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Kybernetika, Vol. 13 (1977), No. 1, (23)--40

Persistent URL: http://dml.cz/dmlcz/125111

KYBERNETIKA — VOLUME 13 (1977), NUMBER 1

How to Handle Fuzzy-Quantities?

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The presented paper deals with the concept of fuzzy-sets, introduced by Zadeh, namely it suggests an application of fuzzy-sets theory to the mathematical models of algebraic operations with non-exactly known quantities the values of which may be described by fuzzy-sets. After introducing the basic ideas of the fuzzy-sets theory and the motivation of this paper, the main text is divided into two parts. The first one is subjected to the preparation of general tools necessary for the solution of the given problem. The second part presents the model of addition and repetition of quantities with non-exactly known values, and it suggests a way how to construct the values of results of such operations. In the last section a modification of the Zadeh's concept of fuzzy-set is suggested and its consequences are briefly discussed.

0. On Fuzzy-Sets and Fuzzy-Papers

The concept of fuzzy-sets was introduced by Zadeh in his well-known paper [1], and it was investigated in many further works. Many interesting results were concentrated e.g. in [7]. The fundamental idea of fuzzy-set is quite simple. If X is a non-empty set then any classical subset A of X is defined by its characteristic function $\varphi_A : X \to \{0, 1\}$, where $\varphi_A(x) = 0$ iff $x \notin A$, and $\varphi_A(x) = 1$ iff $x \in A$. The problem appearing in many actual applications of mathematics is that it is not usually exactly known whether some $x \in X$ belongs to the set A or not. In such a case Zadeh suggests to define a new model of the set A in the following way. A *fuzzy-subset* A of X is a real-valued function $f_A : X \to \langle 0, 1 \rangle$ such that the value $f_A(x)$ is the closer to 1 the more x belongs (or is supposed to belong) to A.

The motivation of the fuzzy-sets concept is also obvious. They enable us to construct mathematical models of non-exactly known facts. The advantages and disadvantages of fuzzy-sets were discussed in other papers and they were accepted by many authors for expressing non-exact knowledge which is included in many mathematical models. By means of fuzzy-sets we may express also such notions as "x is approximately equal to y", "x is about a", "a few", "x perhaps belongs to the set A", "some quantity", e.t.c. Namely the non-exact relations like "x is approximately equal to..." appear relatively often and their modeling by fuzzy-sets may be useful.

For simplification of the further explanations we introduce here three concepts defined by Zadeh in his work. The *maximal fuzzy-subset* of X, is the function $f_X \equiv 1$. If f_A and f_B are two fuzzy-subsets of X then their *intersection* and *union* $f_{A\cap B}$ and $f_{A\cup B}$ are also fuzzy-subsets of X defined by

$$f_{A\cap B}(x) = \min \{f_A(x), f_B(x)\} ,$$

$$f_{A\cup B}(x) = \max \{f_A(x), f_B(x)\} .$$

Not only the relations defined above, but all elementary set theoretical concepts are reformulated for fuzzy-sets. Moreover, some further relations between fuzzy-sets were introduced, so that it is possible to formulate many mathematical concepts based on the set theoretical terminology in their "fuzzy" analogies.

As the concept of set is the fundamental one in mathematics, a lot of fuzzy analogies of known mathematical notions could be defined, and there setted in a real explosion of fuzzy-theoretical papers. Those papers promissed to bring new ideas about classical mathematical concepts and some of them really did it. However, a lot of them, especially the ones concerning applied branches of mathematics, consisted of many definitions reformulating classical concepts, and of few results only. Most of the results were of auxiliary nature, they showed, for example, equivalence or other relations between two fuzzy definitions, or some obvious relations analogous to classical set theoretical operations. In spite of the existence of a few interesting theorems, mostly generalizing some classical results into fuzzy theoretical terminology, the disproportion between the number of definitions and results in fuzzy theoretical papers, especially in papers oriented towards applications, is remarkable. The often used justification of such papers that they bring "new philosophy" or "another point of view" may be hardly accepted in so many cases. Mathematics, like any other science, ought to bring new knowledge about the world. That knowledge is, in mathematics, expressed in new statements about the defined concepts. The formulation of an exact model of some phenomenon is the first step of mathematical research only. It is necessary to be able to handle that model, and to derive its further non-elementary properties. In this sense, many of published fuzzy-papers give us less than we could excpect from them.

