = K_{1i}x_{ri} + K_{2i}y_i + K_{3i}v_i\,, \quad i = 1, \ldots, N$$

such that the closed-loop system (3.4), (4.1) satisfies the following specifications:

(I) The closed-loop system (3.4), (4.1) is asymptotically stable.

(II) Asymptotic regulation occur in the closed-loop system, i.e.

$$y_i(t) - v_i(t) \to 0 \quad \text{as} \quad t \to \infty\,,$$

where the command inputs $v_i(t)$ and the disturbances $p_i(t)$ entering the system at arbitrary points belong to a class of admissible external signals.

(III) The I/O-behaviour of the closed-loop system is well-suited according to given requirements, for example, on the step responses.

(IV) The design requirements (I)–(III) are satisfied in spite of model uncertainties.

According to the Internal Model Principle, the controller has to include a dynamical part which must be fixed in relation to the admissible external signals. This part is called servocompensator. If the closed-loop system is stable and the controller includes such an internal model of the external signals then asymptotic regulation occur [50]. Hence, we will not consider the design specification (II) further but

assume that a corresponding servocompensator is used. In Chapter 12 decentralized PI-control will be investigated, which possess the internal model for step signals and step disturbances.

The design specifications (I), (III) and (IV) look very similar to those prescribed for centralized control systems. In connection with interconnected plants two extensions are often made. First, due to the conceptual difficulties listed in the introductory chapter it is impossible to give the closed-loop system 'optimal' properties. The aim of designing decentralized controllers should be to produce a 'reasonable' rather than an 'optimal' performance. This does not exclude the possibility to use optimization methods (for example the LQ-design principle) as a *tool* for determining reasonable control laws. But in any case, the optimization of the overall system behaviour is not the ultimate goal.

Second, in comparison to decentralized control, the design specifications are extended. An example for additional requirements is connective stability. That is, the overall closed-loop system has to remain stable if some control stations are disconnected from the plant or if some subsystems of the plant including the corresponding decentralized control station is disconnected from the whole compound. This design specification will be considered in Chapters 8, 9, and 12. Another example of design specifications typical of large scale systems is the desire of a far-reaching decoupling of the subsystems by appropriately designing the feedback controller. In connection with decentralized control without information exchange this requirement concerns the degree of excitation of the subsystem j by disturbances entering subsystem i. Command following should be reached locally, that is, the output of the subsystem i should follow the command signals of this subsystem but remain virtually unchanged if command inputs are imposed on the other subsystems.

5. DECENTRALIZATION OF THE DESIGN PROBLEM

Decentralization concerns the information structure inherent in the solution of a decision problem. To explain the information structure of our control problem in detail consider this problem from the point of view of decision theory. Accordingly, we have to find control inputs $u_i(t)$, $t \in (0, \infty)$ on the basis of our a-priori knowledge concerning the plant properties described by a model $\mathscr{S}$ of the plant and the control aim $\mathscr{A}$ given in the form of the design requirements (I)–(IV), and the a-posteriori information $\mathscr{X}$ about the actual state $x(t)$ and $\mathscr{V}$ about the command signal $v(t)$. The design problem is completely described by the information $\{\mathscr{S}, \mathscr{A}, \mathscr{X}, \mathscr{V}\}$.

Since $u_i(t)$ is determined by a fixed control law (4.1) the decision process has two phases:

(1) Design phase: Determine appropriate control laws (4.1) on the basis of the a-priori information $\{\mathscr{S}, \mathscr{A}\}$ about the plant (3.4) and the design specifications (I)–(IV).

(2) Working phase: Determine control inputs $u_i(t)$ on the basis of the a-posteriori information $\{\mathcal{X}, \mathcal{V}\}$ about the system state $x(t)$ delivered by $y_i(t)$ and the current command signal $v_i(t)$.

In the working phase, the decision is made by using the control law (4.1). Since the controller (4.1) represents local feedbacks of y_i towards u_i and no cross-coupling between y_i and u_j $(i \neq j)$ can be produced, this decision process possesses a decentralized information structure. Each control station receives only the local information $\{y_i, v_i\}$. The existence of decentralized fixed modes [113] and the investigation of the structural reasons for the appearance of such modes [3], [25], [87] make clear that the decentralization of the information flow in the working phase brings about new problems and difficulties. However, our intension is to impose a decentralized information structure to the design phase too.

To describe the motivation of decentralization of this phase clearly, we look at a centralized design process first. The problem stated in Chapter 4 can be solved by determining appropriate controller parameters A_{ri}, B_{ri}, ..., K_{3i} on the basis of a complete knowledge $\{\mathcal{S}, \mathcal{A}\}$ of the whole plant (3.4) and all the design specifications (I)−(IV), see Fig. 4.1. Since, roughly speaking, 'all' is known, the elaboration of methods for determining appropriate controller parameters requires merely an extension of the design principles known from multivariable control. This has been really done, and methods for pole placement and optimal decentralized control etc. are known which have regard to the decentralization of the information flow through the controller (4.1) [58], [62], [98].

There are several motivations for the decentralization of the design phase:

— In interconnected systems, whose subsystems are widely distributed in space
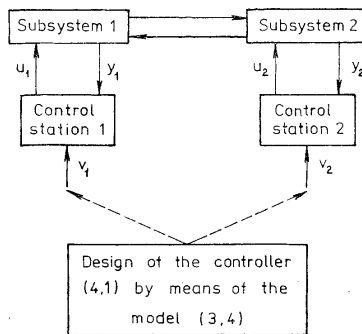

Fig. 4.1. Centralized design.

or represent components of different technological structure and aims, different authorities are responsible for the performance of the subsystems. Then the solution of control problems is decentralized as far as possible, and centralized authorities have the responsibility only for those performance criteria which describe the cooperation of the subsystems.

— If the overall systems can be decomposed into weakly coupled subsystems then the solution of the decentralized control problem can be approximately solved by designing centralized controllers for the isolated subsystems. This in turn is much easier than the design of a structurally constraint control law for the overall system.

— A typical phenomenon encountered in large scale systems is the change of the system structure during operation. The subsystems must have a certain autonomy because they have to be robust against the influences such changes may have on the performance of the overall system. Then the decentralized control station, which belongs to a given subsystem, should be designed from the subsystem point of view in accordance with the effects structural perturbations of the overall system may have.

— The design specifications of decentralized control may not be summarized within a single objective function for the overall system. There are several partially contradictory aims. For example the attainuation of a subsystem disturbance by distributing its effects over the whole plant may be advantageous from the subsystem point of view whereas it is undesirable from the overall system point of view.
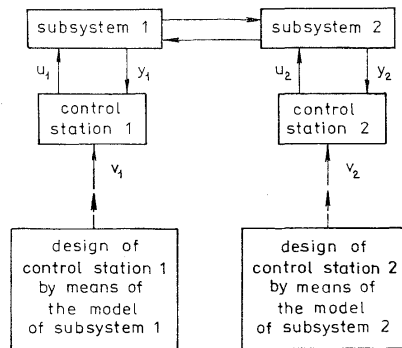

Fig. 4.2. Decentralized design.

Decentralization of the design process means that the control stations (Eq. (4.1) for fixed i) are designed independently by means of different plant models $\mathscr{S}_i$ and design specifications $\mathscr{A}_i$. That is, in contrast to the centralized design of decentralized controllers discussed above, the designer of control station i does not possess a complete model $\mathscr{S}$ of the whole plant and a complete knowledge of the design specifications $\mathscr{A}$ of the other control stations. Nevertheless, the decentralized controller obtained in this way should satisfy the design specifications (I)–(IV), see Fig. 4.2.

In the following chapters, we describe several concepts which aim at decentralizing the overall design problem and propose schemes in which the different control stations are determined independently as far as possible. The main difficulty which must be circumvented in one or another way is raised by the *non-classical information structure* of the decentralized design process. The notion of non-classical information structure has been introduced in decision theory [35] and investigated in connection with decentralized design by, for example, Bailey [7], Li and Singh [56], Tenney and Sandell [100], [101], Tsitsiklis and Athans [104] and Fiorio and Villa [29]. In these references the reason for non-classical (non-causal) information processes and ways to solve decision processes under such an information structure are described in detail. For our purpose, a rough understanding of this issue will be sufficient.

From the point of view of decision theory, decentralized control is a team problem, in which each decision maker has to determine the control input $u_i(t)$ in dependence upon a local model $\mathscr{S}_i$, decision aim $\mathscr{A}_i$, $\mathscr{V}_i$ and state information $\mathscr{X}_i$. If the best possible solution should be received, information exchange among the different decision makers is necessary, because the effect of the control input $u_i(t)$ on the performance of the plant depends on the input signals $u_j(t)$ $(i \neq j)$ imposed at the same moment by the other control stations.

Decision theory provides some ways to circumvent the difficulties brought about by this lack of information:

— In a game theoretical approach the decision makers may have contradictory aims $\mathscr{A}_i$, and methods are given to determine an 'equilibrium solution'. This is a control strategy $u_i(y_i)$ that gives the best possible solution if it is used by all decision makers [28], [76], [105]. For decentralized control two conceptual difficulties arise for the application of this result. First, a deterministic equilibrium may not exist. Then game theory proposes to use mixed strategies which cannot be implemented by a decentralized controller (4.1). Second, the solution of the game necessitates the knowledge of the objective functions $\mathscr{A}_i$ of all decision makers and a global model $\mathscr{S}$ of the process to be controlled. Both is not available in decentralized control.

— It is possible to derive objective functions $\mathscr{A}_i$ for the subsystems from an objective function $\mathscr{A}$ of the overall system to end up with completely independent decision problems, whose solution is globally optimal. This way, however, necessitates the solution of a difficult problem on the overall system level on the basis of

a complete information $\{\mathscr{S}, \mathscr{A}\}$ about the whole design problem. It is not acceptable for decentralized control.

— The decision problem can be reformulated as a problem with nested information structure if the decision makers have to act in a prescribed order. Then each decision maker acts alone on a system whose properties are fixed. The performance of the plant may be completely described by a model $\mathscr{S}_i$ with input u_i and output y_i. This model includes the control stations (4.1) that have already been implemented. It can be received, for example, by identifying the system from the subsystem point of view. This methodology is used in the sequential design of decentralized controllers which will be explained in Chapters 7 and 12.

These decision theoretical considerations show that a complete decentralization of the design phase is only possible if the design problem is reformulated as a problem with nested information structure. This is possible, as mentioned above, if a design sequence is prescribed or, as will be discussed in Chapters 6 and 7, if the overall system has particular structural properties. In all other cases the decentralization of the design process cannot be complete in the sense that no coordination between the decision makers is necessary or in the sense that no approximations have to be made to get independent design problems for the different control stations.

6. DESIGN FOR HIERARCHICALLY DECOMPOSED SYSTEMS

A substantial reduction of the computational complexity of the design can be gained from a hierarchical decomposition of the plant model. As pointed out in Chapter 3.2, such a decomposition is possible whenever the overall system consists of subsystems that are virtually interconnected only in parallel and in series. Then the interconnection matrix L in Eq. (3.3) can be brought into block triangular form. Under this condition the overall design problem can be solved as a sequence of smaller design problems. To explain this in more detail, suppose a chain connection structure according to Fig. 3 where each subsystem is supposed to be controllable and observable. This system structure enables us to completely decentralize the design of decentralized controllers. The different control stations of the decentralized controller are designed independently and only at the subsystem level. Furthermore, such a decomposition makes it possible to deal with dynamical requirements on the closed-loop system, the uncertainty of the plant model and reliability properties of decentralized control configurations, see [11]−[15]. The decomposition of chain connected subsystems into independent subsystems can be considered from the point of view of the decomposition of dynamical requirements on the system. These requirements can or cannot be decomposed into individual subsystem requirements. In the first case fully decentralized design can be performed, as it will be described now, in the second the influence of previous subsystems on the current subsystem must be referred to in the design. It results in a sequential design as described in Chapter 7.

6.1. Complete decentralization of the design for serially interconnected systems

Problem 6.1. Consider the plant

$$(6.1) \qquad \mathscr{S}_p: \quad \dot{x}_i = A_i x_i + B_i u_i + F_i p_{i-1}, \qquad x_i(0) = x_{i0},$$
$$y_i = C_i x_i + D_i u_i + H_i p_{i-1},$$
$$p_i = C_{pi} x_i + D_{pi} u_i + H_{pi} p_{i-1}, \quad p_0 = 0, \quad i = 1, ..., N,$$

where, respectively, x_i, u_i, y_i, p_i denote the ith subsystem state vector, control vector, output vector, interconnection output vector, see Fig. 5.

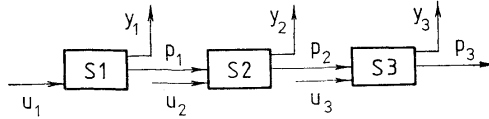**Fig. 5.**

Find a decentralized controller

$$(6.2) \qquad\qquad\qquad u_i = K_i y_i, \quad i = 1, ..., N$$

such that the closed-loop system (6.1), (6.2) satisfies the requirements (I)−(IV) given in Chapter 4.

Solution. Because of the serial connection of the subsystems, the problem of designing the complete decentralized controller, see Eq. (6.2) for all i, is decomposed into N subproblems. Each control station is designed independently in connection with its own subsystem only. Based on this way of the solution, we have to tackle the problems of *decomposition of dynamical requirements* and *the inclusion of the model uncertainties*. To solve the first problem, we suppose that the dynamical requirements on the I/O-behaviour of the overall system can be decomposed for the specific problem into the requirements on the I/O-behaviour of the independent subsystems. The model uncertainties are considered by means of upper bounds.

Evaluation of robustness. Instead of the exact description of the original system (6.1), we consider an approximate model with model error bound, and the triple $(\hat{A}_i, \hat{B}_i, \hat{C}_i)$ controllable-observable, of the form

$$(6.3) \qquad \mathscr{S}: \quad \dot{x}_i = \hat{A}_i x_i + \hat{B}_i u_i + \hat{E}_i s_i + \hat{F}_i p_{i-1}, \quad x_i(0) = x_{i0},$$
$$y_i = \hat{C}_i x_i + \hat{D}_i u_i + \hat{G}_i s_i + \hat{H}_i p_{i-1},$$
$$z_i = \hat{C}_{zi} x_i + \hat{D}_{zi} u_i + \hat{G}_{zi} s_i + \hat{H}_{zi} p_{i-1},$$
$$p_i = \hat{C}_{pi} x_i + \hat{D}_{pi} u_i + \hat{G}_{pi} s_i + \hat{H}_{pi} p_{i-1},$$

and

$$(6.4) \qquad\qquad |s_i| \leqq V_i * |z_i| + r_{0i}(x_{i0}, t), \quad t \geqq 0, \quad i = 1, ..., N,$$

27

where, respectively, s_i, z_i, V_i. r_{0i} are interconnection input vector, interconnection output vector to the error model (6.4), impulse response matrix, free motion of the error model. Eq. (6.4) describes an upper bound of the model error, $*$ denotes convolution operation. Suppose $\mathscr{S}$ to be stabilizable. The closed-loop system (6.2), (6.3) is described by the equations

$$(6.5) \qquad \mathscr{S}_{pc}: \quad \dot{x}_i = \tilde{A}_i x_i + \tilde{E}_i s_i + \tilde{F}_i p_{i-1} \,,$$

$$y_i = \tilde{C}_i x_i + \tilde{G}_i s_i + \tilde{H}_i p_{i-1} \,,$$

$$z_i = \tilde{C}_{zi} x_i + \tilde{G}_{zi} s_i + \tilde{H}_{zi} p_{i-1} \,,$$

$$p_i = \tilde{C}_{pi} x_i + \tilde{G}_{pi} s_i + \tilde{H}_{pi} p_{i-1} \,, \quad i = 1, \dots, N \,,$$

and Eq. (6.4). This system is analysed using the following lemma.

Lemma 6.1. [11].

(I) A sufficient condition for the stability of the system (6.4), (6.5) is:

a) The isolated systems (6.4), (6.5) are stable.

b) The following inequality holds

$$(6.6) \qquad \lambda_M \big[\textstyle\int_0^\infty V_i \, \mathrm{d}t \int_0^\infty \big(|\tilde{G}_i| \, \delta(t) + |\tilde{C}_{zi} \, \mathrm{e}^{\tilde{A}_i t} \, \tilde{E}_i|\big) \, \mathrm{d}t\big] < 1 \,,$$

where $\lambda_M(\cdot)$ denotes the maximal eigenvalue of the corresponding non-negative matrix.

(II) The I/O-behaviour of the system (6.4), (6.5) is approximately described by Eq. (6.5) with $s_i = 0$. If Eq. (6.6) is satisfied, the upper bound of the model error is given by the inequality

$$(6.7) \qquad |y_i - \hat{y}_i| \le |\tilde{G}_i \, \delta(t) + \tilde{C}_i \, \mathrm{e}^{\tilde{A}_i t} \, \tilde{E}_i| * \tilde{V}_i * |\tilde{H}_{zi} \, \delta(t) + \tilde{C}_{zi} \, \mathrm{e}^{\tilde{A}_i t} \, \tilde{F}_i| * |p_{i-1}| \,,$$

where

$$\tilde{V}_i = V_i + V_i * |\tilde{G}_{zi} \, \mathrm{e}^{\tilde{A}_i t} \, \tilde{E}_i| * \tilde{V}_i$$

and $\hat{y}_i$ denotes the output of the approximate model, i.e. Eq. (6.5) for $s_i = 0$.

Approximate model control design. Suppose that the feedback matrix K_i for all i is designed by means of standard LQ-procedures for the approximate model of the subsystem i, see Eq. (6.3) for $s_i = 0$ and $p_{i-1} = 0$. The LQ-problem has the form

$$(6.8) \qquad J_i = \textstyle\int_0^\infty (x_i^\mathsf{T} q_i Q_i x_i + u_i^\mathsf{T} R_i u_i) \, \mathrm{d}t \to \min_{u_i}$$

subject to Eq. (6.3) for $s_i = 0$, $p_{i-1} = 0$. q_i is a scalar used to reach the robustness requirements, see e.g. [8], [64], [75]. The optimal control approach with prescribed eigenvalues for the state feedback matrix generation can also be used [8]. The derived design procedure can be summarized in the following

Algorithm 6.1.

1) Initialize N, x_0, $p_0 = 0$, $\hat{A}_i, \ldots, \hat{H}_{pi}$, $q_i = 1$, V_i.
2) Test controllability and observability: $(\hat{A}_i, \hat{B}_i), (\hat{A}_i, \hat{C}_i)$, $\forall i$. If it is not satisfied goto 8).
3) $i = 1$.
4) Specify Q_i, R_i and solve (6.3), (6.8) for $s_i = 0$, $p_{i-1} = 0$ using standard LQ-procedure. If the desired response is not reached modify Q_i, R_i.
5) Test the condition (6.6). If it is not satisfied modify q_i and goto 4).
6) Evaluate the closed-loop error bound (6.7), (6.8). If it is too broad modify q_i and goto 4).
7) $i = i + 1$. If $i \leq N$ then goto 4).
8) End.

Example. Consider a string of vehicles described in [83]. The objective is to design the controller so that the distance between the vehicles and the velocity of vehicles of the string are constant. The control has to satisfy the requirement that the distance deviation between two succeding vehicles would be decreased monotonically, i.e. without overshoot. This requirement means mathematically that

$$(6.9) \qquad x_{i2}\dot{x}_{i2} < 0, \quad \forall i,$$

where x_{i2} is the distance of vehicle i from the vehicle $i - 1$.

Solution. The ith subsystem can be described in the form (6.3) with $x_i^T = (x_{i1}, x_{i2})$, where, respectively, x_{i1}, x_{i2} is the velocity, the distance. Supposing that the mass of vehicle can be changed during the operator of controller, upper and lower mass bounds are given: $m_i \in \langle 0.8, 1.2 \rangle$. Then the complete description of the ith vehicle is given by the equations

$$(6.10) \qquad
\begin{aligned}
\dot{x}_i &= \begin{pmatrix} -1 & 0 \\ -1 & 0 \end{pmatrix} x_i + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u_i + \begin{pmatrix} 0 \\ -1 \end{pmatrix} s_i + \begin{pmatrix} 0 \\ 1 \end{pmatrix} p_{i-1}, \\
y_i &= x_i + \begin{pmatrix} 1 \\ 0 \end{pmatrix} s_i, \\
z_i &= u_i, \\
p_i &= (1 \quad 0) x_i + s_i
\end{aligned}$$

and

$$(6.11) \qquad |s_i| \leq |1.25\, e^{-1.25t} - e^{-t}| * |z_i|.$$

The dynamical requirement (6.9) can be decomposed into the subsystem level dynamical requirements by the following lemma proved in [11].

Lemma 6.2. A sufficient condition for inequality (6.9) to hold is: The impulse response g_i between p_i and p_{i-1} is non-negative.

Using Lemma 6.2 the considered global problem is decomposed into the subsystem

level design problems: The objective is to find controllers (6.2) such that all closed-loop subsystems (6.2), (6.10), (6.11) are stable, all have non-negative impulse response functions and satisfy further dynamical requirements which may be specified on the subsystem level. Applying Algorithm 6.1 for

$$Q_i = \text{diag}(0{\cdot}05 \ \ 0{\cdot}05), \quad R_i = I_1$$

the following controller parameters

$$K_i^T = (-0{\cdot}22 \ \ 0{\cdot}22)$$

have been received. The simulation results of the robustness analysis are shown

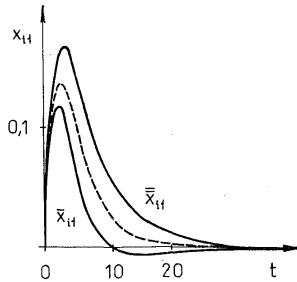

Fig. 6. The results of the robustness analysis: $\bar{x}_{i1}$, $\bar{\bar{x}}_{i1}$ denotes the lower, upper bound, respectively. The middle curve is the closed-loop approximate ith subsystem response.

in Fig. 6. The tolerance band makes clear that the impulse response of the closed-loop ith subsystem is for all masses m_i (nearly) non-negative.

6.2. Decentralized multi-controller configuration

Suppose $\mathscr{S}$ is a given plant (6.1) consisting of serially interconnected subsystems. The objective is to find a decentralized controller (6.2) satisfying the requirements (I)$-$(IV). Suppose further that some failures of the controller (6.2) can occur. These failures must be considered in the design process.

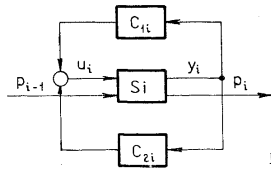There are two basic approaches to multi-controller configurations: with active

Fig. 7.

and passive redundancy [108]. To explain the basic idea of multi-controller con-figuration with passive redundancy consider the ith subsystem $\mathscr{S}_i$ with two controllers $\mathscr{C}_{i1}$, $\mathscr{C}_{i2}$ in parallel (Fig. 7). In the normal mode both controllers are operational. Should either controller fail the system is still stable, even though other properties might be changed adversely [95]. Vidyasagar and Viswanadham [108] solved the problem of reliable stabilization, where the normal mode controller $\mathscr{C}_{i1} + \mathscr{C}_{i2}$ is obtained as a by-product. In contrast, the problem solved in this paper deals with the case, when the normal mode controller is obtained as the solution of Problem 6.1 which further includes the additional requirement on the closed-loop system stability if either controller fails. Suppose that $\mathscr{C}_{i1} = \mathscr{C}_{i2}$

(6.12)
$$\mathscr{C}_{ij}: \quad u_i = 0{\cdot}5K_i y_i .$$

The closed-loop system (6.3), (6.12) is described by the equations

(6.13)
$$\dot{x}_i = \bar{A}_i x_i + \bar{E}_i s_i + \bar{F}_i p_{i-1} ,$$
$$y_i = \bar{C}_i x_i + \bar{G}_i s_i + \bar{H}_i p_{i-1} ,$$
$$z_i = \bar{C}_{zi} x_i + \bar{G}_{zi} s_i + \bar{H}_{zi} p_{i-1} ,$$
$$p_i = \bar{C}_{pi} x_i + \bar{G}_{pi} s_i + \bar{H}_{pi} p_{i-1}$$

and Eq. (6.4).

Problem 6.2. Consider Problem 6.1, where the controller $\mathscr{C}_i$ (6.2) consists of two identical controllers $\mathscr{C}_{i1}$, $\mathscr{C}_{i2}$ so that $\mathscr{C}_i = \mathscr{C}_{i1} + \mathscr{C}_{i2}$. Find a decentralized con-troller satisfying the requirements (I)−(IV) in Problem 6.1 and the requirement

(V) Should either controller $\mathscr{C}_{ij}$, $j = 1, 2$, fail the closed-loop system (6.4), (6.13) is stable.

The solution of Problem 6.2 includes the solution of Problem 6.1. It means that we have to tackle the problems of decomposition of the dynamical requirements and the inclusion of the model uncertainties. To satisfy the requirements (I)−(IV) in Problem 6.2 the solution of Problem 6.1 is used. To satisfy the requirement (V) Lemma 6.1 is used, but not for the closed-loop system with controller $\mathscr{C}_i$. The analysis is performed for the closed-loop system with controllers $\mathscr{C}_{i1}$, $\mathscr{C}_{i2}$. Of course, if the stability test is not satisfied for the closed-loop system (6.4), (6.13) then the design of the closed-loop system with controller $\mathscr{C}_i$ must be modified so that the stability of the closed-loop system with controller $\mathscr{C}_{ij}$ be satisfied. Therefore, Problem 6.2 is solved using the following lemma.