The main reason of this fact is, from author's point of view, the following one. Fuzzy sets are, in their philosophy, a generalized analogy of classical sets. In that sense they may be used for introducing generalized analogies of other mathematical objects. But, in their nature, fuzzy-sets are functions of a specialized type. If we want to derive any useful result concerning new fuzzy-objects, namely if we want to do so in mathematical branches concentrated to applications, we have to be able to handle those functions in desired way. If necessary, we have to construct new tools proper for such handling. After it we shall be able to derive, step by step, the useful fundamental properties of fuzzy-objects and, by means of them, their more complicated features.

The presented paper is subjected to one problem of that type, namely to the problem of modelling values of some non-exactly known quantities (we shall call them fuzzy-quantities) and of modelling values of results of algebraical operations with such fuzzy-quantities. It was already said above that fuzzy-sets are suitable for modelling such notions like "approximately...", "about ..." e.t.c. The problem, solved here, is which fuzzy-sets represent the results of addition of two or more approximate values, of n-times repeated addition of the same approximate value, or "a few times" repeated addition of that value.

We shall see that even such elementary problem needs the application of non-trivial mathematical tools for its solving, and that it provokes some interesting questions concerning the essential properties of fuzzy-sets. The more important appears the preparation of adequate mathematical apparatus for solving further, less trivial, fuzzy-theoretical problems connected with the applicability of fuzzy-sets theory.

PART I: TOOLS

In this part the general mathematical model of concepts used in this paper is introduced and its main properties are investigated. Namely, the set of possible values of non-exactly known quantities is defined as a fuzzy-subset of a measurable group with Haar measure, and the convolutions of functions on that group are studied.

1. Investigated Measure Space

In all sections of this paper we suppose that a measure space $(X, \mathscr{X}, \mu)$ is given, where X is an algebraic additive group with group operation "+" and with topology defined by class of all open subsets of X. It means that

$$(1.1)\qquad \forall(x, y, z \in X) \quad x + (y + z) = (x + y) + z ,$$

$$(1.2)\qquad \exists(O \in X)\, \forall(x \in X) \quad x + O = O + x = x ,$$

$$(1.3)\qquad \forall(x \in X)\, \exists(-x \in X) \quad x + (-x) = -x + x = O .$$

Let us denote by $M + x$ and $x + M$, where $M \subset X$, $x \in X$, the sets

$$M + x = \{y \in X : \exists(z \in M),\ y = z + x\} ,$$

$$x + M = \{y \in X : \exists(z \in M),\ y = x + z\} .$$

 We suppose that $\mathscr{X}$ is a σ-algebra of subsets of X, i.e.

(1.4) $$\forall(M \subset X) \quad M \in \mathscr{X} \Rightarrow X - M \in \mathscr{X} ,$$

(1.5) $$X \in \mathscr{X} ,$$

(1.6) $$M_i \in \mathscr{X}, \quad i = 1, 2, \ldots \Rightarrow \bigcup_{i=1}^{\infty} M_i \in \mathscr{X} ,$$

and moreover,

(1.7) $\mathscr{X}$ contains all closed subsets of X .

Then, consequently,

(1.8) $$\forall(M \in \mathscr{X}) \, \forall(x \in X) \quad x + M \in \mathscr{X} , \quad M + x \in \mathscr{X} .$$

Finally we suppose that μ is σ-finite Haar measure on the space $(X, \mathscr{X})$, i.e. μ is a mapping from $\mathscr{X}$ into real line such that

(1.9) $$\mu(\emptyset) = 0 , \quad \text{where } \emptyset \text{ is the empty set} ,$$

(1.10) $$\forall(M \in \mathscr{X}) \quad \mu(M) \geqq 0 ,$$

(1.11) $$M_i \in \mathscr{X} , \quad M_i \cap M_j = \emptyset , \quad i = 1, 2, \ldots, \quad j = 1, 2, \ldots, \quad i \neq j , \quad \Rightarrow$$
$$\Rightarrow \mu\Big(\bigcup_{i=1}^{\infty} M_i\Big) = \sum_{i=1}^{\infty} \mu(M_i) ,$$