Lemma 6.2.

(I) A sufficient condition for stability, and the I/O-behaviour of the system (6.4), (6.5) are given by Lemma 6.1.

(II) A sufficient condition for stability of the system (6.4), (6.13) is:
a) The isolated systems (6.4), (6.13) are stable.

b) The following inequality holds

(6.14) $\qquad \lambda_M \left[\int_0^\infty V_i \, dt \int_0^\infty \left(\left| \bar{G}_i \right| \delta(t) + \left| \bar{C}_{zi} e^{\bar{A}_i t} \bar{E}_i \right| \right) dt \right] < 1$.

(III) The I/O-behaviour of the system (6.4), (6.13) can be approximated by Eq. (6.13) with $s_i = 0$. If Eq. (6.14) is satisfied, the upper bound of the model error is given by the inequality

(6.15) $\qquad \left| y_i - \hat{y}_{fi} \right| \leqq \left| \bar{G}_i \delta(t) + \bar{C}_i e^{\bar{A}_i t} \bar{E}_i \right| * \bar{V}_i * \left| \bar{H}_{zi} \delta(t) + \bar{C}_{zi} e^{\bar{A}_i t} \bar{F}_i \right| * \left| p_{i-1} \right|$,

where

$$\bar{V}_i = V_i + V_i * \left| \bar{G}_{zi} \delta(t) + \bar{C}_{zi} e^{\bar{A}_i t} \bar{E}_i \right| * \bar{V}_i$$

and $\hat{y}_{fi}$ denotes the output of the approximate model (6.13) for $s_i = 0$.

Proof. Part (I) is Lemma 6.1. Parts (II), (III) are direct application of Lemma 6.1. to the system (6.4), (6.14), all on the subsystem level [11], [12], [14]. $\qquad \square$

The derived design procedure can be summarized in

Algorithm 6.2.

1) Initialize N, x_0, $p_0 = 0$, $\hat{A}_i, \dots, \hat{H}_i$, $q_i = 0$, V_i.
2) Test controllability and observability: $(\hat{A}_i, \hat{B}_i)$, $(\hat{A}_i, \hat{C}_i)$, $\forall i$. If it is not satisfied goto 10).
3) $i = 1$.
4) Specify Q_i, R_i and solve (6.3) (6.8) for $s_i = 0$, $p_{i-1} = 0$. using standard LQ-procedure. If the desired response is not reached modify Q_i, R_i.
5) Test the condition (6.6). If it is not satisfied modify q_i and goto 4).
6) Evaluate the closed-loop error bound (6.7) (6.8). If it is too broad modify q_i and goto 4).
7) Test the condition (6.14). If it is not satisfied modify q_i and goto 4).
8) Evaluate the closed-loop error bound (6.15).
9) $i = i + 1$. If $i \leqq N$ then goto 4).
10) End.

Note that the presented procedure for reliable decentralized design of decentralized controllers can be simply generalized to the case of three or more parallel connected controllers. Then, more than two stability conditions have to be satisfied simultaneously.

7. SEQUENTIAL DESIGN OF DECENTRALIZED CONTROLLERS

The assumptions made in Chapter 6 are restrictive in two respects. First, we considered hierarchically interconnected systems where the one-directional interaction among the subsystems enabled us to design the control stations independently. Second, we assumed that the design requirements could be completely reformulated

as separate design requirements on the subsystems. Now, we extend our considerations in that we allow the design specifications (III) to refer to the interaction of the subsystems too. The assumption concerning the hierarchical structure of the plant remains valid, but, as will be seen in Chapter 12, sequential design methods can be used for strongly coupled subsystems (in the sense of Definition 3.1) too.

Sequential design means that the control station $1, \ldots, i - 1$ have already been designed and implemented when the ith control station is to be designed. The closed-loop subsystems $1, \ldots, i - 1$ are considered as disturbances entering the ith subsystem. Therefore, the dynamical requirements on the global system cannot be decomposed into the independent subsystem requirements.

Two problems are included in the following: linear state regulator problem [83] and linear robust output controller [12].

Problem 7.1. Find a decentralized controller for the following model

$$(7.1) \qquad \int_0^\infty \left(x_i^\mathsf{T} Q_i x_i + u_i^\mathsf{T} R_i u_i \right) \mathrm{d}t \to \min_{K_i}, \quad i = 1, \ldots, N$$

subject to

$$(7.2) \qquad \mathscr{S}_1: \quad \dot{x}_1 = A_1 x_1 + B_1 u_1, \qquad x_1(0) = x_{10},$$
$$y_1 = C_1 x_1,$$
$$z_1 = y_1,$$

$$(7.3) \qquad \mathscr{S}_i: \quad \dot{x}_i = A_i x_i + B_i u_i + E_i s_i, \quad x_i(0) = x_{i0},$$
$$y_i = C_i x_i,$$
$$z_i = C_{zi} x_i, \qquad\qquad i = 2, \ldots, N$$

and

$$(7.4) \qquad s_i = L_i z_i,$$

where $E_i = \operatorname{diag}\left(E_{1i}, \ldots, E_{i-1,i}\right)$, $C_{zi} = \left(C_1, \ldots, C_{i-1}\right)^\mathsf{T}$, $L_i = \operatorname{diag}\left(L_{1i}, \ldots, L_{i-1,i}\right)$. x_i, u_i, y_i, s_i, z_i are respectively the vectors of the states, controls, outputs, interconnection inputs, interconnection outputs. $A_i, B_i, E_i, C_i, C_{zi}, L_i$ are constant matrices, $Q_i = Q_i^\mathsf{T} \geqq 0$, $R_i = R_i^\mathsf{T} > 0$. Suppose the triples $\left(A_i, B_i, C_i\right)$ controllable and observable.

Solution. Implementation of subsystem control requires accessibility to states of that subsystem and preceding subsystems. Therefore, two subproblems must be solved for each subsystem: first, the local linear state regulator problem and then the preceding subsystems influence problem which is respected by a feedforward term.

Solve the local linear state regulator problem

$$(7.5) \qquad \int_0^\infty \left(x_i^\mathsf{T} Q_i x_i + u_i^\mathsf{T} R_i u_i \right) \mathrm{d}t \to \min$$

subject to

$$(7.6) \qquad \dot{x}_i = A_i x_i + B_i u_i.$$

Its solution can be found as a solution of the Riccati equation

$$(7.7) \qquad - P_i A_i - A_i^\mathsf{T} P_i - Q_i + P_i B_i R_i^{-1} B_i^\mathsf{T} P_i = 0.$$

A feedforward term from $\mathscr{S}_1$ is obtained from the Riccati equation for the whole problem using lower triangular matrix structure of the system matrix. The equation which is derived from this procedure for P_{21} has the form

$$(7.8) \qquad P_{21}\tilde{A}_1 + \tilde{A}_2^{\mathrm{T}} P_{21}^{\mathrm{T}} = -P_2 E_{12} L_{12} C_1 ,$$

where

$$\tilde{A}_i = A_i + B_i K_i , \quad K_i = -R_i^{-1} B_i^{\mathrm{T}} P_i .$$

This term is implemented as $-R_2^{-1} B_2^{\mathrm{T}} P_{21}$. Denote the feedforward gain from $\mathscr{S}_j$ to $\mathscr{S}_l$ $(j < l)$ by K_{j1} and the local gain of $\mathscr{S}_i$ by K_{ii}. The augmented system at the ith stage of design has the form

$$(7.9) \qquad \int_0^\infty \left(\bar{x}_i^{\mathrm{T}} Q_i \bar{x}_i + u_i^{\mathrm{T}} R_i u_i \right) \mathrm{d}t \to \min$$

subject to

$$(7.10) \qquad \dot{\bar{x}}_i = \bar{A}_i \bar{x}_i + \bar{B}_i u_i ,$$

where $\bar{x}_i = \left(x_1^{\mathrm{T}}, \ldots, x_i^{\mathrm{T}} \right)^{\mathrm{T}}$,

$$(7.11) \qquad \bar{Q}_i = \begin{pmatrix} 0 & 0 \\ 0 & Q_i \end{pmatrix}, \quad \bar{B}_i = \begin{pmatrix} 0 \\ \cdot \\ 0 \\ B_i \end{pmatrix},$$

$(7.12) \quad \bar{A}_i =$

$$= \begin{pmatrix} \tilde{A}_1 & 0 & 0 & \ldots & 0 \\ E_{12}L_{12}C_1 + B_2 K_{12} & \tilde{A}_2 & 0 & & 0 \\ E_{13}L_{13}C_1 + B_3 K_{13} & E_{23}L_{23}C_2 + B_3 K_{23} & \tilde{A}_3 & & 0 \\ \ldots & & & \cdot & 0 \\ E_{1,i-1}L_{1,i-1}C_1 + E_{i-1,j-1}K_{1,i-1} & E_{2,i-1}L_{2,i-1}C_2 + B_{i-1}K_{2,i-1} & \cdot & & \tilde{A}_{i-1} \\ E_{1i}L_{1i}C_1 & E_{2i}L_{2i}C_2 & & & E_{i-1,i}L_{i-1,i}C_{i-1}A_i \end{pmatrix}$$

Therefore P^i denotes the solution of the Riccati equation

$$(7.13) \qquad -P^i \bar{A}_i - \bar{A}_i^{\mathrm{T}} P^i - \bar{Q}_i + P^i \bar{B}_i R_i^{-1} \bar{B}_i^{\mathrm{T}} P^i = 0 ,$$

where the structure of P^i is supposed to be partitioned into blocks according to the subsystem state dimensions

$$P^i = \left(P_{kl}^i \right), \quad k, l = 1, \ldots, i$$

and the control is given by

$$(7.14) \qquad u_i = -R_i^{-1} B_i^{\mathrm{T}} \sum_{l=1}^i P_{il}^i x_l .$$

The following theorem solves Problem 7.1.

Theorem 7.1. [83]. The solution of Problem 7.1 exists, is unique and it is given by Eq. (7.14). P_{ii}^i are obtained as a solution of free subsystems, $P_{i,i-1}^i$ as a solution of the matrix equation

$$(7.15) \qquad P_{i,i-1}^i \tilde{A}_{i-1} + \tilde{A}_i^{\mathrm{T}} P_{i,i-1}^i = -P_{ii}^i \left(E_{i-1,i} L_{i,i} + C_{i-1} \right)$$

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and P_{ij}^i for $j < i - 1$ from the equation

$$(7.16) \qquad P_{ij}^i \tilde{A}_j + \tilde{A}_i^T P_{ij}^i = - \sum_{l=j+1}^{i-1} \left[P_{il}^i (E_{jl} L_{jl} C_j + B_l K_{jl}) \right] - P_{ii}^i E_{ji} C_j \, .$$

We shall summarize this solution in the algorithm.

Algorithm 7.1.

1) Initialize $A_i, B_i, E_i, C_i, C_{zi}, L_i, Q_i, R_i, x_{i0}, \ i = 1, \ldots, N$.
2) Test controllability and observability of the triple (A_i, B_i, C_i) for all i. If it is not satisfied goto 10).
3) Compute K_i as the solution of Eq. (7.8) for all i.
4) $i = 1$.
5) $i = i + 1$. If $i \leq N$ goto 10).
6) Compute $P_{i,i-1}$ by solving Eq. (7.15).
7) $j = i - 2$. If $j = 0$ goto 5).
8) Compute P_{ij}^i by solving Eq. (7.16).
9) $j = j - 1$. If $j = 0$ goto 5) else goto 8).
10) End.

Problem 7.2. Consider a plant $\mathscr{S}$ in the form

$$(7.17) \qquad \mathscr{S}: \quad \dot{x}_i = A_i x_i + B_i u_i + F_i p_{i-1} \, , \qquad x_i(0) = x_{i0} \, ,$$

$$y_i = C_i x_i + D_i u_i + H_i p_{i-1} \, ,$$

$$p_i = C_{pi} x_i + D_{pi} u_i + H_{pi} p_{i-1} \, , \quad p_0 = 0 \, , \quad i = 1, \ldots, N$$

with the same meaning as in Eq. (6.1). The objective is to find a decentralized controller such that the closed-loop system satisfies the requirements (I)–(IV) given in Chapter 4, where we consider step signals v_i.

Solution. Because of requirement (II), PI-controllers are used

$$(7.18) \qquad \dot{x}_{ei} = e_i \, , \quad e_i = y_i - v_i \, , \quad u_i = K_{1i} e_i + K_{2i} x_{ei} \, ,$$

where $K_i = (K_{1i}, K_{2i})$ is a constant matrix with $\det (I - K_{1i} D_i) \neq 0$. The asymptotic regulation occurs if the closed-loop system defined by Eq. (7.18) is stable.

A complete decomposition of dynamical requirements as in [11] is in general impossible. For instance, the I/O-behaviour of subsystem 2 for step inputs v_1 is influenced by both control stations 1 and 2. A very small dependence of y_2 upon v_1 can only be influenced by designing control station 1 so as to receive a low dependence of the coupling signal p_1 upon v_1 as well as by choosing control station 2 such that the 'disturbance' p_1 is diminished as far as possible. In the sequential design we have to tackle the problems of designing the control station i for the approximate model of free subsystem i as well as of analysing the closed-loop subsystem i for given command steps v_1 and interconnection signals p_{i-1}. The second problem must refer to the model uncertainties of subsystem i.

Consider the ith subsystem model description in the form of Eqs. (6.3), (6.4). The controller for the ith subsystem, see Eq. (7.18), is designed by means of LQ-procedures for the approximate model, i.e. for $s_i = 0$, $p_{i-1} = 0$ in Eq. (6.3). The integral part of controller is considered as a part of an extended subsystem in the form

$$(7.19) \qquad \dot{\tilde{x}}_i = \begin{pmatrix} \dot{x}_i \\ \dot{x}_{ei} \end{pmatrix} = \begin{pmatrix} \hat{A}_i & 0 \\ \hat{C}_i & 0 \end{pmatrix} \begin{pmatrix} x_i \\ x_{ei} \end{pmatrix} + \begin{pmatrix} \hat{B}_i \\ \hat{D}_i \end{pmatrix} u_i + \begin{pmatrix} 0 \\ -I \end{pmatrix} v_i ,$$

$$\tilde{y}_i = \begin{pmatrix} \hat{y}_i \\ \hat{e}_i \end{pmatrix} = \begin{pmatrix} \hat{C}_i & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} x_i \\ x_{ei} \end{pmatrix} + \begin{pmatrix} \hat{D}_i \\ 0 \end{pmatrix} u_i .$$

The the controller

$$(7.20) \qquad u_i = (K_{1i}, K_{2i})\, \tilde{y}_i$$

is designed as the solution of the problem

$$(7.21) \qquad J_i = \int_0^\infty (\tilde{y}_i^{\mathrm{T}} q_i Q_i \tilde{y}_i + u_i^{\mathrm{T}} R_i u_i)\, \mathrm{d}t \to \min$$

subject to Eq. (7.19). q_i is a scalar used to satisfy robustness requirements (IV). The matrices Q_i, R_i must be iteratively designed so that the closed-loop subsystem i satisfies the requirements (I)−(III), see [12], [55], [65], [69].

Evaluation of robustness. We have to check that the original closed-loop subsystem (6.3), (6.4), (7.18) for fixed i satisfies requirements (I)−(IV). This system is described in the form

$$(7.22) \qquad \dot{x}_i = \tilde{A}_i x_i + \tilde{B}_i v_i + \tilde{E}_i s_i + \tilde{F}_i p_{i-1} ,$$

$$y_i = \tilde{C}_i x_i + \tilde{D}_i v_i + \tilde{G}_i s_i + \tilde{H}_i p_{i-1} ,$$

$$z_i = \tilde{C}_{zi} x_i + \tilde{D}_{zi} v_i + \tilde{G}_{zi} s_i + \tilde{H}_{zi} p_{i-1} ,$$

$$p_i = \tilde{C}_{pi} x_i + \tilde{D}_{pi} v_i + \tilde{G}_{pi} s_i + \tilde{H}_{pi} p_{i-1}$$

and Eq. (6.4).

System (7.22) must be investigated concerning its I/O-behaviour with inputs v_i, p_{i-1} and outputs y_i, p_i. We assume that step inputs v_i are used. Information on p_{i-1} is received from the design of the $(i\text{-}1)$th control station. Supposing that $p_{i-1}(t)$ is approximated by a function $\tilde{p}_{i-1}(t)$, the following lemma is used for the analysis.

Lemma 7.2. [12].
(I) A sufficient condition for the stability of system (6.4), (7.22) is:
a) The approximate closed-loop system (7.22) is stable.
b) The following inequality is satisfied

$$(7.23) \qquad \lambda_M \big[\int_0^\infty V_i\, \mathrm{d}t \int_0^\infty \big(|\tilde{G}_{zi}|\,\delta(t) + |\tilde{C}_{zi}\, \mathrm{e}^{\tilde{A}_i t}\, \tilde{E}_{zi}|\big)\, \mathrm{d}t \big] < 1 .$$

(II) The I/O-behaviour of system (6.4), (7.22) can be approximated by Eq. (7.22) for $s_i = 0$. If Eq. (7.22) is satisfied, the upper bound of the model error is given by inequalities

$$(7.24) \qquad |y_i - \hat{y}_i| \leqq V_{ysi} * \bar{V}_i * V_{zvi} * |v_i| + V_{ysi} * \bar{V}_i * V_{zpi} * |\tilde{p}_{i-1}|,$$

$$|p_i - \hat{p}_i| \leqq V_{psi} * \bar{V}_i * V_{zvi} * |v_i| + V_{psi} * \bar{V}_i * V_{zpi} * |\tilde{p}_{i-1}|,$$

where

$$V_{ysi} = \left|\tilde{G}_i\right| \delta(t) + \left|\tilde{C}_i \, \mathrm{e}^{\tilde{A}_i t} \, \tilde{E}_i\right|, \qquad V_{psi} = \left|\tilde{G}_{pi}\right|\delta(t) + \left|\tilde{C}_{pi} \, \mathrm{e}^{\tilde{A}_i t} \, \tilde{E}_i\right|,$$

$$\bar{V}_i = V_i + V_i * \bar{V}_{zsi} * V_i, \qquad V_{zpi} = \left|\tilde{H}_{zi}\right| \delta(t) + \left|\tilde{C}_{zi} \, \mathrm{e}^{\tilde{A}_i t} \, \tilde{G}_i\right|,$$

$$V_{zsi} = \left|\tilde{G}_{zi}\right| \delta(t) + \left|\tilde{C}_{zi} \, \mathrm{e}^{\tilde{A}_i t} \, \tilde{E}_i\right|, \qquad G_i = \begin{pmatrix} \hat{B}_i(I - K_{1i}\hat{D}_i)^{-1} \, \hat{H}_i + \hat{F}_i \\ (I - \hat{D}_i K_{1i})^{-1} \, \hat{F}_i \end{pmatrix}.$$

$$V_{zvi} = \left|\tilde{D}_{zi}\right| \delta(t) + \left|\tilde{C}_{zi} \, \mathrm{e}^{\tilde{A}_i t} \, \tilde{B}_i\right|.$$

The function p_i should describe the influence of command steps at subsystems $1, \ldots, i$ on subsystem $i + 1$. Because of the uncertainties, we have got two tolerance bands which must be added. As the analysis of the ith closed-loop subsystem for a band of possible inputs is very complicated, we propose the following choice of $\tilde{p}_{i-1}$

$$(7.26) \qquad\qquad \tilde{p}_i = \hat{p}_i + \bar{p}_i\sigma(t),$$

where $\hat{p}_i$ is the function representing the middle of the tolerance band and $\bar{p}_i$ is the maximum width of the band. To use this step as a representation of the uncertainties of the signal p_i, $\tilde{p}_i$ can be considered as the implementation of the worst possible interconnection signal for PI control of the ith subsystem. The derived procedure can be summarized in the

Algorithm 7.2.

1) Initialize N, x_0, $p_0 = 0$, $\hat{A}_i, \ldots, \hat{H}_{pi}$, V_i, $q_i = 1$ for all i.
2) Test controllability and observability of $(\hat{A}_i, \hat{B}_i)$, $(\hat{A}_i, \hat{C}_i)$ for all i. If it is not satisfied goto 9).
3) $i = 1$.
4) Specify Q_i, R_i and solve (7.19), (7.20) using standard LQ-procedures. If requirements (I) − (III) on the behaviour are not satisfied modify Q_i, R_i and solve (7.19), (7.20).
5) Test condition (7.23). If it is not satisfied decrease q_i and goto (4).
6) Evaluate the closed-loop error bound (7.24) for step input v_i and interconnection input $\tilde{p}_{i-1}$. If the bands are too broad so that the requirements (III), (IV) are not satisfied, modify q_i and goto 4).
7) Determine $\tilde{p}_i$ (cf. Eq. (7.26)).
8) $i = i + 1$. If $i = N$ then goto 4).
9) End.

Example. Consider the string of vehicles described in Chapter 6.1. Find a decentralized controller such that the velocity of the string and the distance between the vehicles have given values. Moreover, set-point changes of the velocity should be followed without overshoot, the distance between the vehicles should at no time be smaller than a give value and the controller has to satisfy these requirements for a given range of vehicle load. Therefore, Eq. (6.10) is considered for $i = 1, 2, 3$. Instead of Eq. (6.11) the following model is used

$$(7.27) \qquad\qquad s_i = h_i z_i \quad \text{for} \quad |h_i| \leqq 0 \cdot 208 .$$

Solution. Since the command signals can be approximated by steps signals, decentralized PI-controllers are used. To solve the control task, the first vehicle controls its velocity, while the other vehicles control the distance to the preceding vehicle. The design problem is solved using the algorithm described above. The robustness can be analyzed by determining the smallest possible tolerance bands of the step response of the closed-loop subsystems [72]. The first control station represents a state feedback of model (7.19). Applying $Q_1 = 0·1I$, $R_1 = 1$ the controller parameters $k_{11} = -0·316$, $k_{12} = -0·316$ are computed using optimal state feedback control design procedure.

Fig. 8a illustrates the tolerance band of the step response of the closed-loop subsystem 1, i.e. control of velocity for $v_1 = \sigma(t)$. The middle of this band is used as function $\hat{p}_1$ of Eq. (7.26) for the analysis of coupling between subsystems 1 and 2 in connection with a step of amplitude $\bar{p}_1 = 0·05$.
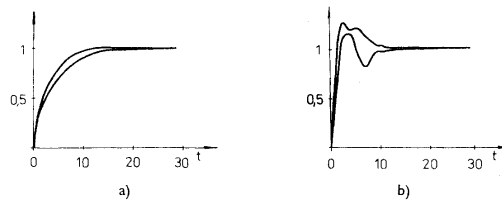

Fig. 8. Command step response of the vehicle control problem.

The second controller is designed as optimal output feedback for (7.19) with $Q_2 = 0·1I$, $R_2 = 1$. It results in $k_{21} = -0·9$, $k_{22} = -0·17$. The analysis of the closed-loop subsystem 2 must be done with respect to command steps $v_2 = \bar{v}_2 \, \sigma(t)$ as well as to the coupling signal $\tilde{p}_1$. Fig. 8b illustrates the tolerance band — control of distance between vehicle 1 and vehicle 2 — of the response of closed-loop subsystem 2 to command step $\sigma(t)$. A similar band can be obtained for the behaviour of y_2 subject to the 'disturbance' p_1.

Finally, Fig. 9 illustrates the behaviour of the global decentralized closed-loop system for $\hat{m}_i = 1$, i.e. for the approximate global closed-loop model, for a command step at vehicle 1.
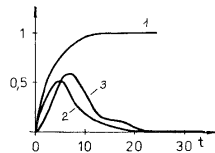

Fig. 9. Command step responses: 1 — velocity of vehicle, 2 — distance between vehicles 1 and 2, 3 — distance between vehicle 2 and 3.

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8. DESIGN ON THE BASIS OF DISJOINT DECOMPOSITION

8.1. The design philosophy

In this chapter we assume that the overall system (3.4) is decomposed into N disjoint subsystems (3.1) which are interconnected via Eq. (3.2). For the design of the decentralized controller (4.1) the interconnections among the subsystems are neglected. Then each control station (4.1) represents a centralized controller of the isolated subsystem and can be designed by methods known from multivariable control theory. The question arises how the overall system behaves under the influence of the subsystem interactions.

The design methods which will be described in the following are motivated by the observation that the behaviour of each subsystem depends mainly on the properties of this free subsystem and is affected by the other subsystems merely to a lower extent. Therefore, the analysis of a single subsystem must have regard to a detailed description of this subsystem, whereas the analysis of the whole system must take into consideration essentially the interconnection structure and only the main properties of every subsystem. Therefore, the methods described below consist of two steps:

1) Separate considerations of the isolated subsystems and the interconnection structure to design the control stations and to find out the main properties of the closed-loop subsystems.

2) Combination of the results of step 1) to conclude the properties of the composite system.

In contrast to hierarchical schemes for determining the solution of a problem with high dimension we do not use complete coordination of the solutions received on the subsystem level. This will become obvious in the algorithms presented later. The main merit of this way of solution is the possibility to analyse and design the system by means of adequate models at every level. The crucial point of this methodology is the evaluation of the overall system properties in step 2. It has been thoroughly elaborated for stability analysis but may be extended to a comprehensive methodology for qualitative analysis.

The stability analysis starts with the investigation at the subsystem level resulting in a rather coarse description of the stability properties of the isolated subsystems $\mathscr{S}i$ by means of very simple auxiliary subsystems $\mathscr{S}i'$, see Fig. 10.

Typically, Lyapunov functions or the norms of the subsystem-operators serve as models $\mathscr{S}i'$. The interconnection structure $\mathscr{K}$ is brought into a simpler form $\mathscr{K}'$ characterizing the strength of the couplings. In the second step the overall system stability is examined by means of the system $\mathscr{K}'$ composed of the subsystems $\mathscr{K}i'$ through the interconnections $\mathscr{K}'$. In most stability tests the system $\mathscr{K}'$ is a linear static or a dynamical system, whose dynamical order corresponds to the number of subsystems. Hence the stability examination of $\mathscr{S}'$ is very simple indeed.

In this chapter we want to solve the following problems:

The overall system (3.4) with $D = 0$

$$(8.1) \qquad \dot{x} = Ax + Bu,$$
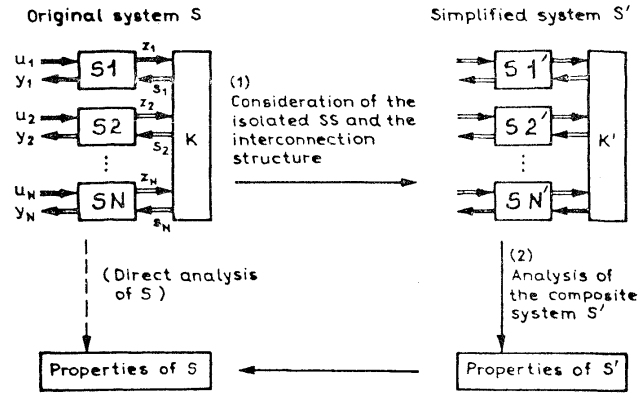$$y = Cx$$

Original system S

Fig. 10. General methodology in the analysis of large scale systems.

is assumed to be decomposed into subsystems

$$(8.2) \qquad \dot{x}_i = A_{ii}x_i + B_iu_i + \sum_j A_{ij}x_j, \quad i = 1, \ldots, N.$$

The pairs (A_{ii}, B_i) are assumed to be controllable. Denote

$$(8.3) \qquad \dot{x}_i = A_{ii}x_i + B_iu_i$$

the ith free subsystem. We consider the following problems.

Problem 8.1. Design a decentralized controller

$$(8.4) \qquad u_i = K_ix_i, \quad i = 1, \ldots, N$$

so that the overall closed-loop system (8.1), (8.4) is stabilized by assigning given spectra $\mathscr{L}_i$ to the closed-loop system (8.3), (8.4).

Problem 8.2. Design a decentralized controller

$$(8.5) \qquad u_i = K_ix_i, \quad i = 1, \ldots, N$$

by solving the problem

$$(8.6) \qquad \sum_i J_i \to \min$$

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where

(8.7)
$$J_i = 0.5 \int_0^\infty \left(\|x_i\|^2_{Q_i} + \|u_i\|^2_{R_i} \right) \mathrm{d}t$$

subject to Eq. (8.2).

Q_i, R_i is a symmetric nonnegative definite matrix, positive definite matrix, respectively. Evaluate the influence of the interconnections on the system performance.

8.2. Stability criteria for composite systems

As a preliminary result for the solution of Problems 8.1 and 8.2 we derive stability criteria for the closed-loop overall system (8.1), (8.2). This system can be described in compact form by

(8.8)
$$\dot{x} = \bar{A}x$$

its elements $\bar{a}_{ij}$ are considered in the form

(8.9)
$$\bar{e}_{ij} = -\delta_{ij}\alpha_i + \bar{e}_{ij}\alpha_{ij}, \quad i, j = 1, ..., N,$$

where

$$\alpha_i > 0, \quad \alpha_{ij} \geqq 0, \quad \alpha_i > \alpha_{ii}.$$

$\bar{e}_{ij}$ are the elements of fundamental matrix E.

Definition 8.1. The fundamental interconnection matrix $\bar{E} = (\bar{e}_{ij})$ has the elements defined by

(8.10)
$$\bar{e}_{ij} = 1 \quad \text{if} \quad x_j \quad \text{influences} \quad \dot{x}_i$$

$$\bar{e}_{ij} = 0 \quad \text{if} \quad x_j \quad \text{has no influence on } \dot{x}_i.$$

By the substitution of the original matrix $\bar{E}$ by $E = (e_{ij})$, $e_{ij} \in \langle 0, \bar{e}_{ij} \rangle$ structural perturbations of the interconnection are introduced.

Definition 8.2. The equilibrium $x = 0$ of the system (8.8) is connectively asymptotically stable in the large if it is asymptotically stable for all interconnection matrices E.

The class of the systems $\mathscr{S}$ can be simply enlarged to the class of the nonliner time-varying bounded functions of the form

(8.11)
$$\dot{x} = \tilde{A}(t, x) x,$$

where

(8.11)
$$\tilde{A}(t, x) = (\tilde{a}_{ij}(t, x)),$$

$$\tilde{a}_{ij}(t, x) = -\delta_{ij} \beta_i(t, x) + e_{ij}(t) \beta_{ij}(t, x)$$

are related to $\bar{a}_{ij}$ by

$$\beta_i(t, x) \|x_i\| \geqq \alpha_i \gamma_i(\|x_i\|), \quad \beta_{ij}(t, x) x_j \leqq \alpha_{ij} \gamma_j(\|x_j\|),$$

where $\gamma_i \colon \mathbb{R}^+ \to \mathbb{R}^+$, $\gamma_i(0) = 0$, $\gamma_i(\varrho_1) < \gamma_i(\varrho_2)$, $\forall \varrho_1, \varrho_2$,
$0 \leqq \varrho_1 < \varrho_2 < +\infty$. The functions $\gamma(\cdot)$ are called the comparison functions, $\|\cdot\|$

denotes the Euclidean norm. Assuming the constraints

$$(8.12) \qquad \gamma_i(\|x_i\|) = \|x_i\| , \quad \forall i ,$$

the following classes of the continuous functions can be defined

$$(8.13) \qquad \mathscr{B}_i = \{\beta_i(t, x): \beta_i(t, x) \geqq \alpha_i\} ,$$

$$\mathscr{B}_{ij} = \{\beta_{ij}(t, x): \|\beta_{ij}(t, x)\| \leqq \alpha_{ij}\} .$$

Definition 8.3. The equilibrium $x = 0$ of the system (8.8) is connectively and absolutely exponentially stable if and only if there are two positive constants C and c such that for arbitrary initial condition x_0 the relation

$$(8.14) \qquad \|x(t, x_0)\| = C\|x_0\| \, \mathrm{e}^{-ct} , \quad \forall t$$

is valid for all $\beta_i \in \mathscr{B}_i$, $\beta_{ij} \in \mathscr{B}_{ij}$ and all interconnection matrices $E(t)$.

To check the stability the following theorem is used.

Theorem 8.1. $[94]$. The equilibrium of the system (8.8) is connectively exponentially stable if the matrix $\bar{A} = (\bar{a}_{ij})$ satisfies the condition

$$(8.15) \qquad \det \begin{vmatrix} \bar{a}_{11} \ldots \bar{a}_{1i} \\ \vdots \qquad \vdots \\ \bar{a}_{i1} \ldots \bar{a}_{ii} \end{vmatrix} > 0 , \quad i = 1, \ldots, n .$$

To characterize a complex plant, it is sufficient that some qualitative quantities of a dynamic process do not overflow some given boundaries. It means that it is not necessary to obtain an exact mathematical model of a plant, but it is sufficient to have some simplified comparison model. The following condition is principal for a comparison model: Denoting a chosen qualitative quantity of a mathematical model by $\mathscr{J}_s$ and an analogous quantity of a comparison model by $\mathscr{J}_m$, then the inequality $\mathscr{J}_s > \mathscr{J}_m$ must hold.

Derive the principle of Lyapunov functions construction for the model (8.2). Supposing the ith subsystem model for $u_i = 0$

$$(8.16) \qquad \dot{x}_i = A_i x_i ,$$

a quadratic Lyapunov function for it has the form

$$(8.17) \qquad g_i(x_i) = x_i^\mathrm{T} H_i x_i ,$$

where H_i is a symmetric positive definite matrix. It follows from the matrix Lyapunov equation

$$(8.18) \qquad A_i^\mathrm{T} H_i + H_i A_i = -G_i ,$$

where G_i is a given symmetric positive definite matrix (often given as a unit matrix). To construct a comparison model, Krasovski's theorem is used $[79]$, $[94]$, $[112]$.

Theorem 8.2. $[79]$. If any solution $x_i(t; t_0, x_0)$ of (8.16) satisfies the condition for exponential stability

$$\|x_i(t; t_0, x_0)\| \leqq M\|x_{i0}\| \, \mathrm{e}^{-\alpha_i(t - t_0)}$$

for $x_{i0} \in \mathbb{R}^{n_i}$ of the equilibrium $x = 0$,
then there exist Lyapunov functions $g_i(x_i)$ in $\mathbb{R}^{n_i}$ satisfying

$$(8.19) \qquad c_{1i}^2 \|x_i\|^2 \leqq g_i(x_i) \leqq c_{2i}^2 \|x_i\|^2 , \quad \dot{g}_i(x_i) = -c_{3i}^2 \|x_i\|^2 ,$$
$$\|\operatorname{grad} g_i(x_i)\| \leqq c_{4i}^2 \|x_i\| ,$$

where c_{ij} are real positive numbers,

$$\dot{g}_i(x_i) = \frac{\mathrm{d} g_i(x_i)}{\mathrm{d} t} ,$$

$$\operatorname{grad} g_i(x_i) = \left(\frac{\partial g_i}{\partial x_{i1}}, \ldots, \frac{\partial g_i}{\partial x_{1,n_i}} \right)^{\mathsf{T}} .$$

The comparison model $\mathscr{S}i'$ has the form

$$(8.20) \qquad \mathscr{S}i': \quad x_i' = x_{i0}'\, \mathrm{e}^{-\alpha_i(t-t_0)} , \quad x_{i0}' = M \|x_{i0}\|$$

Using a quadratic Lyapunov function (8.17) and Eqs. (8.18), (8.19) we obtain

$$(8.21) \qquad c_{1i}^2 = \lambda_m(H_i) , \quad c_{2i}^2 = \lambda_M(H_i) ,$$
$$c_{3i}^2 = \lambda_m(G_i) , \quad c_{4i}^2 = \lambda_M(H_i) ,$$

where $\lambda_m(\cdot)$. $\lambda_M(\cdot)$ denotes the minimum, maximum eigenvalue of a matrix. The comparison model (8.20) has then the form

$$(8.22) \qquad x_{i0}' = \frac{c_{2i}^2}{c_{1i}^2} \|x_{i0}\|^2 \exp\left[-\frac{c_{3i}^2}{c_{2i}^2}(t - t_0) \right] .$$

The Lyapunov function $g_i(x_i)$ is a Lipschitz function

$$(8.23) \qquad |g_i(x_i) - g_i(x_i')| = L_i \|x_i - x_i'\| ,$$

where the Lipschitz constant $L_i = \lambda_M(H_i)/\lambda_m(H_i)^{1/2}$ and the interconnection satisfies

$$(8.24) \qquad \|A_{ij}x_j\| \leqq \xi_{ij} \|x_j\| ,$$

where $\xi_{ij} = \lambda_M(A_{ij}^{\mathsf{T}} A_{ij})^{1/2}$. Further, following the inequality

$$(8.25) \qquad \dot{g}_i(x_i)|_{(8.7)} \leqq \dot{g}_i(x_i)|_{(8.15)} + L_i \|A_{ij}x_j\| , \quad \forall i ,$$

we obtain

$$(8.26) \qquad \dot{g}_i(x_i)|_{(8.7)} \leqq \psi_i(\|x_i\|) + 2L_i \sum_{j=1}^{N} \lambda_M(H_i)^{1/2} \lambda_m^{-1}(G_j) \xi_{ij} \psi_j(\|x_j\|) ,$$

where $\psi_i(\|x_i\|) = 0{\cdot}5\lambda_M(H_i)^{-1/2} \lambda_m(G_i) \|x_i\|$.

Finally, the $N \times N$ aggregated matrix S has the elements

$$(8.27 \qquad s_{ij} = \begin{cases} -1 & \text{for} \quad i = j \\ 2L_i\lambda_M(H_j)^{1/2} \lambda_m(G_j)^{-1} \xi_{ij} & i \neq j . \end{cases}$$

Theorem 8.3. [44]. The equilibrium $x = 0$ of the system (8.1) is connectively exponentially stable in the large if the matrix $S = (s_{ij})$ given by Eq. (8.26) satisfies

the inequalities

$$(8.28) \qquad \det \begin{vmatrix} s_{11} & \dots & s_{1i} \\ \vdots & & \vdots \\ s_{i1} & \dots & s_{ii} \end{vmatrix} > 0 , \quad i = 1, \dots, N .$$

8.3. Aggregation-decomposition method

The aggregation-decomposition method proceeds in two steps: 1. Design of the control station i for the isolated subsystem. 2. Test of overall stability.

Design of the control station i for the isolated subsystem

Let us consider the system $\mathscr{S}$ given by Eq. (8.1). There are several way of pole assignment computations. Let us present one of numerically simple procedure. Suppose that the overall system is input-decentralized. We transform the ith subsystem into the controllable companion form which simplifies spectrum assignment. Denote this transform Q_i, then the original ith subsystem is transformed into the form

$$(8.29) \qquad \dot{\hat{x}}_i = \hat{A}_i \hat{x}_i + \hat{B}_i u_i + \sum_j \hat{A}_{ij} \hat{x}_j ,$$

so that

$$\hat{x}_i = Q_i x_i , \quad \hat{A}_i = Q_i A_i Q_i^{-1} , \quad \hat{B}_i = Q_i B_i , \quad \hat{A}_{ij} = Q_i A_{ij} Q_j^{-1}$$

with

$$\hat{A}_i = \begin{pmatrix} 0 & 1 & \dots & 0 \\ 0 & 0 & & 0 \\ . & . & \dots & 1 \\ -a_1^i & -a_2^i & \dots & -a_{n_i}^i \end{pmatrix} , \quad \hat{B}_i = \begin{pmatrix} 0 \\ . \\ . \\ 1 \end{pmatrix}$$

for dim $B_i = m_i > 1$, see e.g. [94]. Considering a decentralized single-stage control design the interconnection class $\hat{A}_{ij} = (\hat{a}_{pq}^{ij})$ is supposed to be constrained so that for $i \neq j$, $\forall i, j$, it has zero elements below the diagonal

$$(8.30) \qquad \hat{a}_{pq}^{ij} = 0 \quad \text{for} \quad p < q .$$

We design the local control

$$(8.31) \qquad u_i = -\hat{K}_i \hat{x}_i , \quad \forall i ,$$

so that the matrix $\hat{A}_i - \hat{B}_i \hat{K}_i$ has assigned a set of, for the simplicity real, eigenvalues λ_p^i defined by

$$(8.32) \qquad \mathscr{L}_i(c) = \{ c\lambda_1^i, \dots, c\lambda_{n_i}^i \} , \quad \forall i ,$$

for $c \geq 1$, $\lambda_p^i < 0$, $\forall p$. The constant c is to be determined so that the overall system is stable.

Consider the isolated ith subsystem

$$\dot{\tilde{x}}_i = A_i \tilde{x}_i ,$$

where
$$\tilde{x}_i = T_i^{-1} Q_i^{-1} x_i ,$$

$$T_i = R_i \hat{T}_i , \quad R_i = \text{diag} \left(1, c, \ldots, c^{n_i - 1}\right) ,$$

$$\hat{T}_i = \begin{pmatrix} 1 & 1 & \ldots & 1 \\ -\lambda_1^i & -\lambda_2^i & \ldots & -\lambda_{n_i}^i \\ \cdot & \cdot & \ldots & \cdot \\ (-\lambda_1^i)^{n_i - 1} & (-\lambda_2^i)^{n_i - 1} & \ldots & (-\lambda_{n_i}^i)^{n_i - 1} \end{pmatrix} ,$$

(8.33)
$$\Lambda_i = \text{diag} \left(c\lambda_1^i, \ldots, c\lambda_{n_i}^i\right) = T_i^{-1}(\hat{A}_i - \hat{B}_i \hat{K}_i) \, T_i ,$$

$$\tilde{A}_{ij} = T_i^{-1} \hat{A}_{ij} T_j , \quad \forall i, j ,$$

then the subsystem is stabilized with a degree of exponential stability

(8.34)
$$c_i = \min_p c\lambda_p^i .$$

Choosing the Lyapunov function $g_i : \mathbb{R}^{n_i} \to \mathbb{R}^+$

(8.35)
$$g_i(\tilde{x}_i) = \left(\tilde{x}_i^T \tilde{H}_i \tilde{x}_i\right)^{1/2} ,$$

where

(8.36)
$$\Lambda_i^T \tilde{H}_i + \tilde{H}_i \Lambda_i = -2\eta_i \Lambda_i , \quad \tilde{H}_i = \eta_i I_i ,$$

$\eta_i > 0$ is a constant, I_i is the $n_i \times n_i$ unit matrix.

Stability analysis of the overall closed-loop system

Let us explain first the concept of decentralization in this case. The subsystem model design results in local feedback matrices K_i which are determined only using the ith subsystem data, i.e. matrices A_i, B_i and required spectrum $\mathcal{L}_i$. Therefore, all subsystem are designed independently. It means for instance that parallel processing of subsystems design can be applied. Further, on the overall system level the stability test is performed by means of information on the subsystems, i.e. constants c_i for all subsystems, and on the interconnection given by $\lambda_M(\tilde{A}_{ij}^T \tilde{A}_{ij})$, see Fig. 11 for $N = 2$. This test is performed on the lower order matrix, whose order is given by the number of subsystems. Therefore, no coordination is necessary when considering decentralization and the stability test represents only a sufficient condition. It means that only a rough information on subsystems and interconnection is required for the stability test.

Let us construct the aggregation matrix $\bar{S}$ for

(8.37)
$$\bar{s}_{ij} = (1 - \delta_{ij}) \lambda_M(\tilde{A}_{ij}^T \tilde{A}_{ij})^{1/2} ,$$

then

(8.38)
$$\bar{s}_{ij} = -c_i \delta_{ij} + (1 - \delta_{ij}) \lambda_M(\tilde{A}_{ij}^T \tilde{A}_{ij})^{1/2} .$$

The off-diagonal elements of the matrix $\bar{S}$ given by Eq. (8.37) are bounded in c. Note only that if $\bar{S}$ satisfies Eq. (8.28) then multiplying it by positive vector $\mu = \text{diag}(\mu_1, \ldots, \mu_N)$ so that $\bar{S} = \mu S$ the matrix $\bar{S}$ satisfies also the diagonal dominance

condition, i.e. Eq. (8.28) for $\bar{S}$. Using Theorem 8.2 on the matrix $\bar{S}$ given by Eqs. (8.37), (8.38) we obtain the stability test for the overall system for the chosen c. Further, note that only real distinct eigenvalues are considered. Multiple eigenvalues do not influence the derived procedure in principle. The algorithm can be formulated now.
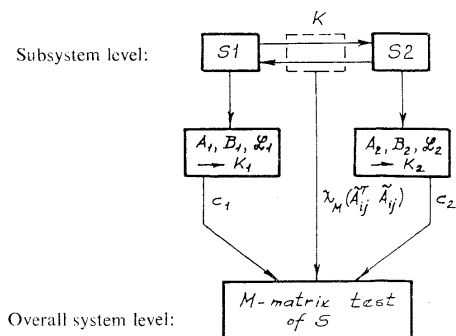

Fig. 11.

Algorithm 8.1.

1) Initialize $A_i, B_i, A_{ij}, \mathscr{L}_i, \varDelta > 0, \forall i, j.$
2) Test controllability (A_i, B_i). If it is not satisfied goto 9, $\forall i$.
3) Compute Q_i, check $\hat{A}_{ij} = \|\hat{a}_{pq}^{ij}\|, \hat{a}_{pq}^{ij} = 0$ for $p < q, \forall i, j, p, q.$
4) Compute $\hat{K}_i, T_i$.
5) Compute $\bar{S} = (\bar{s}_{ij}), \bar{s}_{ij} = -c_i \delta_{ij} + (1 - \delta_{ij}) \lambda_M (\tilde{A}_{ij}^T \tilde{A}_{ij})^{1/2}.$
6) Test $(-1)^p \begin{vmatrix} \bar{s}_{11} \dots \bar{s}_{1p} \\ \vdots \qquad \vdots \\ \bar{s}_{p1} \dots \bar{s}_{pp} \end{vmatrix} > 0, \quad p = 1, \dots, N.$
7) If 6) is not satisfied goto 8).
8) Compute $u = (u_1^T, \dots, u_N^T)^T, u_i = -K_i x_i, K_i = \hat{K}_i Q_i^{-1}.$
9) End.

Therefore, in Algorithm 1 the steps 1)−5), 6)−7) correspond to the subsystem level, overal system level, respectively.

Optimality of the decentralized control

Let us continue to solve Problem 8.2. The decentralized or local control on the subsystem level has been designed as the stabilizing control only in the previous case. We deal with the optimal stabilization now. Consider the solution of the problem (8.2), (8.6). Denote the overall cost performance function value $J^*(t_0, x(t_0))$ and the performance function value $J(t_0, x(t_0))$ when considering the solution of the

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problem (8.3), (8.6). These solutions are given by algebraic Riccati equations

$$(8.38) \qquad P^*A + AP^{\mathsf{T}*} - P^*BR^{-1}P^* + Q = 0$$

and

$$(8.39) \qquad P_iA_i + A_iP_i^{\mathsf{T}} - P_iB_iR_i^{-1}B_i^{\mathsf{T}}P_i + Q_i = 0$$

where $P = \operatorname{diag}(P_1, \ldots, P_N)$. The gain matrices are

$$(8.40) \qquad K^* = R^{-1}B^{\mathsf{T}}P^*, \quad K = R^{-1}B^{\mathsf{T}}P$$

and

$$(8.41) \qquad J^*(t_0, x(t_0)) = \|x_0\|_{P^*}^2, \quad x(t_0) = x_0, \quad J(t_0, x(t_0)) = \|x_0\|_P^2.$$

We define the concept of connective suboptimality index ε to evaluate the interaction effect on the criterion value.

Definition 8.4. [94]. The system (8.2), (8.6), (8.7) is positive, neutral, negative suboptimal with *index* ε if the following relation holds

$$(8.42) \qquad J^*(x_0) \leqq (1 + \varepsilon) J(x_0)$$

for $\varepsilon > 0$, $\varepsilon = 0$, $\varepsilon < 0$, respectively.

The interconnections are then classified as nonbeneficial, neutral, beneficial for $\varepsilon > 0$, $\varepsilon = 0$, $\varepsilon < 0$, respectively.

Consider further the positive suboptimality index.

Theorem 8.4. [43], [93]. If there are nonnegative numbers

$$\tilde{s}_{ij} = (1 - \delta_{ij}) \lambda_M(A_{ij}^{\mathsf{T}}A_{ij})$$

and

$$Z_i = P_iB_iR_i^{-1}B_i^{\mathsf{T}}P_i + Q_i, \quad \forall i, j,$$

then

$$(8.43) \qquad \varepsilon \leqq \frac{1}{2} \frac{\bar{S}}{\bar{\bar{S}} - \bar{S}},$$

where

$$(8.44) \qquad \bar{S} = \sum_j \sum_i \tilde{s}_{ij}, \qquad \bar{\bar{S}} = \frac{\min\limits_i \lambda_m(Z_i)}{\max\limits_i \lambda_M(P_i)}.$$

The design procedure can be formulated as follows.

Algorithm 8.2.

1) Initialize $A_i, B_i, A_{ij}, Q_i, R_i, \forall i$.
2) Test controllability: $(A_i, B_i), \forall i$. If it is not satisfied goto 6).
3) Compute $P_i, \forall i$ by solving Eq. (8.39).
4) Compute ε using (8.43), (8.44).
5) Compute $u = (u_1^{\mathsf{T}}, \ldots, u_N^{\mathsf{T}})^{\mathsf{T}}$ using Eq. (8.40).
6) End.

This procedure evaluates the bounds on the suboptimality index, but it does not give the way how to reduce it. To obtain the beneficial effect the overall control design

$$(8.45) \qquad u^g = (M - K^g)\, x$$

can be used. $K^g = B^+ A$, where B^+ denotes the Moore-Penrose pseudoinverse, neutralizes the interaction effect. M is a suitable chosen matrix. The condition under which the beneficial effect occurs forms an open problem recently.