(1.12) $$\exists(\{M_i\}_{i=1}^{\infty}) , \quad M_i \in \mathscr{X} , \quad \mu(M_i) < \infty , \quad X = \bigcup_{i=1}^{\infty} M_i ,$$

(13) $$\forall(M \in \mathscr{X}) \, \forall(x \in X) \quad \mu(M + x) = \mu(x + M) = \mu(M) .$$

2. Convolutions

Let us suppose that f is an $\mathscr{X}$-measurable and μ-integrable function on X. Its integral over some set $M \in \mathscr{X}$ will be denoted by

$$\int_M f \, d\mu = \int_M f(x) \, d\mu(x) .$$

If $M = X$, we write also abbreviately

$$\int f \, d\mu = \int f(x) \, d\mu(x) .$$

For any function f on X we denote the set

(2.1) $$S_f = \{x \in X : f(x) \neq 0\}$$

and we call it the *support set* of function f.

If f and g are $\mathscr{X}$-measurable functions on X then their *convolutions* $f * g$ and $g * f$ are functions on X defined by

$$[f * g](z) = \int f(z - y)\, g(y)\, d\mu(y) = \int f(x)\, g(-x + z)\, d\mu(x), \tag{2.2}$$

$$[g * f](z) = \int f(-y + z)\, g(y)\, d\mu(y) = \int f(x)\, g(z - x)\, d\mu(x), \tag{2.3}$$

where we write abbreviately $z - y$ and $z - x$ instead of $z + (-y)$ and $z + (-x)$, respectively. Convolutions (2.2) and (2.3) are defined for all $z \in X$ for which the respective integrals exist.

Statement 2.1. If f and g are integrable and $\mathscr{X}$-measurable functions then $f * g$ and $g * f$ are also integrable and $\mathscr{X}$-measurable functions.

Proof. The proof is analogous to the one of the first part of Lemma 24, Chapter VIII, sec. 1, in [3]. The function

$$h(y, z) = f(z - y)\, g(y)$$

is a measurable function on the Cartensian product of spaces

$$(X, \mathscr{X}, \mu) \times (X, \mathscr{X}, \mu),$$

and the desired statement for $f * g$ follows from Fubini theorem and Tonelli theorem (cf. [3], Chapter III, sec. 11). Analogously we may prove the statement for the convolution $g * f$. □

Statement 2.2. Let f and g be $\mathscr{X}$-measurable, integrable and bounded functions on $(X, \mathscr{X}, \mu)$, and let at least one of the support sets S_f and S_g be bounded. Then $f * g$ is a bounded, $\mathscr{X}$-measurable and integrable function and $S_{f * g}$ is a bounded set.

Proof. The statement follows immediately from Statement 2.1 and from [6], Chapter III, § 2, Theorem 3. □

Statement 2.3. If the group X is commutative and if f and g are functions on X then

$$[f * g](x) = [g * f](x)$$

for all $x \in X$ for which the convolutions exist.

Proof. The proof is completely analogous to the one given in [3], Lemma 25, Chapter VIII, sec. 11.

$$[f * g](z) = \int f(z - y)\, g(y)\, d\mu(y) = \int f(x)\, g(z - x)\, d\mu(x) = [g * f](z). \quad \square$$

If the group X is not commutative then the convolutions $f * g$ and $g * f$ may be generally different, as follows from the following simple example.