8.4. I/O-analysis of the closed-loop system

The aggregation-decomposition method was mainly concerned with the stability of the overall system. The suboptimality index gives only a rough measure of the I/O-behaviour. Now, the method of designing the decentralized stations separately for the isolated subsystem and analysing the overall system by means of global subsystem models will be extended so that the overall system I/O-behaviour can be evaluated more precisely. Note, however, that we do not use the complete overall system to avoid the dimensionality problem, to be able to have regard to structural perturbations as above and to consider model uncertainties.

Comparison system. The key notion of our approach are comparison systems. Instead of the exact system description (8.1) a linear system

$$(8.46) \qquad r(t) = V_v * w = \int_{-\infty}^{+\infty} V(t - \tau)\, w(\tau)\, d\tau$$

is used to describe the behaviour of the original system. In Eq. (8.46) $V(t)$ denotes the $r \times m$ impulse response matrix and $r(t)$, $w(t)$ the input and output vectors respectively. The star $*$ denotes the convolution operation. The impulse response matrix $V(t)$ is to be determined in such a way that for every bounded input $u(t)$ in Eq. (8.1) and $w(t) = |u(t)|$ in Eq. (8.46) the output $r(t)$ of the system (8.46) majorizes the absolute value of the output $y(t)$ of the system (8.1)

$$(8.47) \qquad r(t) = V * |u| \geqq |y(t)| \quad \text{for all} \quad t \geqq 0.$$

Here and in the following the symbol $|\cdot|$ signifies that all elements of the vector or matrix are replaced by their absolute values. The relation $\leqq$ applies for all elements of the vectors or matrices, respectively.

Definition 8.5. A comparison system is a linear multi-input multi-output system (8.46) for which Eq. (8.47) holds for all $u(t)$.

This definition extends the notion of the comparison system introduced in [2], [27], [63], [66], [103] to systems (8.46) with more than one input and output and with possibly high dynamical order and delays.

It can be shown that every comparison system possesses the property

$$(8.48) \qquad V(t) \geqq 0 \quad \text{for all} \quad t.$$

If the comparison system is I/O-stable the matrix of its reinforcement

$$(8.49) \qquad W = \int_0^\infty V(t)\, dt$$

exists and presents a useful characteristic of the comparison system.

Theorem 8.5. [66]. The comparison system (8.46) for the linear system

$$(8.50) \qquad y(t) = G * u(t) \quad \text{with} \quad G(t) = C\, e^{At}\, B$$

is given by Eq. (8.46) with

$$(8.51) \qquad V(t) \geqq |G(t)| \,.$$

Analysis of the overall system. The closed-loop system (8.1), (8.4) can be decomposed into

$$(8.52) \qquad \begin{aligned} \dot{x}_i &= \bar{A}_i x_i + \bar{E}_i s_i + \bar{F}_i v_i \,, \\ y_i &= \bar{C}_i x_i \,, \\ z_i &= \bar{C}_{zi} x_i \end{aligned}$$

and

$$(8.53) \qquad s = Lz \,,$$

where the subsystem (8.52) includes the decentralized controller (4.1). Comparison systems of the subsystems (8.52) are described by

$$(8.54) \qquad \begin{aligned} |y_i(t)| &\leqq r_{yi}(t) = V_{i1} * |u_i| + V_{i2} * |s_i| \,, \\ |z_i(t)| &\leqq r_{zi}(t) = V_{i3} * |u_i| + V_{i4} * |s_i| \,, \end{aligned}$$

where $V_{i1}(t) \geqq |\bar{C}_i\, e^{\bar{A}_i t}\, \bar{F}_i|$ etc. They are used as $\mathscr{S}i'$ in Fig. 10. Their interconnections $\mathscr{K}'$ are estimated from Eq. (8.53)

$$(8.55) \qquad |s(t)| \leqq \bar{L} |z(t)| \quad \text{with} \quad \bar{L} \geqq |L| \,.$$

To analyse the overall system a comparison system of (8.52), (8.53) is derived. Combining the comparison system (8.54) with the interconnection condition (8.55) the inequalities

$$(8.56) \qquad \begin{aligned} |y(t)| &\leqq \text{diag}\, V_{i1} * |u| + \text{diag}\, V_{i2} * |s| \,, \\ &\leqq \text{diag}\, V_{i1} * |u| + \text{diag}\, V_{i2} \cdot \bar{L} * |z| \quad \text{because of Eq. (8.55)}, \\ &\leqq \text{diag}\, V_{i1} * |u| + \text{diag}\, V_{i2} \cdot \bar{L} * r_z \end{aligned}$$

are received. $r_z(t)$ can be estimated using Eq. (8.54)

$$(8.57) \qquad \begin{aligned} r_z(t) &\leqq \text{diag}\, V_{i3} * |u| + \text{diag}\, V_{i4} * |s| \,, \\ &\leqq \text{diag}\, V_{i3} * |u| + \text{diag}\, V_{i4} \cdot \bar{L} * |z| \,, \\ &\leqq \text{diag}\, V_{i3} * |u| + \text{diag}\, V_{i4} \cdot \bar{L} * r_z \,. \end{aligned}$$

The vector $\bar{r}_z(t)$ is defined reading Eq. (8.57) as equality

$$(8.58) \qquad \bar{r}_z(t) = V * |u|$$

with
(8.59)
$$V(t) = \text{diag } V_{i3} + \text{diag } V_{i4} \cdot \bar{L} * V(t) \,.$$

Eq. (8.58) describes a comparison system of (8.57)

(8.60)
$$\bar{r}_z(t) \geqq r_z(t)$$

if the matrix

(8.61)
$$P = (I - \text{diag } V_{i4} \cdot \bar{L})$$

is an M-matrix, where

$$W_{i4} = \int_0^\infty V_{i4}(t) \, dt$$

(cf. Eq. (8.50)). A comparison system of the system (8.52), (8.53) can be derived from (8.56), (8.57), (8.58)

(8.62)
$$|y(t)| \leqq r(t) = V_{yu} * |u|$$

with

(8.63)
$$V_{yu}(t) = \text{diag } V_{i1} + \text{diag } V_{i2} \cdot \bar{L} * V$$

and $V(t)$ from Eq. (8.59).

Stability analysis. According to Eq. (8.62) the original system is proved to be stable, if its comparison system is stable.

Theorem 8.6. [66]. Assume that the given system (8.52), (8.53) is described by comparison systems (8.54), (8.55). The overall system is I/O-stable if all comparison systems are I/O-stable and if the matrix P in Eq. (8.61) is an M-matrix.

Because of the properties of M-matrices, Theorem 8.6 provides a test for connective stability.

Compared with known stability conditions, Theorem 8.6 presents a less conservative stability test, because each subsystem is dealt with as a *MIMO*-system and the comparison systems include more information about the subsystems as first order Lyapunov functions (8.17). Nevertheless the stability test presented here is as simple as the others. Only a matrix P of order N must proved to be an M-matrix.

I/O-behaviour of the subsystems. The approach presented so far can be applied to determine the I/O-behaviour of a certain subsystem, say $\mathscr{S}\mathscr{S}1$. The behaviour of this subsystem depends on the properties of this subsystem as well as the influence of the other subsystems through the input signals $z_1(t)$ and $s_1(t)$, see Fig. 10. Therefore, the investigation of the isolated $\mathscr{S}\mathscr{S}1$ alone fails to give a right picture of the I/O-behaviour of this subsystem. Additionally it must be evaluated how this result can be modified by the influence of the other subsystems (joint together in $\overline{\mathscr{S}\mathscr{S}}2$). This can be done by means of a comparison system of $\overline{\mathscr{S}\mathscr{S}}2$. For $s_1 = 0$ the model (8.52) provides an approximation y_1 of the output of Subsystem 1

(8.63)
$$\dot{x}_1 = \bar{A}_{11}x_1 + \bar{F}_1 v_1 \,,$$
$$\hat{y}_1 = \bar{C}_1 x_1 \,.$$

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To determine the error $y_1(t) - \hat{y}_1(t)$ of this approximation, $\overline{\mathscr{S}\mathscr{S}}2$ may be known to the extent that a comparison system

$$(8.64) \qquad\qquad |z_2(t)| \leqq V_2 * |s_2|$$

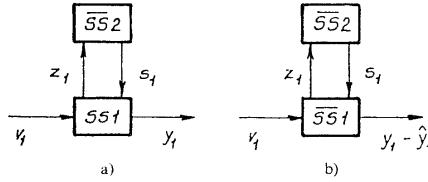can be set up. The approximation error $y_1(t) - \hat{y}_1(t)$ depends on $V_1(t)$ and $s_1(t)$

a)   b)

Fig. 12.

and can be represented by

$$(8.65) \qquad \overline{\mathscr{S}\mathscr{S}}1: \qquad \dot{x}_1 = \bar{A}_{11}x_1 + \bar{E}_1 s_1 \,,$$

$$y_1 - \hat{y}_1 = \bar{C}_1 x_1 \,,$$

$$\dot{\tilde{x}}_1 = \bar{A}_{11}\tilde{x}_1 + \bar{E}_1 s_1 + \bar{F}_1 v_1 \,,$$

$$z = \bar{C}_{z1}\tilde{x}_1 \,.$$

It is the output of a **feedback structure composed by** $\overline{\mathscr{S}\mathscr{S}}2$ and the system (8.65), see Fig. 12, and can be estimated using the results presented above. To do this, the comparison systems of $\overline{\mathscr{S}\mathscr{S}}1$ are derived from the models (8.65) yielding

$$(8.66) \qquad\qquad |y_1(t) - \hat{y}_1(t)| \leqq V_{ys} * |s_1| \,,$$

$$|z_1(t)| \leqq V_{zu} * |u| + V_{zs} * |s_1| \,,$$

where the impulse response matrices are determined according to Theorem 8.5. $\overline{\mathscr{S}\mathscr{S}}1$ and $\overline{\mathscr{S}\mathscr{S}}2$ are interconnected through

$$(8.67) \qquad\qquad \begin{pmatrix} s_1 \\ s_2 \end{pmatrix} = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} .$$

Then a comparison system of the form (8.62) can be derived from (8.66), (8.67)

$$(8.68) \qquad\qquad |y_1(t) - \hat{y}_1(t)| \leqq V_{zu} * V * V_{ys} * |v_1| \,,$$

with

$$(8.69) \qquad\qquad V(t) = V_2 + V_2 * V_{zs} * V$$

provided that $P = (I - V_2 V_{zs})$ is an M-matrix.

Theorem 8.7. [66]. The I/O-behaviour of the overall system (8.52), (8.53) can be approximated by Eq. (8.63). If the stability condition presented in Theorem 8.6 is satisfied, an upper bound of the model error $y_1 - \hat{y}_1$ is given by Eqs. (8.68), (8.69).

8.5. Extensions

The design philosophy stated in Chapter 8.1 and elaborated in Chapters 8.2 and 8.3 is used in many design methods, which have been described in the literature in the last decade. To name a few, Darwish, Soliman and Fantin [21], and Mahalanabis and Singh [77] considered the question which additional design requirements have to be imposed on the subsystem level in order to ensure that the separately designed control station satisfies the stability test for the overall closed-loop system. Veselý et al. [106] decomposed the optimality condition given by the Bellman-Lyapunov equation to derive a method for the separate design of the control stations.

A parallel frequency domain method can be derived from the Direct Nyquist Array method proposed by Rosenbrock [89] as shown by Nwokah [81], [82], see also [75]. An extension to multi-dimensional interconnected signals has been made by Bennett and Baras [17] on the basis of block diagonal dominance.

9. DESIGN ON THE BASIS OF OVERLAPPING DECOMPOSITION

In this chapter, we use overlapping decomposition as explained in Section 3.4 for the reduction of the design complexity. As far as stability is concerned, overlapping decomposition leads to a complete decentralization of the design task, because the stability of the expansion implies the stability of the original system (cf. Theorem 3.4). Therefore, the stability of the whole system can be proved by means of stability tests for the low order subsystems of the expanded system.

In Section 9.1, decentralized control is studied on the basis of overlapping decomposition of the plant. Sections 9.2 and 9.3 deal with particular structures of feedback control of large scale system. Multicontroller configurations are considered which should satisfy given requirements even if some of the parallel subcontrollers are not in operation. In this context, overlapping decomposition can be considered as a useful means for the design of a class of reliable controllers, see for instance [36], [40], [41], [45], [95].

9.1. Decentralized control

As shown in Theorem 3.4 overlapping decomposition can be used for a conceptual simplification of the stability analysis. On this basis, we consider the decentralized stabilization.

Problem 9.1. Consider the system (3.17), (3.18), which satisfy the condition for the expansion-contraction relation of Theorem 3.3. Find the controllers $u = -Kx$ for Eq. (3.17) and $u = -\overline{K}\overline{x}$ for Eq. (3.18) as a solution of the optimal control problem

$$(9.1) \qquad J = \tfrac{1}{2}\int_0^\infty \left(\|x\|_Q^2 + \|u\|_R^2 \right) \mathrm{d}t \to \min$$

subject to (3.17), and

(9.2) $$\bar{J} = \tfrac{1}{2} \int_0^\infty \left(\| \bar{x} \|_Q^2 + \| u \|_R^2 \right) dt \to \min$$

subject to (3.18).

We want to formulate the conditions under which K, $\bar{K}$ satisfy the expansion-contraction relation for the closed loop system (3.17), (3.18). and specify the properties of the expansion-contraction relation for the problems (9.1), (3.17) and (9.2), (3.18).

To set up the relations between K, $\bar{K}$ for (3.17), (3.18), we denote

(9.3) $$A_c = A - BK , \quad \bar{A}_c = \bar{A} - \bar{B}\bar{K} .$$

$\bar{A}_c$ should be an expansion of A_c (cf. Theorem 3.3). It has the form

(9.4) $$\bar{A}_c = TA_c T^+ + M_c ,$$

where M_c is some complementary matrix satisfying the equations

(9.5) $$T^+ M_c^i T = 0 \quad \text{for all} \quad i = 1, \dots, \bar{n} ,$$

with $\bar{n}$ overlapping subsystems. This restricts the class of the gain matrices $\bar{K}$ that can be used to stabilize the openloop expansion $\bar{\mathscr{S}}$. This class is specified by the following theorem.

Theorem 9.1. [94]. Under the conditions of Theorem 3.3 $\bar{A}_c$ is an expansion of the system A_c if $\bar{K}$ has the form

(9.6) $$\bar{K} = KT^+ + F$$

with $(\cdot)^+$ denoting the pseudoinverse, where the complementary matrix F satisfies the conditions

(9.7) $$FM^{i-1}T = 0 , \quad FM^{i-1}N = 0 \quad \text{for all} \quad i = 1, \dots, \bar{n} .$$

As an example consider the system $\mathscr{S}$

(9.8) $$\mathscr{S}: \quad \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} B_{11} & 0 \\ 0 & 0 \\ 0 & B_{32} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} ,$$

where x_1, x_2, x_3 are n_1-, n_2-, n_3-dimensional state vectors, and u_1, u_2 are m_1-, m_2-dimensional control vectors, respectively. The dotted lines denote the decomposition structure. The local control laws are supposed to have the form

(9.9) $$u_1 = -K_1 x_1 - K_2 x_2 , \quad u_2 = -K_3 x_2 - K_4 x_3 .$$

Therefore, the matrix K has the form

(9.10) $$K = \begin{pmatrix} K_{11} & K_{12} & 0 \\ 0 & K_{22} & K_{23} \end{pmatrix} .$$

We use the same transformation T and M as in the example (3.23) to obtain

$$(9.11) \qquad \mathscr{S}: \quad \begin{pmatrix} \dot{\bar{x}}_1 \\ \hline \dot{\bar{x}}_2 \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} & 0 & A_{13} \\ A_{21} & A_{22} & 0 & A_{23} \\ \hline A_{21} & 0 & A_{22} & A_{23} \\ A_{31} & 0 & A_{32} & A_{33} \end{pmatrix} \begin{pmatrix} \bar{x}_1 \\ \hline \bar{x}_2 \end{pmatrix} + \begin{pmatrix} B_{11} & 0 \\ 0 & 0 \\ \hline 0 & 0 \\ 0 & B_{32} \end{pmatrix} \begin{pmatrix} u_1 \\ \hline u_2 \end{pmatrix},$$

where $\bar{x}_1 = \left(x_1^{\mathrm{T}},\ x_2^{\mathrm{T}}\right)^{\mathrm{T}}$, $\bar{x}_2 = \left(x_2^{\mathrm{T}},\ x_3^{\mathrm{T}}\right)^{\mathrm{T}}$. The decentralized control law has the form

$$(9.12) \qquad\qquad u_1 = -\bar{K}_1 \bar{x}_1, \quad u_2 = -\bar{K}_2 \bar{x}_2,$$

where

$$\bar{K} = \begin{pmatrix} \bar{K}_1 & 0 \\ 0 & \bar{K}_2 \end{pmatrix} = \begin{pmatrix} \bar{K}_{11} & \bar{K}_{12} & 0 & 0 \\ 0 & 0 & \bar{K}_{23} & \bar{K}_{24} \end{pmatrix}$$

holds. Choosing in Eq. (9.6) the complementary matrix F as

$$(9.14) \qquad\qquad F = \begin{pmatrix} 0 & \frac{1}{2}\bar{K}_{12} & -\frac{1}{2}\bar{K}_{12} & 0 \\ 0 & -\frac{1}{2}\bar{K}_{23} & \frac{1}{2}\bar{K}_{23} & 0 \end{pmatrix}$$

the matrix K satisfies Eq. (9.6) and we have

$$(9.15) \qquad\qquad \bar{K} = \begin{pmatrix} \bar{K}_{11} & \bar{K}_{12} & 0 \\ 0 & \bar{K}_{23} & \bar{K}_{23} \end{pmatrix}.$$

Although we have designed two independent control stations (9.12) for the isolated subsystems of the expansion (9.11), the overall system controller (8.15) is no completely decentralized controller. This result is received only because we used state feedback controllers (9.12). If we had designed incomplete state feedback for the isolated subsystems, where $\bar{K}_{12} = \bar{K}_{22} = 0$, we would get the completely decentralized controller (9.15). For both cases, the control stations in Eq. (9.12) can be designed independently, because the expansion $\mathscr{S}$ has the structure (9.11) whose disjoint decomposition yields completely independent design problems (Fig. 12).
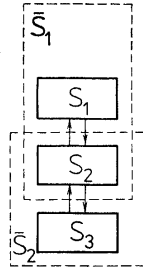

Fig. 12.

Let us deal with the relation of the problem (9.1), (3.17) to (9.2), (3.18) in the context of expansion-contraction.

Definition 9.1. The pair $(\bar{\mathscr{S}}, \bar{J})$ (cf. Eqs. (9.2), (3.18)) *includes* the pair $(\mathscr{S}, J)$ (cf. Eqs. (9.1), (3.17)) if there exists a matrix T satisfying the condition in Definition 3.2 and if $J(x_0, u) = \bar{J}(\bar{x}_0, u)$ holds for any fixed $u(t)$.

Under the transformation (3.20) the conditions for $\bar{A}$, $\bar{B}$ are given by Eq. (3.21). We have to find the relations between $\bar{Q}$ and $\bar{R}$ for $(\bar{\mathscr{S}}, \bar{J})$ and $(\mathscr{S}, J)$ to be in the relation expansion-contraction. Introducing the relation

$$(9.16) \qquad \bar{Q} = (T^+)^{\mathrm{T}} Q T^+ + M_Q, \quad \bar{R} = R + N_R,$$

where M_Q, N_R are complementary matrices of appropriate dimensions, we can prove the following assertion.

Theorem 9.2. [40]. The pair $(\bar{\mathscr{S}}, \bar{J})$ includes the pair $(\mathscr{S}, J)$ if either

$$(9.17) \qquad 1) \quad MT = 0, \quad N = 0, \quad T^{\mathrm{T}} M_Q T = 0, \quad N_R = 0$$

or

$$(9.18) \qquad 2) \quad T^+ M^i T = 0, \quad T^+ M^{i-1} N = 0, \quad M_Q M^{i-1} T = 0,$$

$$M_Q M^{i-1} N = 0, \quad N_R = 0, \quad i = 1, ..., \bar{n}.$$

Note that (9.17) and (9.18) are two different sets of conditions for $(\bar{\mathscr{S}}, \bar{J})$ to be the expansion of $(\mathscr{S}, J)$ and for $(\mathscr{S}, J)$ to be a contraction of $(\bar{\mathscr{S}}, \bar{J})$.

Theorem 9.2 specifies the conditions for a expansion $(\bar{\mathscr{S}}, \bar{J})$ of the original problem $(\mathscr{S}, J)$. Applying a disjoint decomposition on $(\bar{\mathscr{S}}, \bar{J})$ we get as the ith subsystem of the expansion

$$(9.19) \qquad \mathscr{S}i: \dot{\bar{x}}_i = \bar{A}_i \bar{x}_i + \bar{B}_i u_i + \sum_j \bar{A}_{ij} \bar{x}_j + \sum_j \bar{B}_{ij} u_j,$$

or for the overall system

$$(9.20) \qquad \bar{\mathscr{S}}: \dot{\bar{x}} = \bar{A}_D \bar{x} + \bar{B}_D u + \bar{A}_C \bar{x} + \bar{B}_C u.$$

Ignoring the interactions among the subsystems leads to

$$(8.21) \qquad \bar{\mathscr{S}}D: \dot{\bar{x}} = \bar{A}_D \bar{x} + \bar{B}_D u.$$

The solution of the problem (9.2), (9.21) is given as the symmetric positive definite solution of the Riccati equation with block diagonal structure

$$(9.22) \qquad \bar{A}_D^{\mathrm{T}} \bar{P} + \bar{P} \bar{A}_D - \bar{P} \bar{B}_D R^{-1} \bar{B}_D \bar{P} + \bar{Q} = 0.$$

It means that the expansion $(\bar{\mathscr{S}}, \bar{J})$ can be dealt with by the methods of Chapter 8.2 including the investigations of the suboptimality index, etc.

We summarize the basic steps into the design algorithm for optimal control problem. Suppose $\bar{N}$ overlapping subsystems.

Algorithm 9.1.

1) Initialize A, B, Q, R, x_0.
2) Specify $T, M, N, L, F, M_Q, N_R, \bar{N}$.

3) Compute $\bar{A}_i, \bar{B}_i, \bar{Q}_i, \bar{R}_i, \bar{A}_{ij}, \bar{B}_{ij}, \forall i, j$.
4) Test the condition (9.17). If it is satisfied goto 7) else test the condition (9.18). If it is satisfied goto 6) else goto 5).
5) Specify new $T, M, N, L, F, M_Q, N_R, \bar{N}$ and goto 4).
6) Compute $\bar{K}$ using Algorithm 8.2 (cf. Chapter 8).
7) Compute $K, J(x_0)$.
8) End.

Algorithm 9.1 is directed to the optimal control design. Of course, its modification for pole placement is simply possible. A systematic way for specifying the transformation matrix T and complementary matrices in step 3) has to be elaborated yet. The decentralized design is performed by Algorithm 8.2 in step 6).

9.2. Overlapping decompositions for serially interconnected systems

In this chapter, we consider the system depicted in Fig. 13. It consists of three subsystems in a chain connection and two decentralized controllers. Obviously, no disjoint decomposition is appropriate to divide the design problem. As will be explained now, an overlapping decomposition simplifies the design of the two control stations.
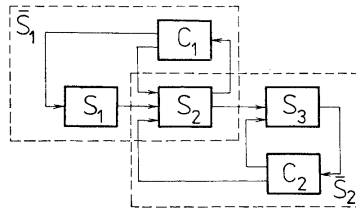

Fig. 13.

We treat an overall system with N subsystems. A control structure is supposed where the ith controller depends on the states $x_i, x_{i-1}, \ldots, x_{i-s}$ for given $s < i - 2$. The sample situation of this type is the case of 3 subsystems both with $s = 1$, see Fig. 13. Three subsystems $\mathscr{S}1, \mathscr{S}2, \mathscr{S}3$ can be decomposed using overlapping decomposition into two subsystems $\bar{\mathscr{S}}1, \bar{\mathscr{S}}2$ of expansion $\bar{\mathscr{S}}$.

Problem 9.2. Consider the plant

(9.23) $\qquad \mathscr{S}p: \quad \dot{x}_i = A_i x_i + B_i u_i + F_i p_{i-1}, \quad x_i(0) = x_{i0}$

$\qquad\qquad\qquad y_i = C_i x_i + D_i u_i + H_i p_{i-1},$

$\qquad\qquad\qquad p_i = C_{pi} x_i + D_{pi} u_i + H_{pi} p_{i-1}, \quad i = 1, \ldots, N,$

where respectively x_i, u_i, y_i, p_i denote the ith subsystem state vector, control vector, output vector, interconnection vector. Find a decentralized controller

$$(9.24) \qquad\qquad u_i = K_i y_i + K_{i+1} y_{i+1}$$

so that the closed loop system (9.23), (9.24) satisfies the requirements (I)−(IV) given in Chapter 4.

The solution is based on the overlapping decomposition which results in an expansion with an upper triangular state matrix. That is, the expansion is a serially interconnected system (6.1), (6.2). Therefore, the solution of Problem 6.1 can be used here.

We consider the expanded plant in the form

$$(9.25) \qquad \bar{\mathscr{S}}p: \quad \dot{\bar{x}}_k = \bar{A}_k \bar{x}_k + \bar{B}_k \bar{u}_k + \bar{F}_k \bar{p}_{k-1}, \quad \bar{x}_k(0) = \bar{x}_{k0},$$

$$\bar{y}_k = \bar{C}_k \bar{x}_k + \bar{D}_k \bar{u}_k + \bar{H}_k \bar{p}_{k-1},$$

$$\bar{p}_k = \bar{C}_{pk} \bar{x}_k + \bar{D}_{pk} \bar{u}_k + \bar{H}_{pk} \bar{p}_{k-1}, \quad k = 1, \ldots, N-1,$$

where $\bar{x}_k = \left(x_k^{\mathrm{T}}, x_{k+1}^{\mathrm{T}}\right)^{\mathrm{T}}$, $\bar{u}_k = \left(u_k^{\mathrm{T}}, u_{k+1}^{\mathrm{T}}\right)^{\mathrm{T}}$, $\bar{p}_{k-1} = p_{k-1}$, $\bar{p}_k = p_{k+1}$. The matrices $\bar{A}_k, \ldots, \bar{H}_{pk}$ are uniquely given by Eq. (9.23) and the relations between state, control and interconnection variables of the systems (9.23) and (9.25). Therefore, we expanded the system (9.23) so that the state x_k is contained in the expanded subsystems $\bar{\mathscr{S}}(k-1)$, $\bar{\mathscr{S}}k, k = 1, \ldots, N-1$. The transformation T has the form

$$(9.26) \qquad T = \begin{pmatrix} I_{n_1} & & & & & \\ \hline & I_{n_2} & & & & \\ & I_{n_2} & & & \mathbf{0} & \\ & \hline & I_{n_3} & & & \\ & & I_{n_3} & & & \\ & & \hline & I_{n_4} & & \\ & \mathbf{0} & & & \ddots & \\ & & & & & I_{n_N} \end{pmatrix}.$$

An appropriately chosen matrix M satisfying the conditions (9.7) has the form

$$(9.27) \qquad\qquad M = \operatorname{diag}\left(M_1, \ldots, M_{N-1}\right),$$

where

$$(9.28) \qquad\qquad M_i = \begin{pmatrix} 0 & \frac{1}{2}F_i & -\frac{1}{2}F_i & 0 \\ 0 & \frac{1}{2}A_{i+1} & -\frac{1}{2}A_{i+1} & 0 \\ 0 & -\frac{1}{2}A_{i+1} & \frac{1}{2}A_{i+1} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Because the matrices $MT = 0, N = 0, F = 0$ in Eqs. (3.21), (9.6), the conditions of Theorem 9.1 are satisfied, see e.g. [40]. It means that (9.25) is an expansion of the system (9.23).

The problem (9.25) with

$$(9.29) \qquad \bar{u}_k = \bar{K}_k \bar{y}_k, \quad k = 1, \ldots, N - 1,$$

is equivalent with Problem 6.1. Therefore, its solution can be used to solve (9.25), (9.29). It means that fully decentralized control design can be performed on the expansion of the system (9.23), (9.24).

9.3. Reliable controllers and overlapping

We have described the basic situation of control design using overlapping decomposition in Chapter 9.1. The case of two parallel controllers acting on the part of the plant has been considered. We extend it to the case of more than two controllers. It means that we consider the whole system for the control design. We ask, how the designed controller can be divided into two controllers. This question has been answered in Section 6.2 by division into two equivalent subcontrollers. We formulate this problem for an overall sytsem, but instead of the overall system the ith subsystem (6.3) or (9.25) of serial connection structure of subsystems can be considered. Suppose a model $\mathscr{S}$ and N_p identical parallel controllers $\mathscr{C}_i$. Then the overall controller $\mathscr{C}$ has the form $\mathscr{C} = \sum_{i=1}^{N_p} \mathscr{C}_i$.

Problem 9.3. Consider the plant

$$(9.30) \qquad \dot{x} = Ax + Bu,$$

$$y = Cx + Du.$$

Find a controller

$$(9.31) \qquad \mathscr{C}: \quad u = Ky, \quad K = \sum_{i=1}^{N_p} K_i$$

such that the closed-loop system (9.30), (9.31) satisfies the requirements (I)−(IV) given in Chapter 4 and is reliable against the failure of $N_{pq} < N_p$ controllers. That is if N_{pq} controllers fail, the closed-loop system (9.30), (9.31) is stable.

The solution is based on the analogy with the solution of Problem 6.2., where the ith subsystem has been considered and $N_{pq} = N_p/2$. The case of the system with $N_p = 3$ is given in Fig. 14, where $\overline{\mathscr{S}}1$, $\overline{\mathscr{S}}2$, $\mathscr{S}3$ are the subsystems of an expansion. If $N_{pq} = 1$ holds, then the closed-loop system behaviour will be "better" than if only two controllers would be considered with the assumption that one of them can fail.

Reliability of multicontroller configurations. Suppose the simplest possible structure which is given by Eq. (9.8). We can interpret the state equation for x_2 as a plant $\mathscr{P}$ and two identical state equations for x_1 and x_3 as controllers. We obtain

parallel structure of controllers which act against controller failures. If one of these two controllers fail the second still operates. Therefore, the controller design is performed in a decentralized way using the expansion.

In terms of reliability theory, decentralized controllers represent serial connections of controllers having no redundancy against failure of any controller. To make
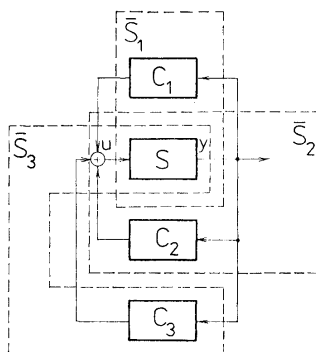

Fig. 14.

control systems more reliable against controller failures, the redundancy of controllers is introduced into control schemes. In terms of reliability theory such multiple control systems represent parallel connections of controllers. The concept of structural reliability of control systems is introduced in the following way [95]. We suppose that each controller either operaters or fails. We use a binary variable c_i to signify that the ith controller is functioning $(c_i = 1)$ or fails $(c_i = 0)$. For the scheme (9.8) we have the vector $c = \{c_1, c_2\}$. Further, we introduce a binary indicator $\Phi = \Phi(c)$ describing the structure of controllers, that is $\Phi(c) = \prod_{i=1}^{N_p} c_i$ of $\Phi(c) = 1 - \prod_{i=1}^{N_p} (1 - c_i)$ for N_p controllers in series or in parallel for these basic cases, respectively. For a parallel structure given by Eq. (9.8) we get

$$(9.32) \qquad \Phi(c) = 1 - (1 - c_1)(1 - c_2) = \max\{c_1, c_2\} .$$

If we suppose that each controller operates of fails at random and independently of each other, we characterize the ith controller state by a random variable with the probability

$$(9.33) \qquad P\{c_i = 1\} = p_i = E(c_i), \quad i = 1, 2,$$

where $E(c_i)$ denotes the mean value of the random variable c_i and call it the reliability

of the ith controller. The overall system reliability is given by the reliability function χ

(9.34)
$$\chi = \chi(p),$$

with $p = \{p_1, p_2\}$. Of course, various controller structures are characterized by their corresponding reliability functions.

We have dealt with reliability considerations in a 1-out-of-2 basis. Consider now the reliability of a system with three parallel controllers, shown in Fig. 14. We suppose that each of controllers has reliability p. Then various failure structures can be considered. Denote respectively $\chi_1(p)$, $\chi_2(p)$, $\chi_3(p)$, the reliability function for two parallel controllers (cf. Fig. 12), three parallel controllers (cf. Fig. 14), a 2-out-of-3 structure of controllers (cf. Fig. 14), that is $N_{pq} = 2$, $N_p = 3$, we obtain

(9.35)
$$\chi_1(p) = 1 - (1 - p)^2, \quad \chi_2(p) = 1 - (1 - p)^3,$$
$$\chi_3(p) = 3p^2(1 - p) + p^3.$$

It means that a desirable reliability of a control system can be achieved by a sufficient number of controllers with low reliability in this way.

10. DESIGN OF SINGULARLY PERTURBED SYSTEMS

We have dealt with a weak coupling in Chapter 8. That approach is based on the notion of a nonsingular perturbation, i.e. a perturbation term in the right-hand side of a differential equation. A further kind of weak interconnections is encountered in systems whose subsystems have very slow and very fast time constants, i.e. in an approach which is based on the notion of a singular perturbation, that is a perturbation in the left-hand side of a differential equation. The basis for the systematic investigation of this property has been explained ih Section 3.5. The aim of this chapter is to extend these results and to use them for the design of decentralized controllers, see for instance [43], [44]–[49], [51], [52], [85], [90], [91].

In Section 10.1 further results concerning systems with slow and fast modes will be presented. Emphasis is laid on the questions how the two-time scale property can be detected and which simplifications of the analysis are possible. In Section 10.2 feedback controllers are designed for singularly perturbed systems by means of the principles of pole assignment and optimal control. Section 10.3 extends these results to the solution of a two-time-scale multiparameter perturbations problem.

10.1. Analysis of two-time scale systems

This chapter is organized as follows. First, we characterize a two-time scale property for explicit and implicit systems. Then we present a decomposition procedure for pole placement and optimal control for singularly perturbed systems. The last

part generalizes these results to multimodel control systems, where each decision maker knows only his simplified model.

We analyze the conditions for a two-time-scale property. Two cases are considered: explicit and implicit linear singularly perturbed systems. Let us deal with explicit systems first.

Considered a linear time invariant system

$$\dot{x} = Ax \,. \tag{10.1}$$

Definition 10.1. A system (10.1) has a *two-time-scale property* if it can be decomposed into two subsystems

$$\begin{pmatrix} \dot{x}_s \\ \dot{x}_f \end{pmatrix} = \begin{pmatrix} A_s & 0 \\ 0 & A_f \end{pmatrix} \begin{pmatrix} x_s \\ x_f \end{pmatrix} \tag{10.2}$$

and if for the largest eigenvalues $\lambda_M(A_s)$ of A_s and the smallest eigenvalue $\lambda_m(A_f)$ of A_f

$$|\lambda_M(A_s)| \ll |\lambda_m(A_f)| \tag{10.3}$$

holds.

For example, we suppose that (10.1) is a closed-loop system. It should be not only stable, but also well damped, e.g. $|\text{Im}\,\lambda(A)| \leq 5 |\text{Re}\,\lambda(A)|$. If A is a nonsingular matrix, then $|\lambda_M(A)| \leq \|A\|$, $|\lambda_m(A)|^{-1} \leq \|A^{-1}\|$. Therefore, the system (10.1) has the two-time-scale property (10.3) if

$$\|A_f^{-1}\| \ll \|A_s\|^{-1} \,. \tag{10.4}$$

In order to determine A_s and A_f for the system (10.1), consider the system (10.1) in the partitioned form

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{11} & A_{12} \end{pmatrix} \begin{pmatrix} x_1 \\ x_1 \end{pmatrix}, \tag{10.5}$$

where $\dim x_1 = \dim x_s = n_1$, $\dim x_2 = \dim x_f = n_2$. Further let

$$x_1 = x_s + M x_f \,, \tag{10.6}$$
$$x_2 = N x_s + (I + NM) x_f = x_f + N x_1 \,.$$

If the $n_2 \times n_1$ (resp. $n_1 \times n_2$) matrix N (resp. M) exists as a real root of equations

$$A_{22}N - NA_{11} - NA_{12}N + A_{21} = 0 \,, \tag{10.7}$$
$$(A_{11} + A_{12}N) M + M(A_{22} - NA_{12}) + A_{12} = 0 \,,$$

we have

$$A_s = A_{11} + A_{12}N \,, \tag{10.8}$$
$$A_f = A_{22} - NA_{12} \,.$$

An appropriate test of two-time-scale property is given in the following lemma.

Lemma 10.1. [20]. Suppose that A_{22} is a nonsingular matrix in Eq. (10.5). Denote

$$N_0 = -A_{22}^{-1}A_{21} \,, \quad A_0 = A_{11} + A_{12}N_0 \,. \tag{10.9}$$

If

(10.10)
$$\|A_{22}^{-1}\| \ll \frac{1}{3(\|A_0\| + \|A_{12}\| \|N_0\|)} ,$$

then the sequence N_k defined by

(10.11)
$$N_{k+1} = -A_{22}^{-1}(A_{21} - N_k A_{11} - N_k A_{12} N_k)$$

with N_0 given by Eq. (10.9), converges to a real bounded root (10.7). Further

(10.12)
$$\|N_{k+1} - N\| \leqq \|N_k - N\| , \quad k = 0, 1 \ldots$$

holds. We get $N_1 - N_0 = A_{22}^{-1} N_0 A_0$ from Eqs. (10.9), (10.12) for $k = 0$ and hence $\|N_1 - N_0\| \leqq \varepsilon \|N_0\|$ if

(10.13)
$$\|A_{22}\| \leqq \varepsilon \|A_0\|^{-1}$$

where ε is a small positive number.

Lemma 10.2. [20]. The system (10.5) has the two-time-scale property if Eqs. (10.10)--(10.12) hold and Eq. (10.13) is satisfied for $\varepsilon \ll 1$, that is,

(10.14)
$$\|A_{22}^{-1}\| \ll \|A_0\|^{-1} .$$

It follows from Lemmas 10.1 and 10.2 that a sufficient condition for a system to possess the two-time-scale property is

(10.15)
$$\|A_{22}^{-1}\| \ll (\|A_0\| + \|A_{12}\| \cdot \|N_0\|)^{-1} .$$

Approximate expressions for x_1 and x_2 are given in the following lemma. $O(\varepsilon)$ denotes an element of "order of ε".

Lemma 10.3. [20]. If the inequality (10.14) holds, then

(10.16)
$$x_1(t) = x_{s0}(t) - A_{12} A_{22}^{-1} x_{f0}(t) + O(\varepsilon) ,$$
$$x_2(t) = -A_{22}^{-1} A_{21} x_{s0}(t) + x_{f0} + O(\varepsilon) ,$$

where x_{s0} and x_{f0} are obtained from simplified subsystems

(10.17)
$$\begin{pmatrix} \dot{x}_{s0} \\ \dot{x}_{f0} \end{pmatrix} = \begin{pmatrix} A_0 & 0 \\ 0 & A_{22} \end{pmatrix} \begin{pmatrix} x_{s0} \\ x_{f0} \end{pmatrix} .$$

It means that for $x_s = x_{s0} + O(\varepsilon)$, $x_f = x_{f0} + O(\varepsilon)$ we obtain $O(\varepsilon)$ approximation for the system (10.17). Note that if A is not in the form satisfying Eq. (10.15), the two-time-scale property can be tested using an appropriate transformation, e.g. permutation or scaling of the state variables. Considering a property of the eigenvalues of Eq. (10.1), we use the following lemma.

Lemma 10.4. [20]. The eigenvalues of a two-time-scale system consist of n_1 small eigenvalues $\lambda(A_s)$ and n_2 large eigenvalues $\lambda(A_f)$, which are approximated by

(10.18)
$$\lambda(A_s) = \lambda(A_0) (1 + O(\varepsilon)) ,$$
$$\lambda(A_f) = \lambda(A_{22}) (1 + O(\varepsilon)) .$$

Let us turn our attention to the time scale decomposition of linear *implicit singularly perturbed systems*. Consider such a system in the form

$$(10.19) \qquad \dot{x} = A(\varepsilon)\,x\,,$$

where $A(0)$ is singular and $A(\varepsilon)$ is analytic at $\varepsilon = 0$, that is, $A(\varepsilon) = \sum_{i=0}^{\infty} \varepsilon^i \bar{A}_i$. Suppose that $A_0 \neq 0$ and $\det A(\varepsilon) \neq 0$ for all ε.

Definition 10.2. [46]. An $n \times n$ matrix M of rank $r < n$ has a *semisimple null structure* (SSNS) if M has r nonzero eigenvalues.

Suppose that $\bar{A}_0$ has SSNS with rank r. Choose an $(n - r) \times n$ matrix P_1 so that $P_1 \bar{A}_0 = 0$. Further choose an $r \times n$ matrix P_2 so that the $n \times n$ matrix $P^{\mathrm{T}} = (P_1^{\mathrm{T}}, P_2^{\mathrm{T}})$ is nonsingular and

$$(10.20) \qquad P \bar{A}_0 P^{-1} = \begin{pmatrix} 0 & 0 \\ \tilde{A}_{21} & \tilde{A}_{22} \end{pmatrix}$$

holds, where $\tilde{A}_{22}$ is a nonsingular $r \times r$ matrix. Using the state transformation $\bar{x} = Px$ of Eq. (10.19) we obtain

$$(10.21) \qquad \dot{\bar{x}}_1 = \varepsilon^m [A_{11}(\varepsilon)\,\bar{x}_1 + A_{12}(\varepsilon)\,\bar{x}_2]\,,$$
$$\dot{\bar{x}}_2 = A_{21}(\varepsilon)\,\bar{x}_1 + A_{22}(\varepsilon)\,\bar{x}_2\,,$$

where m is a finite integer, $A_{ij}(\varepsilon)$ are analytic at $\varepsilon = 0$ with $A_{22}(0) = \tilde{A}_{22}$. The two-time scale property of Eq. (10.21) can be recognized using the decoupling transformation $\tilde{x}_1 = (I - \varepsilon MN)\,\bar{x}_1 - \varepsilon M \bar{x}_2$ and $\tilde{x}_2 = N\bar{x}_1 + \bar{x}_2$, where $N(\varepsilon)$, $M(\varepsilon)$ satisfy the equations (ε in brackets is dropped)

$$(10.22) \qquad 0 = A_{21} - A_{22}N + \varepsilon^m N(A_{11} - A_{12}N)\,,$$
$$0 = A_{12} - M(A_{22} + \varepsilon^m N A_{12}) + \varepsilon^m(A_{11} - A_{12}N)\,M\,.$$

$N(\varepsilon)$ and $M(\varepsilon)$ exist for sufficiently small ε and they are analytic at $\varepsilon = 0$ with $N(0) = A_{22}^{-1}(0)\,A_{21}(0)$, $M(0) = A_{12}(0)\,A_{22}^{-1}(0)$. The decoupling transformation yields

$$(10.23) \qquad \dot{\tilde{x}}_1 = \varepsilon^m [A_{11}(\varepsilon) - A_{12}(\varepsilon)\,N(\varepsilon)]\,\tilde{x}_1\,,$$
$$\dot{\tilde{x}}_2 = [A_{22}(\varepsilon) + \varepsilon^m N(\varepsilon)\,A_{12}(\varepsilon)]\,\tilde{x}_2\,.$$

Therefore, we have shown that for a given matrix $A(\varepsilon)$, analytic at $\varepsilon = 0$, with $A(0)$ singular and SSNS, we can find a matrix $S(\varepsilon)$, with $S(\varepsilon)$ and $S(\varepsilon)^{-1}$ analytic at $\varepsilon = 0$, such that

$$(10.24) \qquad S(\varepsilon)\,A(\varepsilon)\,S(\varepsilon)^{-1} = \mathrm{diag}\,[\varepsilon^{\bar{m}}\,D_1(\varepsilon),\,D_2(\varepsilon)]\,,$$

where $\bar{m} \geq m$, see [46].

Note that the procedures for explicit and implicit singularly perturbed systems which have been derived for the two-time-scale systems can be extended to multi-time-scale systems. see e.g. [46].

10.2. Control system design

Two simple procedures are considered in this part. The eigenvalue placement and optimal control design for two-time-scale systems are considered in which the feedback design is separated according to the slow and fast subsystems.

Pole assignment in two-time-scale systems

Problem 10.2.1. Find a feedback matrix F for the system

$$(10.25) \qquad \dot{x}_1 = A_{11}x_1 + A_{12}x_2 + B_1 u,$$
$$\dot{x}_2 = A_{21}x_1 + A_{22}x_2 + B_2 u$$

such that the control $u = Fx$ places the eigenvalues at the desired locations which satisfy the two-time-scale property. Assume that the system (10.25) satisfies the following conditions:

1) The matrices A_{ij} in Eq. (10.25) satisfy the two-time-scale property

$$(10.26) \qquad \|A_{22}^{-1}\| \ll \left(\|A_0\| + \|A_{12}\| \cdot \|A_{22}^{-1}A_{12}\|^{-1}\right).$$

2) $\|B_2\|, \|A_{22}\|, \|B_0\|$ and $\|A_0\|$ are all of the same order of magnitude, where

$$(10.27) \qquad A_0 = A_{11} - A_{12}A_{22}^{-1}A_{21},$$
$$B_0 = B_1 - A_{12}A_{22}^{-1}A_{21}.$$

3) The pairs $(A_{22}, B_2), (A_0, B_0)$ are controllable.

An approximate separated eigenvalue placement is realized using the following separation theorem.

Theorem 10.1. [20]. Suppose that the system (10.25) satisfies the conditions $1)-3)$. If the feedback control has the form

$$(10.28) \qquad u = \left[\left(I + F_f A_{22}^{-1}B_2\right)F_s + F_f A_{22}^{-1}A_{21}\right]x_1 + F_f x_2,$$

then the eigenvalues of the closed-loop system (10.1) consist of n_1 small eigenvalues $\lambda(A_s)$ and n_2 large eigenvalues $\lambda(A_f)$ for which the following approximations

$$(10.29) \qquad \lambda(A_s) = \lambda(A_0 + B_0 F_s)\left(1 + O(\varepsilon)\right),$$
$$\lambda(A_f) = \lambda(A_{22} + B_2 F_f)\left(1 + O(\varepsilon)\right).$$

hold.

Denote Λ the desired spectrum of eigenvalues $\Lambda = \{\Lambda_1, \Lambda_2\}$, where Λ_1 (resp. Λ_2) denotes n_1 small (resp. n_2 large) eigenvalues. We summarize the procedure in an algorithm.

Algorithm 10.1.

1) Initialize $A_{ij}, B_i, i, j = 1, 2; \Lambda_1, \Lambda_2$.
2) Test conditions $1)-3)$. If they are satisfied goto 3) else goto 6).

3) Compute F_s such that all $\lambda(A_0 + B_0 F_s) \in \Lambda_1$.

4) Compute F_f such that all $\lambda(A_{22} + B_2 F_f) \in \Lambda_2$.

5) Compute control u using Eq. (10.28) and the closed-loop system (10.1).

6) End.

The matrices F_s, F_f are designed in a fully decentralized way.

Optimal control of two-time-scale systems

Problem 10.2.2. Find a feedback F as a solution of the optimal control problem

$$(10.30) \qquad J = \tfrac{1}{2} \int_0^\infty \left(\|y\|_I^2 + \|u\|_R^2 \right) \mathrm{d}t \to \min, \quad R > 0$$

subject to

$$(10.31) \qquad \dot{x}_1 = A_{11} x_1 + A_{12} x_2 + B_1 u, \quad x_1(0) = x_{10},$$

$$\varepsilon \dot{x}_2 = A_{21} x_1 + A_{22} x_2 + B_2 u, \quad x_2(0) = x_{20},$$

$$y = C_1 x_1 + C_2 x_2.$$

with small ε, $0 < \varepsilon \ll 1$.

The solution is based on the decomposition of the plant (10.30) into a slow and a fast subsystems. First, denote the solution of Problem 10.2.2 by $u_0 = -R^{-1} B^{\mathrm{T}} K x$, where $x = (x_1^{\mathrm{T}}, x_2^{\mathrm{T}})^{\mathrm{T}}$, $B = (B_1^{\mathrm{T}}, \varepsilon^{-1} B_2^{\mathrm{T}})^{\mathrm{T}}$ and K is the positive definite solution of the Riccati equation

$$(10.32) \qquad 0 = -KA - A^{\mathrm{T}} K + KBR^{-1} B^{\mathrm{T}} K - C^{\mathrm{T}} C$$

with $C = (C_1 \ C_2)$ and

$$A = \begin{pmatrix} A_{11} & A_{12} \\ \dfrac{1}{\varepsilon} A_{21} & \dfrac{1}{\varepsilon} A_{22} \end{pmatrix}.$$

Decomposing Problem 10.2.2. into slow and fast subproblems, we obtain the "slow" subproblem

$$(10.33) \qquad J_s = \tfrac{1}{2} \int_0^\infty \left(\|y_s\|_I^2 + \|u_s\|_R^2 \right) \mathrm{d}t \to \min$$

subject to

$$(10.34) \qquad \dot{x}_s = A_0 x_s + B_0 u_s. \quad x_s(0) = x_{s0},$$

$$y_s = C_0 x_s + D_0 u_s,$$

where

$$(10.35) \qquad A_0 = A_{11} - A_{12} A_{22}^{-1} A_{21}, \quad B_0 = B_1 - A_{12} A_{22}^{-1} B_2,$$

$$C = C_1 - C_2 A_{22}^{-1} A_{21}, \quad D_0 = -C_2 A_{22}^{-1} B_2.$$

The "fast" subproblem has the form

$$(10.36) \qquad J_f = \tfrac{1}{2} \int_0^\infty \left(\|y_f\|_I^2 + \|u_f\|_R^2 \right) \mathrm{d}t \to \min$$

subject to

$$(10.37) \qquad \varepsilon \dot{x}_f = A_{22} x_f + B_2 u_f, \quad x_f(0) = x_{f0},$$

$$y_f = C_2 x_f,$$

where $x_f = x_2 - \bar{x}_2, u_f = u - u_s, y_f = y - y_s, \bar{x}_2$ is the state x_2 for $\varepsilon = 0$ in Eq. (10.31).

The solution of the problem (10.33), (10.34) and (10.36), (10.37) is received by means of the well known optimal control techniques. For the slow subproblem (10.33) (10.34) the solution is given by

$$(10.38) \qquad 0 = -K_s(A_0 - B_0 R_0^{-1} D_0^T C_0) - (A_0 - B_0 R_0^{-1} D_0^T C_0)^T K_s +$$
$$+ K_s B_0 R_0^{-1} B_0^T K_s - C_0^T(I - D_0 R_0^{-1} D_0^T) C_0,$$

where $R_0 = R + D_0^T D_0$. Then the control u_s is

$$(10.39) \qquad u_s = -R_0^{-1}(D_0^T C_0 + B_0^T K_s) x_s = F_s x_s.$$

The fast subproblem (10.36), (10.37) has the solution

$$(10.40) \qquad 0 = -K_f A_{22} - A_{22}^T K_f + K_f B_2 R^{-1} B_2^T K_f - C_2^T C_2.$$

Then the control u_f is

$$(10.41) \qquad u_f = -R^{-1} B_2^T K_f x_f = F_f x_f.$$

The existence and uniqueness of these solutions are described by the following theorem.

Theorem 10.2. [21].

1) If the triple (A_0, B_0, C_0) in the problem (10.33), (10.34) is stabilizable-detectable, then Eq. (10.38) has a unique positive semidefinite stabilizing solution K_s.

2) If the triple (A_{22}, B_2, C_2) in the problem (10.36), (10.37) is stabilizable-detectable, then Eq. (10.40) has a unique solution K_f.

The importance of Theorem 10.2 lies in the fact that the conditions for control design of slow and fast subsystems are completely separated. The composite feedback control u_c given by Eq. (10.32) using Eqs. (10.39), (10.41) results in a slow controller $\mathscr{C}_s$ and in a fast controller $\mathscr{C}_f$, see Fig. 15, which has the form

$$(10.42) \qquad u_c = -\left[(I - R^{-1} B_2^T K_f A_{22}^{-1} B_2) R_0^{-1}(D_0^T C_0 + B_0^T K_s) +\right.$$
$$\left. + R^{-1} B_2^T K_f A_{22}^{-1} A_{21}\right] x_1 + R^{-1} B_2 K_f x_2.$$

Consider the relation between optimal control and composite control, i.e. between u_0 and u_c. Denote by J_0 and J_c the value of criterion for optimal control and composite control, respectively. The following theorem holds.
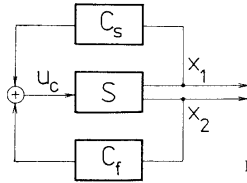

Fig. 15.

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Theorem 10.3. [22]. If the conditions 1) and 2) in Theorem 10.2 are satisfied, then the positive semidefinite stabilizing solution $K = K(\varepsilon)$ of (10.32) is analytic at $\varepsilon = 0$, that is

$$(10.43) \qquad K = \begin{pmatrix} K_1 & \varepsilon K_2 \\ \varepsilon K_2^T & \varepsilon K_3 \end{pmatrix} + \sum_{i=1}^{\infty} \frac{\varepsilon^i}{i!} \begin{pmatrix} K_1^{(i)} & \varepsilon K_2^{(i)} \\ \varepsilon K_2^{(i)T} & \varepsilon K_3^{(i)} \end{pmatrix} .$$

The matrices K_1 and K_3 satisfy

$$(10.44) \qquad\qquad K_1 = K_s , \quad K_3 = K_f$$

and

$$(10.45) \qquad\qquad J_c = J_0 + O(\varepsilon^2) .$$

Therefore, the composite feedback control (10.42) is an $O(\varepsilon^2)$ near-optimal approximation of the optimal control (10.30), (10.31).

Consider now the approximation reached by optimizing only the slow subsystem. That is assuming A_{22} stable we put $K_f = 0$ in Eq. (10.42) and denote the reduced control u_r

$$(10.46) \qquad u_r = -R_0^{-1}(D_0 C_0 + B_0^T K_s) x_1 = Fx .$$

Denoting J_r the corresponding criterion value, we can prove the following lemma.

Lemma 10.5. [21]. If A_{22} is stable, then the constant terms of the power series of J_r and J_0 at $\varepsilon = 0$ are equal, that is,

$$(10.47) \qquad\qquad J_r = J_0 + O(\varepsilon) .$$

For simplicity we rewritte the expression for u_c given by Eq. (10.42) in the form

$$(10.48) \qquad u_c = -R^{-1}B^T \begin{pmatrix} K_s & 0 \\ \varepsilon K_m^T & K_f \end{pmatrix} x = -R^{-1}B^T M_c x ,$$

where

$$(10.49)$$
$$K_m = \left[K_s(B_1 R^{-1} B_2^T K_f - A_{12}) - (A_{21} K_f + C_1^T C_2) \right] (A_{22} - B_2 R^{-1} B_2^T K_f)^{-1} .$$

We summarize the results in an algorithm.

Algorithm 10.2.

1) Initialize $A_{ij}, B_i, C_{ij}, i, j = 1, 2; R; x_0$.
2) Test conditions 1), 2) in Theorem 10.2. If they are satisfied go to 3) else go to 7).
3) Compute F_s by solving Eq. (10.38).
4) Compute F_f by solving Eq. (10.40).
5) Compute u_s, u_f, u_c, u_r which are given by Eqs. (10.39), (10.41), (10.42), (10.46).
6) Compute $J_c = \frac{1}{2} x_0 P_c x_0$, $J_r = \frac{1}{2} x_0 P_r x_0$, where P_c is the positive definite solution of the Lyapunov equation

$$P_c(A - BR^{-1}B^T M_c) + (1 - BR^{-1}B^T M_c)^T P_c = -M_c^T BR^{-1}B^T M_c - C^T C$$

and P_r is positive definite solution of the Lyapunov equation

$$P_r(A - BF) + (A - BF)^{\mathrm{T}} P_r = -F^{\mathrm{T}}R^{-1}F - C^{\mathrm{T}}C .$$

7) End.

Therefore, we can conclude that the existence conditions 1), 2) in Theorem 10.2 enable us to perform decentralized design for slow and fast subsystems, and that the proposed composite and reduced controls do not depend on ε.

10.3. Multimodelling

The main ideas of decentralized control design for singularly perturbed systems are described in Chapter 10.2. We extend this situation to the case of decentralized control design of a system, where each decision maker uses its own simplified reduced order model. The model simplification is based on the assumptions that the fast subsystems are weakly coupled and the slow subsystem is common for all controllers.

Problem 10.3. Consider an optimal control problem

$$(10.50) \qquad J = \tfrac{1}{2} \int_0^\infty \left(\|y\|_I^2 + \|u\|_R^2 \right) \mathrm{d}t \to \min$$

subject to

$$(10.51) \qquad \dot{x}_0 = A_0 x_0 + \sum_{j=1}^N A_{0j} x_j + \sum_{j=1}^N B_{0i} u_j , \quad x_0(0) = x_0$$

$$\varepsilon_i \dot{x}_i = A_{0i} x_0 + A_{ii} x_i + \sum_{i \neq j} \varepsilon_{ij} A_{ij} x_j + B_{ii} u_i , \quad x_i(0) = x_{i0} , \quad i = 1, \dots, N ,$$

where

$$y = \left(x_0^{\mathrm{T}}, x_1^{\mathrm{T}}, \dots, x_N^{\mathrm{T}} \right)^{\mathrm{T}} ,$$
$$u = \left(u_1^{\mathrm{T}}, \dots, u_N^{\mathrm{T}} \right)^{\mathrm{T}} .$$

Suppose that the kth subsystem neglects both the weak coupling parameters and the fast dynamic of all other subsystems. It means that $\varepsilon_j = 0$ for $j \neq k$ and $\varepsilon_{ij} = 0$. Supposing A_{ii} to be a nonsingular matrix, we obtain

$$(10.52) \qquad x_i = -A_{ii}^{-1}(A_{i0} + B_{ii} u_i) , \quad i \neq k , \quad i = 1, \dots, N .$$

Then the kth simplified problem using the substitution (10.52) deletes x_i for all $i \neq k$ in the criterion and model. The k-th simplified problem has then the form

$$(10.53) \qquad J_k = \tfrac{1}{2} \int_0^\infty \left(\|y_k\|_I^2 + \|u\|_R^2 \right) \mathrm{d}t \to \min$$

subject to

$$\dot{x}_{0k} = A_k x_{0k} + A_{0k} x_k + B_{0k} u_k + \sum_{\substack{j \\ j \neq k}} B_{kj} u_j ,$$

$$(10.54) \qquad \varepsilon_k \dot{x}_k = A_{k0} x_k + A_{kk} x_k + B_{kk} u_k ,$$

$$y_k = \left(x_{0k}^{\mathrm{T}}, x_k^{\mathrm{T}} \right)^{\mathrm{T}} = C_{0k} x_{0k} + C_{kk} x_k ,$$

where

$$A_k = A_0 - \sum_j A_{0j} A_{jj} A_{j0} , \quad B_{kj} = B_{0j} - A_{0j} A_{jj}^{-1} B_{jj} .$$

The existence of a joint slow problem for all k is derived under the validity of two assumptions. The first assumption concerns the model, where all perturbations in Eq. (10.51) including $\varepsilon_k = 0$ for (10.54) are neglected. The second assumption concerns the consistence between the criteria J_k given by Eq. (10.53) and $\tilde{J}_k$ using (10.52), where the criterion $\tilde{J}_l$ is the criterion J_l of the lth decision maker known to the decision maker in the form $\tilde{J}_l = \tilde{J}_l(x_0, u_l)$ since x_l is not included in his model (10.54).

The solution is based on the two-time-scale technique. Considering problem (10.53), (10.54), the slow problem and the fast problems can be formulated so that the fast subproblems are independent and the slow problem is of a game type problem. Concerning the relation among various ε_i two possibilities can occur. Problem (10.50), (10.51) can be considered as a multi-time-scale problem if ε_i are of different orders of magnitude, or as a multiparameter perturbations problem if ε_i are of the same order of magnitude. The mathematical formulation of a multiparameter perturbations problem is motivated by various real problems which can be encountered in large systems practice. To illustrate the main ideas of its solution, consider without loss of generality the case for $N = 2$, see Fig. 16.
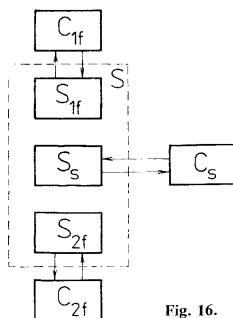

Fig. 16.

The slow subproblem is

$$(10.55) \qquad J_s = \tfrac{1}{2} \sum_{i=1}^{2} w_i \int_0^\infty \left(\| y_s \|_I^2 + \| u_{is} \|_R^2 \right) \mathrm{d}t \to \min$$

subject to

$$(10.56) \qquad \dot{x}_s = A_s x_s + B_s u_s , \quad x_s(0) = x_{s0} ,$$

$$y_s = C_s x_s + D_s u_s ,$$

where $A_s = A_0 - A_{0i} A_{ii}^{-1} A_{i0}$, $B_s = (B_{1s}, B_{2s})$, $B_{is} = B_{0i} - A_{0i} A_{ii}^{-1} B_{ii}$, $u_s = (u_{1s}^\mathsf{T}, u_{2s}^\mathsf{T})^\mathsf{T}$, $C_s = (C_{1s}^\mathsf{T}, C_{2s}^\mathsf{T})^\mathsf{T}$, $C_{is} = C_{0i} - C_{ii} A_{ii}^{-1} A_{i0}$, $D_s = (D_{1s}^\mathsf{T}, D_{2s}^\mathsf{T})^\mathsf{T}$, $D_{is} = -C_{ii} A_{ii}^{-1} B_{ii}$, w_i are weights, $w_1 + w_2 = 1$, $0 < w_i < 1$. Employing Theorem 10.2 and supposing the triple (A_s, B_s, C_s) to be stabilizable-detectable, we obtain

the solution in the form

$$(10.57) \qquad u_{is} = -R_{is}^{-1}(D_{is}^T C_{is} + (1/w_i) B_{is}^T K_s) x_s = F_{is} x_s ,$$

where K_s is the positive semidefinite stabilizing solution of the Riccati equation which is analogous to Eq. (10.38).

On the other hand, the fast kth subproblem is

$$(10.58) \qquad J_{kf} = \tfrac{1}{2} \int_0^\infty \left(\|x_{kf}\|_{Q_k}^2 + \|u_{kf}\|_R^2 \right) dt \to \min$$

subject to

$$(10.59) \quad \dot{x}_{kf} = A_{kk} x_{kf} + B_{kk} u_{kf} , \quad x_{kf}(0) = x_{k0} + A_{kk}^{-1}(A_{k0} x_0 + B_{kk} u_{ks}(0)) ,$$

where $Q_k = C_{kk}^T C_{kk}$.

Its solution is given by Theorem 10.2 under the assumption that the triple (A_{kk}, B_{kk}, C_{kk}) is stabilizable-detectable in the form

$$(10.60) \qquad u_{kf} = -R_k^{-1} B_{kf}^T K_{kf} x_{kf} = F_{kf} x_{kf} ,$$

where K_{kf} is the positive semidefinite stabilizing solution of the Riccati equation which is analogous to Eq. (10.40). The composite control, which is in principle given by Eq. (10.32), has the form

$$(10.61) \quad u_{kc} = -\left[(I - R_k^{-1} B_{kk}^T K_{kf} A_{kk}^{-1} B_{kk}) R_{ks}^{-1}(D_{ks}^T C_{ks} + (1/w_k) B_{ks}^T K_s) + \right.$$
$$\left. + R_k^{-1} B_{kk}^T K_{kf} A_{kk}^{-1} A_{k0} \right] x_s - R_k^{-1} B_{kk}^T K_{kf} x_k , \quad k = 1, 2 .$$

Considering now the difference between the feedback system (10.51), (10.61) and (10.56), (10.57), (10.59), (10.60) for $k = 1, 2$ we obtain the following result.

Theorem 10.4. [44]. If F_{1f}, F_{2f} stabilize the system (10.59) and F_{1s}, F_{2s} stabilize the system (10.56), then there exists a scalar $\sigma > 0$ such that

$$(10.62) \qquad x_0 = x_s + O(\|\varepsilon\|) ,$$
$$x_1 = -A_{11}^{-1}(A_{10} + B_{11} F_{1s}) x_s + x_{1f}(t/\varepsilon_1) + O(\|\varepsilon\|) ,$$
$$x_2 = A_{22}^{-1}(A_{20} + B_{22} F_{2s}) x_s + x_{1f}(t/\varepsilon_2) + O(\|\varepsilon\|) ,$$

hold for all $t \in \langle 0, +\infty), 0 < \|\varepsilon\| < \sigma$.

We summarize this procedure in an algorithm. We denote $m, M, m < M$, positive real numbers and we avoid the costs evaluation, see [44].

Algorithm 10.3.
1) Initialize $A_0, A_{0j}, \ldots, B_{ii}, R, w_i, x_0, N, m, M, \forall i$, see (10.50), (10.51).
2) Test the stabilizability-detectability of the triples $(A_s, B_s, C_s), (A_{ii}, B_{ii}, C_{ii}), \forall i$. If it is satisfied goto 3) else goto 7).
3) Test the multiparameter property of the system (10.51). If

$$m \leqq \varepsilon_i/\varepsilon_1 \leqq M , \quad \forall i,$$

is satisfied, then goto 4) else goto 7).
4) Compute F_s by solving (10.55), (10.56) and compute u_s using Eq. (10.57).

5) Compute F_f by solving (10.58), (10.59) and compute u_f using Eq. (10.60).
6) Compute u_{kl}, x_0, x_i using Eqs. (10.61), (10.62).
7) End.

11. DECENTRALIZED CONTROL OF STRONGLY COUPLED SYMMETRIC COMPOSITE SYSTEMS

Until now, the division of the design task has been done due to some kind of weakness of the subsystem interactions. "Weak" means that the subsystems can be analysed and designed separately and that the global solution lies in the vicinity of the local ones. Weakly coupled are e.g. systems with a hierarchical structure (Chapters 6 and 7), with low magnitude interactions (Chapters 8 and 9) and with slow and fast modes (Chapter 10).

In the following we take up a quite different philosophy for dealing with composite systems. This philosophy is based on the observation that the analysis of strongly coupled systems may be very simple if all the subsystems behave in a similar way. That is, considerable simplifications are gained from the conformity of the subsystem behaviour rather than the weakness of their interactions. Hence, the results apply arbitrarily strongly interconnected systems, for which weak-interaction methods fail.

In more detail, it is assumed that the overall system is composed of identical subsystems, which are symmetrically interconnected. This class of systems will be referred to as *symmetric composite systems* (Definition 11.1).

Systems of this kind are theoretically interesting, because considerable simplifications of the modelling and design problems can be made for arbitrary dynamical properties of the subsystems, strong interconnections and an unrestrained number of subsystems. In Section 11.2 it will be shown that the behaviour of each subsystem operating within the whole system can be exactly modelled by a state space model of order twice the order of the isolated subsystem. This confirms the experience that the subsystems of large-scale systems can, under certain conditions, be reasonably described by low order aggregate models and that such models must have regard to the interactions between the subsystems unless the interactions are weak. Moreover, the existence of decentralized controllers can be tested in terms of the modified subsystem models only (Theorem 11.1).

In Section 11.3 conceptual simplifications of the design problem are described. As far as stability is concerned, the whole design problem of the decentralized controller is equivalent to the design of a robust centralized controller that simultaneously stabilizes two low-order plants (Theorem 11.3). Hence the decentralized controller can be designed in a completely decentralized way. The I/O behaviour can be exactly investigated by means of a model with two decentralized control stations. For systems with a large number of subsystems this model can be further aggregated.

As a further motivation for the study of symmetric composite systems, the results are relevant to technological systems which cannot be decomposed into weakly coupled subsystems but whose subsystems behave similarly as, from a technological point of view, they participate in doing the same task. Multiarea power systems are considered in Section 11.4 to illustrate the results.

11.1. The design problem

The plant consists of N subsystems each of which is described by the state space model

(11.1)
$$\dot{x}_i = Ax_i + Bu_i + Es_i, \quad x_i(0) = x_{i0}$$
$$y_i = Cx_i$$
$$z_i = C_z x_i, \qquad i = 1, 2, ..., N,$$

where x_i, u_i, s_i, y_i, z_i are the vectors of the subsystem states, control inputs, interconnection inputs, control outputs, and interconnection outputs, respectively (Fig. 1). The interconnections are described by the algebraic relation

(11.2)
$$s = Lz,$$

where $s = (s_1^T, ..., s_N^T)^T$, $z = (z_1^T, ..., z_N^T)^T$, and

(11.3)
$$L = \begin{pmatrix} L_d & L_q & \cdots & L_q \\ L_q & L_d & \cdots & L_q \\ \cdots\cdots\cdots\cdots \\ L_q & L_q & \cdots & L_d \end{pmatrix}.$$

Equations $(11.1)-(11.3)$ reflect the assumptions that the subsystems are identical and coupled in a symmetric way. However, no restrictions are imposed on the dynamical properties of the subsystems and the sign and strength of the interactions. The following investigations hold for an arbitrarily large number of subsystems.

Definition 11.1. The system $(11.1)-(11.3)$ is called a *symmetric composite system*.

If the plant is subjected to external disturbances p_i, the term $+ Fp_i$ has to be appended to the first line of Eq. (11.1). It is only for notational convenience that we ignore disturbances and restrict our considerations to the command response of the closed loop and that we do not permit direct throughput from u_i and s_i to y_i and z_i.

Our aim is to design a decentralized controller

(11.4)
$$\dot{x}_{ri} = Fx_{ri} + Gy_i + Hv_i,$$
$$u_i = K_x x_{ri} + K_y y_i + K_v v_i, \quad i = 1, ..., N,$$

where v_i denotes the command input of subsystem i so that the close-loop system $(11.1)-(11.4)$ satisfies the design specifications given in Chapter 4.

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Decentralized controllers with identical control stations are used for two reasons. First, it is desirable to preserve the symmetry of the whole system. Disturbances should be rejected 'locally', i.e. within the subsystems they enter. Command following should be attained with the least possible excitation of the other subsystems. Second, the following investigations will show that the use of identical control stations has considerable methodological advantages.

11.2. Dynamics of symmetric composite systems

By combining Eqs. $(11.1)-(11.3)$ the state space model of the overall system is received

$$\dot{x} = \begin{pmatrix} A + EL_dC_z & EL_qC_z & \ldots & EL_qC_z \\ EL_qC_z & A + EL_dC_z & \ldots & EL_qC_z \\ \hdotsfor{4} \\ EL_qC_z & EL_qC_z & \ldots & A + EL_dC_z \end{pmatrix} x + \text{diag}\,(B, \ldots, B)\, u \,,$$

$$y = \text{diag}\,(C, \ldots, C)\, x \,, \quad x(0) = x_0 \,.$$

To get more insight into this model a similarity transformation

$$\tilde{x} = Tx$$

with

$$T = \frac{1}{N} \left(\begin{array}{cccc|c} (N-1)I & -I & \ldots & -I & -I \\ -I & (N-1)I & \ldots & -I & -I \\ \hdotsfor{5} \\ -I & -I & \ldots & (N-1)I & -I \\ \hline I & I & \ldots & I & I \end{array} \right),$$

$$T^{-1} = \left(\begin{array}{cccc|c} I & 0 & \ldots & 0 & I \\ 0 & I & \ldots & 0 & I \\ \hdotsfor{5} \\ \hline -I & -I & \ldots & -I & I \end{array} \right)$$

is carried out, where the left upper hyper block of T and T^{-1} consists of $(N-1) \times (N-1)$ identity matrices I of dimension n (in part multiplied by $(N-1)$ or -1, respectively).

$$\dot{\tilde{x}} = \left(\begin{array}{ccc|c} A_s & & & \\ & A_s & & \\ \hdotsfor{4} \\ & & A_s & \\ \hline & & & A_0 \end{array} \right) \tilde{x} + \frac{1}{N} \left(\begin{array}{cccc|c} (N-1)B & -B & \ldots & -B & -B \\ -B & (N-1)B & \ldots & -B & -B \\ \hdotsfor{5} \\ -B & -B & \ldots & (N-1)B & -B \\ \hline B & B & \ldots & B & B \end{array} \right) u \,,$$

(11.5)

$$\tilde{x}(0) = Tx_0 \,, \quad y = \left(\begin{array}{cccc|c} C & 0 & \ldots & 0 & C \\ 0 & C & \ldots & 0 & C \\ & & \ddots & & \\ 0 & 0 & \ldots & C & C \\ \hline -C & -C & \ldots & -C & C \end{array} \right) \tilde{x} \,,$$

where
(11.6)
$$A_s = A + E(L_d - L_q) C_z,$$
$$A_0 = A + EL_dC_z + (N - 1) EL_qC_z.$$

Obviously, symmetric composite systems posses important properties, which will considerably simplify the analysis. First, as the spectrum of the system matrix consists of the spectrum of A_s with multiplicity $N - 1$ and the spectrum of A_0, the stability of the overall system $(11.1)-(11.3)$ can be easily tested.

Theorem 11.1. A symmetric composite system $(11.1)-(11.3)$ is stable if and only if all the eigenvalues of the matrices
$$A_s = A + E(L_d - L_q) C_z$$
$$A_0 = A + E(L_d + (N - 1) L_q) C_z$$
have negative real parts.

Hence, only two nth order matrices A_s and A_0 that occur in the model (11.5) have to be investigated. In general, the stability of the isolated subsystems with system matrix A is neither necessary nor sufficient for the overal system stability. For symmetric composite system we have

Corollary 11.1. If the interconnection matrix (11.3) has the property $L_d = L_q$, then the stability of the subsystems is a necessary condition for the stability of the symmetric composite system.

Second, since the output y_i of subsystem i depends merely on the subvectors $\tilde{x}_i$ and $\tilde{x}_N$ of the overall system state $\tilde{x}$, it is completely described by the low-order model

(11.7)
$$\dot{\tilde{x}}_i = \begin{pmatrix} A_s & \\ & A_0 \end{pmatrix} \tilde{x}_i + \begin{pmatrix} \frac{N-1}{N} B \\ \frac{1}{N} B \end{pmatrix} u_i + \begin{pmatrix} -\frac{1}{N} B \\ \frac{1}{N} B \end{pmatrix} \sum_{j \neq i} u_j,$$

$$\tilde{x}_i(0) = \begin{pmatrix} \frac{N-1}{N} x_{i0} + \frac{1}{N} \sum_{j \neq i} x_{j0} \\ \sum_{j=1}^{N} x_{j0} \end{pmatrix},$$

$$y_i = (C \quad C) \tilde{x}_i.$$

That is, in a symmetric composite system the performance of a certain subsystem operating within the whole system can be *exactly* described by a low order model or order twice the order of the subsystem. Nonetheless, the model (11.7) reflects both the free motion of subsystem i from arbitrary initial states of all subsystems as well as the I/O behaviour of subsystem i subject to all inputs u_j. Equation (11.7) points out in which way the couplings within the whole system influence the subsystem

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behaviour. The model (11.7) consists of two parts in parallel each of which represents a subsystem model with feedback of z_i towards s_i (Fig. 17). For a large number of subsystems a further model simplification is possible because the second part
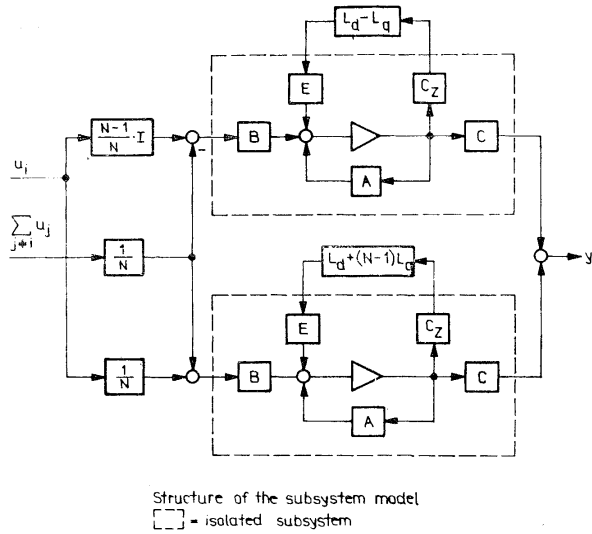

Structure of the subsystem model
$\begin{bmatrix} \text{--} \\ \text{--} \end{bmatrix}$ = isolated subsystem

Fig. 17.

of the state vector will no longer be excited. Then Eq. (11.7) yields the approximate model

(11.8) $$\dot{\hat{x}}_i = A_s \hat{x}_i + B u_i, \quad \hat{x}_i(0) = x_{i0},$$

$$y_i = C \hat{x}_i$$

which has only the same order as the isolated subsystem (11.1). Eq. (11.8) shows that the isolated subsystem (Eq. (11.1) with $s_i = 0$) may be a rather coarse representation of the subsystem behaviour and that merely a little modification of this model can considerably improve the approximation accuracy. Instead of using the isolated subsystem as approximate model (as it is done in the aggregation-decomposition method, Section 8.2) a coarse model

(11.9) $$s_i = (L_d - L_q) z_i$$

should be used to take regard of the influence the other subsystems exercise upon

the subsystem i (Fig. 18). If $L_d = L_q$, Eq. (11.8) coincides with the isolated subsystem.

The analysis of the overall system can be carried out by means of the modified subsystem model (11.7). As far as stability is concerned this has been shown in
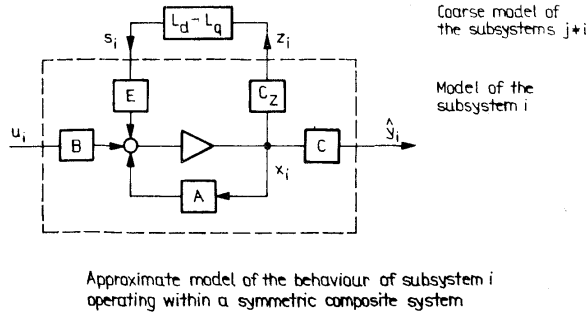

Approximate model of the behaviour of subsystem i
operating within a symmetric composite system

Fig. 18.

Theorem 11.1. For the analysis of observability, controllability and the existence of fixed modes the following result has been proved in [73].

Lemma 11.1 [73]. For the symmetric composite system (11.1)−(11.3) the following statements are equivalent:
  (i) The system (11.1)−(11.3) is completely controllable through $u = (u_1^T, \ldots, u_N^T)^T$ and completely observable through $y = (y_1^T, \ldots, y_N^T)^T$.
 (ii) The pairs (A_s, B) and (A_0, B) are controllable and the pairs (A_s, C) and (A_0, C) are observable with A_s and A_0 from Eq. (11.6).
(iii) The system (11.1)−(11.3) has ho decentralized fixed modes.

The last result ensures that the plant (11.1)−(11.3) is stabilizable by decentralized control and that an arbitrary set of poles can be assigned to the closed-loop system, if the plant is controllable and observable (for definition and significance of fixed modes see [45], [58], [98]).

11.3. Reduction of the design complexity

The model of the closed-loop system is received from Eqs. (11.1)−(11.4). Combining (11.1) and (11.4) we get

$$(11.10) \qquad \begin{pmatrix} \dot{x}_i \\ \dot{x}_{ri} \end{pmatrix} = \begin{pmatrix} A + BK_yC & BK_x \\ GC & F \end{pmatrix} \begin{pmatrix} x_i \\ x_{ri} \end{pmatrix} + \begin{pmatrix} BK_v \\ H \end{pmatrix} v_i + \begin{pmatrix} E \\ 0 \end{pmatrix} s_i ,$$

$$y_i = (C \ \ 0) \begin{pmatrix} x_i \\ x_{ri} \end{pmatrix} , \qquad z_i = (C_z \ 0) \begin{pmatrix} x_i \\ x_{ri} \end{pmatrix} ,$$

or in short

(11.11)
$$\dot{\bar{x}}_i = \bar{A}\bar{x}_i + \bar{B}v_i + \bar{E}s_i,$$
$$y_i = \bar{C}\bar{x}_i,$$
$$z_i = \bar{C}_z\bar{x}_i.$$

Like the plant $(11.1)-(11.3)$ the closed-loop system consists of identical subsystems (11.11) that are symmetrically interconnected via the relations (11.2) and (11.3). In analogy to Eqs. (11.5), (11.6) and (11.7) the following equations can be derived

$$\dot{\bar{x}} = \begin{pmatrix} \bar{A}_s & 0 & .. & 0 \\ 0 & \bar{A}_s & 0 & 0 \\ \multicolumn{4}{c}{\dots\dots\dots} \\ 0 & .. & \bar{A}_s & 0 \\ \hline 0 & .. & 0 & A_0 \end{pmatrix} \bar{x} + \frac{1}{N} \begin{pmatrix} (N-1)\bar{B} & -\bar{B} & \dots & -\bar{B} & -\bar{B} \\ -\bar{B} & (N-1)\bar{B} & \dots & -\bar{B} & -\bar{B} \\ \multicolumn{5}{c}{\dots\dots\dots\dots\dots\dots\dots\dots\dots} \\ -\bar{B} & -\bar{B} & \dots & (N-1)\bar{B} & -\bar{B} \\ \hline \bar{B} & \bar{B} & \dots & \bar{B} & \bar{B} \end{pmatrix} v,$$

(11.12)
$$y = \begin{pmatrix} \bar{C} & 0 & \dots & 0 & \bar{C} \\ 0 & \bar{C} & \dots & 0 & \bar{C} \\ \multicolumn{5}{c}{\dots\dots\dots\dots\dots} \\ 0 & 0 & \dots & \bar{C} & \bar{C} \\ \hline -\bar{C} & -\bar{C} & \dots & -\bar{C} & \bar{C} \end{pmatrix} \bar{x},$$

where

(11.13)
$$\bar{A}_s = \bar{A} + E(L_d - L_q)C_z,$$
$$A_0 = \bar{A} + E(L_d + (N-1)L_q)C_z,$$

and

(11.14)
$$\begin{pmatrix} \dot{\tilde{x}}_i \\ \dot{\tilde{x}}_N \end{pmatrix} = \begin{pmatrix} \bar{A}_s & 0 \\ 0 & \bar{A}_0 \end{pmatrix} \begin{pmatrix} \tilde{x}_i \\ \tilde{x}_N \end{pmatrix} + \begin{pmatrix} \dfrac{N-1}{N}\bar{B} \\ \dfrac{1}{N}\bar{B} \end{pmatrix} v_i + \begin{pmatrix} -\dfrac{1}{N}\bar{B} \\ \dfrac{1}{N}\bar{B} \end{pmatrix} \sum_{j \neq i} v_j,$$
$$y_i = (\bar{C} \quad \bar{C}) \begin{pmatrix} \tilde{x}_i \\ \tilde{x}_N \end{pmatrix}, \quad i = 1, \dots, N.$$

Since the model (11.14) has the same parameters for all indices i, we confine our considerations to subsystem 1 and drop the index of $\tilde{x}_i$. All phenomena encountered in the whole system can be studied by means of the model

(11.15)
$$\dot{\tilde{x}} = \begin{pmatrix} \bar{A}_s & 0 \\ 0 & \bar{A}_0 \end{pmatrix} \tilde{x} + \begin{pmatrix} \dfrac{N-1}{N}\bar{B} \\ \dfrac{1}{N}\bar{B} \end{pmatrix} v_i + \begin{pmatrix} -\dfrac{1}{N}\bar{B} \\ \dfrac{1}{N}\bar{B} \end{pmatrix} \sum_{j \neq i}^{N} v_j,$$
$$y_i = (\bar{C} \quad \bar{C}) \tilde{x}.$$

Theorem 11.2. The command response of the symmetric composite system (11.1) to (11.3) under decentralized control (11.4) is exactly represented by the model (11.15) of dynamical order twice the order of the closed-loop subsystem (11.11).

The model (11.15) can be thought of as the closed-loop system that consists of the plant

$$(11.16) \qquad \dot{x} = \begin{pmatrix} A_s & 0 \\ 0 & A_0 \end{pmatrix} x + \begin{pmatrix} B & 0 \\ 0 & B \end{pmatrix} \tilde{u} \,,$$

$$\tilde{y} = \begin{pmatrix} C & 0 \\ 0 & C \end{pmatrix} x \,,$$

$$y_1 = (C \quad C) \, x \,,$$

with A_s and A_0 from Eq. (11.6) and the decentralized controller

$$(11.17) \qquad \dot{x}_r = \begin{pmatrix} F & 0 \\ 0 & F \end{pmatrix} x_r + \begin{pmatrix} G & 0 \\ 0 & G \end{pmatrix} \tilde{y} + \begin{pmatrix} H & 0 \\ 0 & H \end{pmatrix} \begin{pmatrix} \tilde{v}_1 \\ \tilde{v}_2 \end{pmatrix} \,,$$

$$\tilde{u} = \begin{pmatrix} K_x & 0 \\ 0 & K_x \end{pmatrix} x_r + \begin{pmatrix} K_y & 0 \\ 0 & K_y \end{pmatrix} \tilde{y} + \begin{pmatrix} K_v & 0 \\ 0 & K_v \end{pmatrix} \begin{pmatrix} \tilde{v}_1 \\ \tilde{v}_2 \end{pmatrix} \,,$$

where

$$(11.18) \qquad \tilde{v}_1 = \frac{N-1}{N} v_1 - \frac{1}{N} \sum_{j=2}^{N} v_j \,,$$

$$\tilde{v}_2 = \frac{1}{N} \sum_{j=1}^{N} v_j \,.$$

(cf. Eqs. (11.10), (11.11) and (11.15) and decompose $\tilde{x}$ into plant states and controller states).

Corollary 11.2. The decentralized controller (11.4) satisfies the design specifications for the plant $(11.1) - (11.3)$ if and only if the decentralized controller (11.17), (11.18) meets these requirements in connection with the plant (11.16).

Hence, the symmetry of the system brings about considerable reduction of the design complexity. Only a low order auxiliary plant (11.16) and a decentralized controller with merely two control stations have to be considered. This holds true for an arbitrary number of subsystems and arbitrarily strong interactions among the subsystems. The number N influences merely the parameters of the plant (11.16).

Stability of the closed-loop system

Fig. 19 makes clear that the closed-loop system $(11.16) - (11.18)$ consists of two closed loops in parallel. These loops have different plants but identical feedback.

Theorem 11.3 [74]. The decentralized controller (11.4) ensures the stability of the

closed-loop system $(11.1)-(11.4)$ if and only if the control station

$$(11.19) \qquad\qquad \dot{x}_r = F x_r + G y + H v ,$$

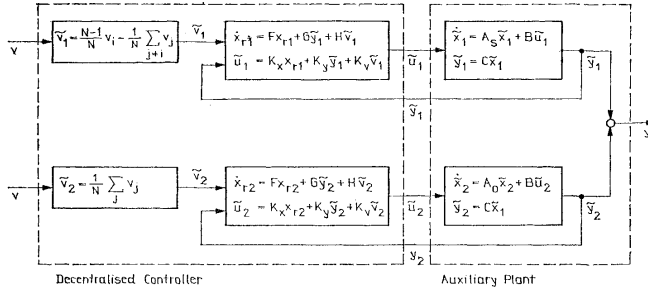$$u = K_x x + K_y y + K_v v$$

Fig. 19. The auxiliary closed-loop system.

(cf. Eq. (11.4) with indices dropped) ensures closed-loop stability simultaneously for the plants

$$(11.20) \qquad\qquad \dot{x} = A_x x + B u ,$$

$$y = C x ,$$

and

$$(11.21) \qquad\qquad \dot{x} = A_0 x + B u ,$$

$$y = C x .$$

This theorem states that in view of the stability requirement (I) the problem of designing the decentralized controller (11.4) for the overall system $(11.1)-(11.3)$ can be replaced by the task of designing a centralized controller (11.19) that simultaneously stabilizes the two auxiliary plants (11.20) and (11.21). This is a problem of *robust control*.

The problem of simultaneous stabilisation has been tackled in [1], [108] (for a survey see [75]). It has been proved in [108] that two plants are simultaneously stabilised by a common compensator if and only if a single auxiliary plant, which is derived from the two original plants, is stabilised. In this way, Theorem 11.3 shows that the decentralized controller (11.4) can be found by designing a centralized controller for an auxiliary plant. A detailed discussion of this design problem and its solution can be found in [74].

Asymptotic regulation

Theorem 11.4. [74]. The decentralized controller (11.4) ensures asymptotic regulation within the closed-loop overall system (11.1)−(11.4) if and only if asymptotic regulation occurs in both closed-loop auxiliary systems (11.20) and (11.21).

This theorem says that each decentralized control station must include a servocompensator as described by the Internal Model Principle [51] and that no reduction of the servocompensator due to the symmetry of the plant is possible.

Simplification of the model (11.16)−(11.18)

For systems with a large number of subsystems, the I/O behaviour can be reasonably approximated by the closed-loop system (11.8), (11.19) where the index i has to be dropped. This becomes obvious from Eqs. (11.16)−(11.18) and Fig. 18. So, we have no longer a decentralized control but merely a centralized control.

This observation suggests to design the decentralized controller (11.4) in relation to the design requirement (III) in 2 *design steps*:

1. Design a centralized controller (11.19) for the approximate model (11.8) so as to satisfy the dynamical requirements (III).
2. Check the dynamical behaviour of the closed-loop system by means of the exact model (11.16)−(11.18).

This design method is especially useful if the plant has many subsystems and the stability requirement (I) makes no real problem during the design process.

Algorithm 11.1.

1) Initialize the model parameters A, B, C, L_d, L_q.
2) Determine the model (11.20), (11.21).
3) Design the decentralized controller (11.19) for the model (11.20).
4) Check the stability of the closed-loop system (11.19), (11.21).
5) Check the I/O behaviour of the closed-loop system (11.16)−(11.18). If the design specifications (III) are not satisfy, modify the controller and goto 3) else goto 6).
6) End.

**11.4. Example: Decentralized control of an multiarea power system**

As an example consider the problem of controlling the voltages of the feeding nodes in an electric power system. The plant consists of several synchronous machines which feed the load through transformers and the distribution net (Fig. 20). The machines including their generator voltage controllers are considered as subsystems the generator voltage V_{gi} and the node voltage V_{ei} being the interconnection outputs and inputs, respectively (see Fig. 20 (b)). The inputs to the subsystems

are the command inputs V_{ci} of the generator voltage controllers. The outputs V_{ei} of devices for measuring the node voltages act as subsystem outputs.
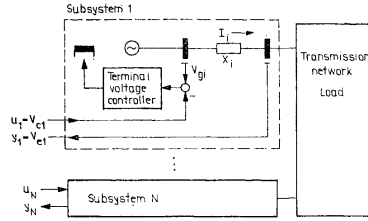
(a) Electric power system as the plant of the decentralised voltage control problem
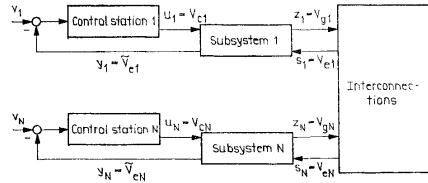
(b) Structure of the decentralised control system

Fig. 20. Decentralized voltage control of an electric power system.

Assuming identical generators and a symmetric net this system can be modelled by Eqs. $(11.1)-(11.3)$ with

(11.22)
$$A = \begin{pmatrix} -1\cdot94 & -0\cdot16 & 0 \\ 2\cdot58 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix}, \quad B = \begin{pmatrix} 0\cdot9 \\ -1 \\ 0 \end{pmatrix}, \quad E = \begin{pmatrix} -0\cdot33 \\ -0\cdot015 \\ 2 \end{pmatrix},$$

$$C = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix},$$
$$C_z = \begin{pmatrix} 2\cdot55 & 0 & 0 \end{pmatrix},$$
$$L_d = 0\cdot655,$$
$$L_q = 0\cdot053.$$

Hence, we have

(11.23)
$$A_s = \begin{pmatrix} -2\cdot45 & -0\cdot16 & 0 \\ 2\cdot56 & 0 & 0 \\ 3\cdot07 & 0 & -2 \end{pmatrix}, \quad A_0 = \begin{pmatrix} -2\cdot49 & -(N-1)\,0\cdot0446 & -0\cdot16 & 0 \\ 2\cdot55 & -(N-1)\,0\cdot002 & 0 & 0 \\ 3\cdot34 & +(N-1)\,0\cdot27 & 0 & 0 \end{pmatrix}.$$

As the set points for the node voltages can be assumed to change stepwise we use decentralized PI controllers.

We use the two-step design method described in Section 11.3. At first, the parameters of the centralized PI controller $\big($Eq. (11.19) without index $i\big)$ are determined for the approximate model (11.8). We do this by calculating the optimal output feedback $u = K_I x_r + K_p y$ of the model (11.8) that has been expanded by the dynamical part of the controller (11.19) $\big(v = 0,\ x_r(0) = x_{r0}\big)$. Denote $x_e = \big(x_r^T, x^T\big)^T$. For the performance index

$$(11.24) \qquad\qquad J = \int_0^\infty \big(\|x_e\|_Q^2 + \|u\|_I^2\big)\,\mathrm{d}t \to \min$$

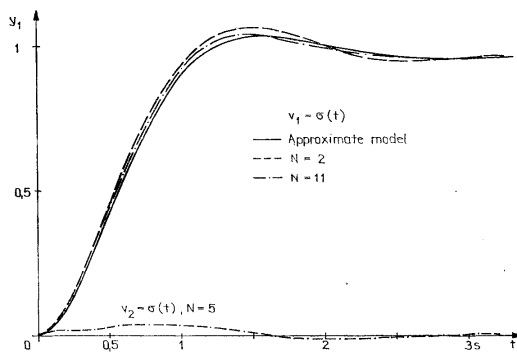different controllers have been received in dependence upon the choice of the matrix

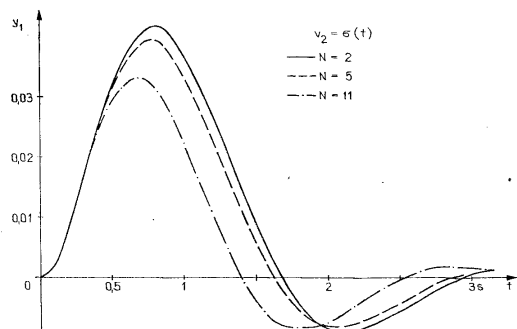Fig. 21 (a). Step response of the closed-loop overal system: $y_1(t)$ for $v_1(t) = \sigma(t)$.

Fig. 21 (b). Step response of the closed-loop overall system: $y_2(t)$ for $v_2(t) = \sigma(t)$.

$Q = \text{diag}(q_{11}, q_{22})$. We select the solution for $Q = 10I$ $(K_I = -3\cdot16, K_P = -2\cdot17)$ because this controller leads to the most suitable command response of the closed-loop approximate model (11.8), (11.19) (Fig. 21).

If we use these controller parameters for the decentralized control of the overall plant, the stability of the closed-loop system can be concluded from Theorem 11.3 because not only the closed-loop approximate model (11.8), (11.19) but also the system (11.19), (11.21) is stable. Since PI controllers are used, asymptotic regulation occurs in the closed-loop overall system as well (cf. Theorem 11.4). To investigate the command response of the closed-loop system exactly we use the model (11.16) to (11.18) with the given parameters. As shown in Fig. 21(a) the command step response of the overall system is very near to the approximation for a wide range of N. The cross-couplings (y_1 for step input at v_2) are small and decrease with increasing number of subsystems (Fig. 21(b)). One of these curves has been drawn into Fig. 21(a) to illustrate the smallness of the cross-couplings in relation to the influence of v_1 on y_1.

11.5. Extensions

The utilisation of symmetry properties of the plant is an alternative way for simplifying the problem of analysing interconnected systems and designing decentralized controllers. As this method uses structural properties of the plant it can be applied even to strongly coupled systems.

Until now, papers on symmetrically interconnected systems are rare even in the fields of modelling and analysis. In [5], [16] and [18] systems composed of first order subsystems are considered. It is demonstrated that symmetry in the interconnection relation may lead to considerable simplifications of the stability analysis. The investigation of a much more general class of symmetric composite systems in this chapter has shown that the design of decentralized controllers can be done on the subsystem level if the interactions among the subsystems are included into the considerations by means of very simple models.

Besides the example in Section 11.4, application studies can be found in [70], [73] and [74].

So far we have presupposed the complete identity of the subsystem dynamics. However, the results on symmetric composite systems can be extended to systems whose subsystems are similar rather than identical. The system is decomposed into the symmetric core and the remaining part. While the analysis of the symmetric core is as simple as described in this chapter and the decentralized controller can be designed in the way described above, the result has to be tested in connection with the remaining part of the plant. This motivates a new way for testing the stability of the closed-loop overall system. Instead of using the method depicted in Fig. 10, the overall system is decomposed according to Fig. 22, where $\mathscr{P}$ denotes the sym-

metric core and $\mathscr{S}$ the remaining part. As the stability of the symmetric core can be tested very easily (cf. Theorem 11.1) the main problem is the stability test of the system $\mathscr{S}'$. Reference [70] gives the solution to this problem and provides two application examples.
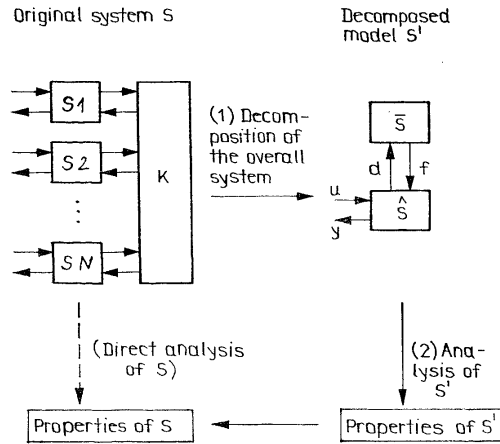

Fig. 22. Stability analysis of systems composed of similar subsystems.

12. ROBUST DECENTRALIZED PI CONTROLLERS

12.1. The design problem

In this chapter, we consider the plant

$$(12.1) \qquad \dot{x} = Ax + Bu, \quad u = (u_1^T, ..., u_N^T)^T,$$
$$y = Cx, \qquad y = (y_1^T, ..., y_N^T)^T$$

under decentralized PI control

$$(12.2) \qquad \dot{x}_{ri} = y_i - v_i,$$
$$u_i = K_{Ii}x_{ri} + K_{Pi}(y_i - v_i).$$

That is, we specify the design requirements (I) − (IV) as given in Chapter 4 as follows:
(II) Asymptotic regulation has to be ensured for step commands $v_i = \bar{v}_i\,\sigma(t)$ and step disturbances $p_i = \bar{p}_i\,\sigma(t)$, where $\sigma(t)$ denotes the step function.

Furthermore, in Section 12.2 we consider the uncertainty of the closed-loop system that results from sensor or actuator failures. For stability considerations, this uncertainty can be represented by scalars e_i that are introduced into Eq. (12.2)

(12.3)
$$\dot{x}_{ri} = y_i - v_i,$$
$$u_i = e_i K_{Ji} x_{ri} + e_i K_{Pi}(y_i - v_i).$$

The scalar e_i indicates whether the control station i is $(e_i = 1)$ or is not $(e_i = 0)$ in operation. Obviously,

(12.4)
$$e = (e_1, \ldots, e_N)^{\mathrm{T}} \in \mathscr{E} = \{e: \text{ with } e_i \in \{0, 1\}\}$$

holds. Then the design requirements (I) and (IV) are combined to yield
(I') The stability of the closed-loop system (12.1), (12.3) has to be ensured for all operating conditions that are described by Eq. (12.4).

12.2. Existence of robust decentralized PI controllers

In this section, it will be investigated under what conditions a decentralized PI controller (12.3) exists that ensures the stability of the closed-loop system (12.1.), (12.3) for all $e \in \mathscr{E}$.

First, we derive preliminary results concerning centralized I control

(12.5)
$$\dot{x}_r = y - v,$$
$$u = K x_r$$

of the plant (12.1). From Eqs. (12.1) and (12.5) we get

$$\begin{pmatrix} \dot{x} \\ \dot{x}_r \end{pmatrix} = \begin{pmatrix} A & BK \\ C & 0 \end{pmatrix} \begin{pmatrix} x \\ x_r \end{pmatrix} + \begin{pmatrix} 0 \\ -I \end{pmatrix} v .$$

This system is stable if and only if all the eigenvalues of

$$\bar{A} = \begin{pmatrix} A & BK \\ C & 0 \end{pmatrix}$$

have negative reals parts. Hence,

(12.6)
$$\det |-\bar{A}| > 0$$

holds. On the other hand, we assume that the plant (12.1) is stable. Therefore, we have

(12.7)
$$\det |-A| > 0 .$$

Using the determinant relation

(12.8)
$$\det \begin{vmatrix} R & S \\ T & U \end{vmatrix} = \det R \cdot \det |U - TR^{-1}S|$$

provided that the submatrix R is invertible, we receive from Eq. (12.6)

$$\det |-A| \cdot \det |CA^{-1}BK| > 0 .$$

Since
$$K_s = -CA^{-1}B$$
represents the matrix of the static reinforcement of the system (12.1) and Eq. (12.7) holds, we get

(12.9) $$\det |-K_s K| > 0$$

as a necessary stability condition of the closed-loop system (12.1), (12.5).

Lemma 12.1. [64], [75]. Assume that the plant (12.1) is stable. Then a necessary condition for the stability of the closed-loop system (12.1), (12.5) is given by the inequality (12.9).

A sufficient stability condition can be derived in a similar way if the controller matrix is used in the form

(12.10) $$K = -a\tilde{K} .$$

Lemma 12.2. [64], [75]. Consider the stable plant (12.1) with static reinforcement K_s and I control (12.5), (12.10). If $\tilde{K}$ satisfies the conditions

(12.11) $$\text{Re } \lambda_i(-K_s\tilde{K}) < 0 ,$$

then there exists some $\bar{a}$ such that the closed-loop system (12.1), (12.10), (12.5) is stable for all $a \in (0, \bar{a})$. λ_i denotes the ith eigenvalue of the given matrix.

Now, these results are applied to the decentralized control system (12.1), (12.3). According to Eq. (12.9) a necessary condition for the stability of the closed loop for $e_i = 1 \, (i = 1, ..., N)$ is given by

(12.12) $$\det \left| -\begin{pmatrix} K_{s11} & K_{s12} & \dots & K_{s1N} \\ K_{s21} & K_{s22} & \dots & K_{s2N} \\ \dots\dots\dots\dots\dots\dots\dots \\ K_{sN1} & K_{sN2} & \dots & K_{sNN} \end{pmatrix} \begin{pmatrix} K_{I1} & & & 0 \\ & K_{I2} & & \\ & & \ddots & \\ 0 & & & K_{IN} \end{pmatrix} \right| > 0 ,$$

where pure I control is considered $(K_{Pi} = 0)$. If $\dim y_i = \dim u_i$ holds Eq. (12.12) can be transformed in the following way. Without loss of generality we assume

(12.13) $$\det |-K_{Ii}| > 0 .$$

Then, Eq. (12.12) is equivalent to

(12.14) $$\det K_s > 0 .$$

If $e_1 = 0$ and $e_i = 1 \, (i = 2, ..., N)$, then instead of Eq. (12.12) the condition

(12.15) $$\det \left| -\begin{pmatrix} K_{s22} & \dots & K_{s2N} \\ \dots\dots\dots\dots \\ K_{sN2} & \dots & K_{sNN} \end{pmatrix} \begin{pmatrix} K_{I2} & 0 \\ & \ddots \\ 0 & K_{IN} \end{pmatrix} \right| > 0$$

has to be considered because the closed-loop system includes feedback only between $y_2, ..., y_N$ and $u_2, ..., u_N$ and the matrix K_s has, thus, to be reduced by deleting

the first hyper row and hyper column. Consequently,

$$(12.16) \qquad \det \begin{vmatrix} K_{s22} & \dots & K_{s2N} \\ \dots\dots\dots\dots & \\ K_{sN2} & \dots & K_{sNN} \end{vmatrix} > 0$$

has to be satisfied. After having considered corresponding conditions for all $e \in \mathscr{E}$ the following result is received.

Theorem 12.1. [68], [75]. Consider a stable linear system (12.1) with static reinforcement K_s. Assume that

$$(12.17) \qquad \det (K_{sii}) > 0$$

holds. A necessary condition for the existence of decentralized PI-controllers (12.3) that ensure closed-loop stability for all $e \in \mathscr{E}$ is given by the requirement that the determinants of K_s and of all matrices that result from K_s by deleting corresponding hyper rows and hyper columns have to be positive.

Here, Eq. (12.13) is replaced by (12.17). Both assumptions are identical because for $e_j = 0$ for all $j \neq i$, Eq. (12.9) leads $\det \left| -K_{sii} K_{Ii} \right| > 0$. Theorem 12.1 states not only a necessary stability condition but an existence condition because the determinant inequalities (12.14), (12.16) etc. are not dependent on the controller matrices K_{Ii}. Theorem 12.1 is not only true for I control but also for PI control since a decentralized PI controller (12.2) with sufficiently small matrices K_{Pi} exists whenever a decentralized I controller exists.

Theorem 12.1 describes the necessary autonomy of the subsystems of the plant (12.1). It is satisfied at least for systems with low static cross-couplings $K_{sij} (j \neq i)$. If the subsystems are stronger connected then no decentralized controller can be found that ensures closed-loop stability for all $e \in \mathscr{E}$.

For systems with dim $y_i =$ dim $u_i = 1$, Theorem 12.1 claims that all principal minors of K_s be positive. In connection with Lemma 12.2 this condition turns out to be sufficient as well because the controller (12.2) with $K_{Pi} = 0$ and

$$(12.18) \qquad K_{Ii} = -a_i K_{sii}^{-1}$$

satisfies the condition (12.11) for the decentralized control system for all $e \in \mathscr{E}$.

Theorem 12.2. [68], [75]. Consider a stable linear system (12.1) with dim $y_i =$ $=$ dim $u_i = 1$. Assume that $K_{sii} > 0$ holds. Then a necessary and sufficient condition for the existence of decentralized PI controllers (12.3) that ensures closed-loop stability for all $e \in \mathscr{E}$ is given by the requirement that all principal minors of K_s have to be positive.

Similar results have been published in [33] and [80]. A quite different approach has been proposed in [60] and [61], where high-gain feedback rather than low-gain feedback as in Lemma 12.2 provides the basis for the design method.

As an *example* consider the voltage control problem described in Section 11.4.

under the ideal assumption that the reactive power/voltage behaviour of the system can be completely split from the active power/frequency behaviour. Then, all admittances are reactive. The static plant model can be set up by the mesh theorem $V_{gi} = I_i X_i + V_{ei}$, where I_i is the reactive current and X_i the admittance of the transformer (Fig. 19). That is, we have

(12.19)
$$u_i = I_i X_i + y_i.$$

The transmission network including the load can be described by

(12.20)
$$\begin{pmatrix} I_1 \\ \vdots \\ I_N \end{pmatrix} = Y \begin{pmatrix} V_{e1} \\ \vdots \\ V_{eN} \end{pmatrix}$$

with Y denoting the admittance matrix. Eqs. (12.19) and (12.20) yield

$$u = (I + \operatorname{diag} X_i Y) y.$$

Hence, we have

(12.21)
$$K_s^{-1} = I + \operatorname{diag} X_i Y.$$

The admittance matrix has the properties

$$Y = (y_{ij}) \quad \text{with} \quad y_{ij} \leq 0 \quad \text{for} \quad i \neq j$$
$$y_{ii} \geq \sum_{j \neq i} |y_{ij}| > 0.$$

Therefore, Y is an M-matrix and so is the matrix K_s^{-1} (for the theory of M-matrices see [75], [79], [94]). Therefore, the plant satisfies the existence condition of Theorem 12.2.

The existence is proved for an arbitrarily large number N of subsystems and it is not affected by switching transmission lines on or off, or by changing the generator parameters. Theorem 12.2 is satisfied due to structural reasons which become obvious in the properties of the admittance matrix.

An extension of this result to real-existing power systems has been published in [31].

12.3. Sequential tuning of PI controllers

Now, the question arises how to determine suitable controller parameters provided that the existence conditions are satisfied. In this section, the method of sequentially tuning the control stations is described. A method for designing the control stations by methods of robust centralized control will be presented in Section 12.4.

If we confine our consideration to a single control station at any instant of time, we have a problem of centralized control. The resulting plant includes all other control stations that are connected to the given plant (Fig. 23). In order to design a certain control station, say control station 1, we need a model of the resulting plant with input u_1 and output y_1. This problem and its solution has, in a modified

way, been considered in Chapter 7. Here, we give a much simpler solution for decentralized PI controllers.

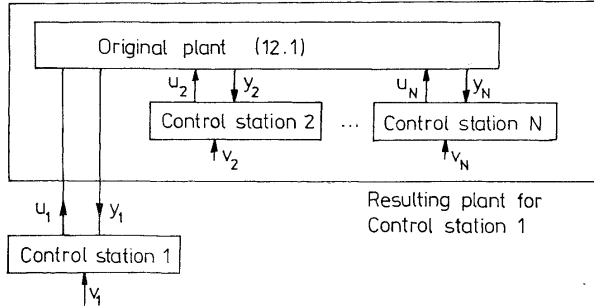According to Lemma 12.2 a stabilising PI control station (Eq. (12.2) for a given

Fig. 23. Decentralized control system from the point of view of subsystem 1.

index i) can be found by using the controller matrix

$$(12.22) \qquad K_{Ii} = -a_i \tilde{K}_{Ii}$$

and

$$(12.23) \qquad K_{Pi} = -b_i \tilde{K}_{Pi}$$

with $\tilde{K}_{Ii}$ satisfying the relation

$$(12.24) \qquad \text{Re } \lambda_i(-\bar{K}_{sii}\tilde{K}_{Ii}) < 0 ,$$

arbitrary $\tilde{K}_{Pi}$ and sufficiently small scalars a_i and b_i. The matrix $\bar{K}_{sii}$ denotes the static reinforcement between u_i and y_i under the influence of the control stations that are already connected to the plant. This suggests the following tuning algorithm, which assumes that the static reinforcement of the subsystem under consideration can be measured in simple experiments.

Sequential tuning algorithm

1) $i = 1$.
2) Determine the static model $y_i = \bar{K}_{sii}u_i$ of the plant including the control stations that have been already brought into operation.
3) Choose controller parameters according to Eqs. (12.22)−(12.24). Make experiments to ensure that a sufficiently large stability degree is ensured. Change the scalars a_i and b_i accordingly.
4) If $i < N$ let $i = i + 1$ and goto 2) else goto 5).
5) End.

This tuning algorithm has been proposed in [22] and [23] and applied to a load and frequency control problem for a large power system in [24]. We refer the interested reader to these references and to [67] for the proof of the following theorems which are derived from Lemma 12.2 and Theorems 12.1 and 12.2.

Theorem 12.3. [67]. If the plant (12.1) satisfies the assumptions and the existence condition of Theorem 12.2, there are scalars $\bar{a}_i$ and $\bar{b}_i$ $(i = 1, ..., N)$ such that the closed-loop system (12.1), (12.3), (12.22)−(12.24) is stable for all $e \in \mathscr{E}$ with $a_i \in \in (0, \bar{a}_i)$ and $b_i \in (0, \bar{b}_i)$.

Hence, such a controller can be found by means of the tuning algorithm presented above if a_i and b_i are chosen sufficiently small.

If we release our requirement that the closed-loop system should be stable for all $e \in \mathscr{E}$ and restrict our claim to the stability in all sequential design steps, a weaker existence condition than that of Theorem 12.1 can be found.

Theorem 12.4. Consider a stable plant with static reinforcement K_s. The controller (12.2) can be designed in the sequential way described in the algorithm above while retaining the stability in all design steps if and only if the conditions

$$(12.25) \qquad \det \begin{vmatrix} K_{s11} & K_{s12} & \ldots & K_{s1k} \\ K_{s21} & K_{s22} & \ldots & K_{s2k} \\ \ldots & \ldots & \ldots & \ldots \\ K_{sk1} & K_{sk2} & \ldots & K_{skk} \end{vmatrix} \neq 0, \quad k = 1, 2, ..., N$$

are satisfied.

Under this condition the controller can be found in the sequential way but it is no longer sure that the closed-loop system remains stable if arbitrary control stations are disconnected from the plant.

From the point of view of the decentralization of the design process, which is emphasised in this paper, the sequential design algorithm has interesting characteristics. First, only the static reinforcement is necessary as information about the behaviour of the plant. This information is available in most applications. Second, only for the test of existence of the controller the whole matrix K_s is necessary. If the existence can be proved by structural reasons, as in the example of Section 12.2, only the subsystem reinforcement must be available. This can be acquired by experiments or, if K_s is available, by calculation [68].

12.4. Decentralized design procedure

In this section, a design method will be presented which is particularly suitable for PI control and proceeds in a completely decentralized manner. It is motivated by the result presented in Theorems 12.1 and 12.2. Accordingly, the decentralized controller (12.3) can ensure closed-loop stability for all the 2^N operating conditions $e \in \mathscr{E}$ only if the subsystems possess a certain autonomy. Therefore, it is reasonable

to design the control station for the subsystem and consider the possible influences of the other control stations as model uncertainty. If all control stations are robust enough to tolerate the uncertainties of the models for which they are designed then the closed-loop system will be stable for all $e \in \mathscr{E}$.

If we consider the plant (12.1) from the point of view of subsystem i we get the model

$$(12.26) \qquad \dot{x} = Ax + B_i u_i ,$$

$$y_i = C_i x ,$$

where B_i and C_i are the ith hyper column of hyper row, respectively, in B and C, which belong to the ith input and output. If other control stations are connected to the plant, this model is no longer true. The deviation of the behaviour of the plant under the influence of these control stations from the behaviour of the model (12.26) depends on the controller parameters and cannot be determined in advance. However, the deviation in the static behaviour turns out to be independent of the controller parameters and merely dependent on which control stations are in operation. In more detail, the static reinforcement $\bar{K}_{sii}$ between u_i and y_i can be determined by means of $K_s = (K_{sij})$ in the following way [67], [68]

$$(12.27)$$
$$\bar{K}_{s11}(e) = K_{s11} + (K_{s12} \dots K_{s1N}) \begin{pmatrix} K_{s22} & \dots & e_2 K_{s2N} \\ \dots\dots\dots\dots\dots\dots \\ e_N K_{sN2} & \dots & K_{sNN} \end{pmatrix}^{-1} \begin{pmatrix} e_2 K_{s21} \\ \vdots \\ e_N K_{sN1} \end{pmatrix} ,$$

$\bar{K}_{sii}$ has the same form for $i > 2$ with exchanged indices. Consequently, the static reinforcement between u_i and y_i is known in advance for all $e \in \mathscr{E}$, no matter which controller parameter will be chosen for the control stations. This suggests to use the model

$$(12.28) \qquad \dot{x}_i = Ax_i + B_i(1 + k_i) u_i ,$$

$$y_i = C_i x_i ,$$

$$|k_i| \leqq \bar{k}_i$$

for the resulting plant of the ith control station. k_i is merely described by an upper bound $\bar{k}_i$ and represents the model uncertainties that are brought about by the connection or disconnection of the other control stations to the plant. For systems with dim $u_i = $ dim $y_i = 1$ the scalar $\bar{k}_i$ can be determined as the maximum value of the second term in Eq. (12.27). For multivariable subsystems, $\bar{k}_i$ represents the maximum value of all entries of the corresponding matrix in (12.27).

The problem of designing the control station i for the plant described by Eq. (12.28) is a problem of robust control since Eq. (12.28) represents a set of plants rather than a single system. This problem can be solved by means of the method proposed in [72].

Decentralized design algorithm

1) Determine the models (12.28) $(i = 1, \ldots, N)$ from the plant model (12.1) and upper bounds of the second term in Eq. (12.27).
2) Design the control stations (12.2) independently of each other for the corresponding model (12.28). The controllers should be robust enough to meet the design specifications for all $|k_i| \leqq \bar{k}_i$.
3) The control stations designed independently are used together as decentralized controller (12.3).
4) End.

Notice that the control stations are designed completely independently. The interaction between them through the plant are considered as model uncertainties. The model (12.28) need not be derived from a overall model (12.1) but can be determined independently for the subsystems. However, in contrast to the model of the isolated subsystem used in the aggregation-decomposition method for the design of the control stations, the model (12.28) has regard to the subsystem interactions. The model uncertainties result merely from the effects that the other control stations may exercise upon the plant.

To sketch the application of this design procedure, let us continue with the example of Sections 11.4 and 12.2. With the matrix K_s for the system with 20 subsystems given in [31] we get

$$0 \cdot 338 \leqq \bar{K}_{s11}(e) \leqq 0 \cdot 375 .$$

or

$$\bar{K}_{s11}(e) = 0 \cdot 356(1 + k_1)$$

with

$$|k_1| \leqq 0 \cdot 06 .$$

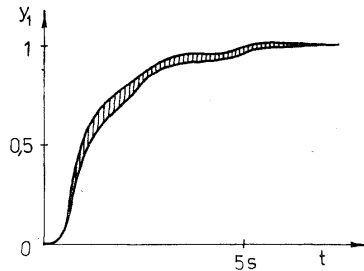Using a model of 25th order and the uncertainties given above the control station 1

Fig. 24. Tolerance band of the command step response of the closed-loop system (12.2), (12.28) $(i = 1)$.

with $K_{I1} = K_{P1} = -0.834$ we get the tolerance band shown in Fig. 24. This band covers all the command step responses of the closed-loop systems (12.2), (12.28) with index $i = 1$. Hence it includes the step response of the overall closed-loop system for all $e \in \mathscr{E}$.

The last assertion is not strictly true insofar as the model (12.28) is only an approximation of the behaviour of subsystem 1 for all $e \in \mathscr{E}$. To show that this approximation is acceptable under the autonomy conditions stated in Theorems 12.1 and 12.2, Fig. 25 shows the step response of the resulting plant of control station 1 (cf. Fig. 22) for different operating conditions e. This step response has the static final value given by Eq. (12.27), but it is reduced to the static value 1 for the purpose of comparison. Fig. 24 makes clear that the step responses for different operating conditions are really approximately given by the model (12.28). Only small deviations are to be seen. This demonstrates the reasonability of the model (12.28).
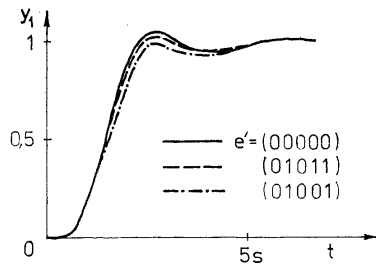

Fig. 25. Step response of the resulting plant of control station 1 for different operating conditions (normed for final value equal to 1).

13. FURTHER TRENDS IN DECENTRALIZED CONTROL DESIGN

Feedback control of large scale systems has to cope with the complexity of the process to be controlled, the information structure constraints, and the uncertainties of the overall system behaviour. As we have seen in this paper, all three aspects bring about uncertainties of the system under consideration. *Approximation* is used as a means for dealing with the complexity of the system. It reduces the dimension of the problems but causes additional modelling errors. This became obvious in all the decomposition methods discussed in Chapter 3 and in Chapters 6—10. All problems that have been stated at the subsystem level were merely approximations of the overall problem and the approximation errors had to be considered in some way.

Information structure constraints impose direct *restrictions* on the information about the plant and the other control stations that is available while designing

a certain control station. Such restrictions, which are originated in the circumstances of the practical applications, have not been considered adequately in theoretical investigations of large scale systems control. Most of the methods assume that a complete model of the overall system is available, and some of them derive equivalent design problems for the subsystems from the overall system problem. The present paper has emphasised these methods and made clear that information structure constraints bring about uncertainties of the design problem and necessitate systematic investigations of the effects that these uncertainties may have.

In addition to that, the model of each large scale system has *primary uncertainties*, which result from an incomplete knowledge of the plant. In order to explain clearly the uncertainties that are caused by the information structure constraints and approximations, we have considered the primary uncertainties only in the Chapters 6 and 8. But the methods used to cope with uncertainties enable us to have regard of primary uncertainties in all other problems too.

In the following, we survey the methodologies that have been described in this paper and compare the kind of uncertainties that are brought about by the decentralization of the design process. Simultaneously, we group them according to the main principle they follow and outline problems that have to be solved by future research.

The design process has been decentralized in three principal ways. The first is *decentralization by restriction of the design process*. If the control stations have to be designed and implemented sequentially, all the design problems can be solved by the corresponding subsystem authorities. Although we have completely specified the subsystem design problems in Chapters 7 and 12, it remains uncertain, whether the control stations will satisfy the design specifications within the overall system, which also includes the control stations that have to be determined yet. Until now, this kind of uncertainty has not been considered in sequential design methods and awaits a systematic treatment. That is, it has to be investigated which design specifications have to be satisfied by the ith control station during the sequential design and implementation process so that the overall decentralized control systems eventually meets the given requirements.

The second way is *decentralization by decomposition* of the design problem. The overall system is teared into subsystems that are weakly coupled. Then, the design problem is decomposed accordingly and solved while ignoring their interdependencies. So, the control stations are obtained from independent design problems and used together as decentralized control of the overall system. Uncertainties occur due to the neglection of the 'weak' couplings between the subsystems and it has to be investigated whether the closed-loop overall system actually satisfies the given design specifications.

Depending on the kind of the weak couplings, different ways of decomposition have been proposed. The hierarchical decomposition in Chapter 6 is based on the absence of certain interactions and results in several structurally strongly coupled subsystems. The separate solution of the design problems for the strongly coupled

subsystems brings about the same kind of uncertainties as the sequential design process.

Disjoint and overlapping decompositions as explained in Chapter 8 and 9 aim at the partitioning of the overall systems into subsystems with low magnitude interactions. There is a lot of methods available for the analysis of the overall system including these weak interactions. The stability criteria presented in Chapter 8 may serve as an example. All these methods have in common that they use only some rough approximation of the subsystem behaviour and the interaction relation and, thus, need no precise model of the overall system. In contrast to that, the question how to apply the proposed decomposition schemes to a practical application has not been completely answered. It is still a matter of engineering intuition to select those parts of the overall system that have to be considered as the subsystems and the overlapping parts. Since the success of such a choice can be seen not until the control stations have been designed and it becomes obvious whether the closed-loop subsystems are weakly coupled, systematic procedures for generating the decomposed system have to be elaborated.

The singular perturbation method for decomposition aims at the separation of subsystems with quite different time constants. The corresponding uncertainties are mainly tackled by qualitative methods that prove the existence of some bounds for the perturbation variable ε up to which the overall system can be considered as weakly coupled and, thus, the subsystems can be designed separately. Such results have been described in Section 3.5 and Chapter 10. They showed that the decentralized design of decentralized controllers for systems with different time scales is based on analysis methods that have been elaborated for singularly perturbed systems. From this point of view, new theoretical research has to be done in order to exploit the sparsity of the system matrices and to find ways for the decomposition of implicit systems into subsystems with different speed of the dynamics.

The third way is *decentralization by exploitation of structural properties*. It is based on the structural analysis of the decentralized control system rather than decomposition. 'Structural' properties are considered here in the broadest sense in which they refer to all phenomena of the system which do not strongly depend on the system parameters. Two results along these lines have been presented in Chapters 11 and 12, where the symmetry of the overall system or the integral part in the control stations, respectively, have been considered as the structural peculiarity of the system under consideration. While the utilisation of symmetry properties led to the reduction of the design problem even for strongly coupled systems, an investigation of systems under PI control fixed a necessary autonomy of the subsystems which in turn suggested a completely decentralized design process. In both cases, the uncertainties of the auxiliary design problems were known to be sufficiently small due to the structural properties of the system.

This survey of the approaches to decentralized design of decentralized controllers makes it obvious that most of the methods start with an exact model of the overall

system. They derive simplified models that have to be used in the independent design processes. This is particularly true for the decentralization by decomposition. In contrasts with the practical restrictions which, in general, make a complete quantitative analysis of the overall system impossible. Therefore, the analysis of the whole system must be restricted to structural considerations. As a result of such a structural analysis it should become obvious in which way the overall design problem can be divided into independent design tasks, which parts of the overall system comprise the 'plant' in such a design task and which design requirements have to be satisfied.

The first steps into this direction have already been done, for example in Chapters 6, 7, 11 and 12. Further success seems to be possible only if structural properties of interconnected systems will be investigated in order to select the typical phenomena of a particular class of systems. These phenomena are quite independent of the parameter values of a given system and can be used to elaborate conceptual simplifications of the design task. Then, a complete model of the overall system is no longer necessary as, for example, in the case of symmetric systems where we had never set up the overall model of the power system. Instead of deriving the approximate model from the overall model, such new methods will prescribe the part of the system or the properties of the overall system that have to be referred to in the independent design tasks. Models that describe these properties can be set up at the subsystem level by the corresponding control authority. In this way, the whole process of modelling, analysing and designing large scale systems observes the restrictions that are imposed by the practical applications.