Example 2.1. Let X be a countable set, let $\mathscr{X}$ be the σ-algebra of all subsets of X, and let μ be defined in the following way

$$\mu(M) = \text{cardinal number of } M\,, \quad \text{if } M \text{ is finite}\,,$$
$$\mu(M) = +\infty\,, \quad \text{if } M \text{ is infinite}\,.$$

Let a, b, k, $m \in X$, and let $a + b = m$, $b + a = k$, $k \neq m$. If f_a and f_b are functions on X such that

$$f_a(a) = f_b(b) = 1\,,$$
$$f_a(x) = 0 \quad \text{for} \quad x \neq a\,, \quad f_b(x) = 0 \quad \text{for} \quad x \neq b\,,$$

then

$$[f_a * f_b]\,(m) = 1\,, \quad [f_a * f_b]\,(x) = 0\,, \quad x \neq m\,,$$
$$[f_b * f_a]\,(k) = 1\,, \quad [f_b * f_a]\,(x) = 0\,, \quad x \neq k\,.$$

Statement 2.4. If f, g and h are functions on X then

$$[[f * g] * h]\,(x) = [f * [g * h]]\,(x)$$

for all $x \in X$ for which the convolutions exist.

Proof. The proof is analogous to the one of Lemma 25, Chapter VIII, sec. 11, in [3].

$$[[f * g] * h]\,(t) = \int\left(\int f(x)\, g(-x + r)\, d\mu(x)\right) h(-r + t)\, d\mu(r) =$$
$$= \int\int g(-x + r)\, f(x)\, h(-r + t)\, d\mu(r)\, d\mu(x) =$$
$$= \int\left(\int g(-x + r)\, h(-r + t)\, d\mu(r)\right) f(x)\, d\mu(x) =$$
$$= \int\left(\int g(y)\, h(-y - x + t)\, d\mu(y)\right) f(x)\, d\mu(x) =$$
$$= \int [g * h]\,(-x + t)\, f(x)\, d\mu(x) = \int f(x)\, [g * h]\,(-x + t)\, d\mu(x) =$$
$$= [f * [g * h]]\,(t)\,. \qquad \square$$

Statement 2.5. Let $f_1, f_2, \ldots, f_n$ be functions on X such that $0 \leqq f_i(x) \leqq 1$ for all $i = 1, 2, \ldots, n$, $x \in X$. Then

$$[f_1 * f_2 * \ldots * f_n](x) \geqq 0$$

for all $x \in X$ for which the convolutions exist. If, moreover,

$$\int f_i(x)\, d\mu(x) \leqq 1$$

for at least $n - 1$ among functions f_i, $i = 1, 2, \ldots, n$, then also

$$[f_1 * f_2 * \ldots * f_n](x) \leqq 1$$

for all $x \in X$ for which the convolutions exist.

Proof. The first inequality of the statement follows immediately from the assumption of non-negativity of functions f_i, $i = 1, 2, \ldots, n$. Let us suppose, now, that

$$\int f_i(x)\, d\mu(x) \leqq 1 \quad \text{for} \quad i = 1, 2, \ldots, j-1, j+1, \ldots, n\,.$$

Then

$$[f_1 * (f_2 * (\ldots * (f_{n-1} * f_n) \ldots))](y) =$$

$$= \int [f_1 * \ldots * f_{n-1}](y - x_n) f_n(x_n)\, d\mu(x_n) =$$

$$= \int \ldots \int [f_1 * \ldots * f_j](y - x_n - x_{n-1} - \ldots - x_{j+1}) f_{j+1}(x_{j+1}) \ldots$$

$$\ldots f_{n-1}(x_{n-1}) f_n(x_n)\, d\mu(x_n)\, d\mu(x_{n-1}) \ldots d\mu(x_{j+1}) =$$

$$= \int \ldots \int [f_2 * \ldots * f_j](-x_1 + y - x_n - \ldots - x_{j+1}) f_1(x_1) f_{j+1}(x_{j+1}) \ldots$$

$$\ldots f_n(x_n)\, d\mu(x_n) \ldots d\mu(x_{j+1})\, d\mu(x_1) =$$

$$= \int \ldots \int f_j(-x_{j-1} - \ldots - x_1 + y - x_n - \ldots - x_{j+1}) f_1(x_1) \ldots$$

$$\ldots f_{j-1}(x_{j-1}) f_{j+1}(x_{j+1}) \ldots f_n(x_n)\, d\mu(x_n) \ldots d\mu(x_{j+1})\, d\mu(x_{j-1}) \ldots$$

$$\ldots d\mu(x_1) \leqq \int \ldots \int f_1(x_1) \ldots f_{j-1}(x_{j-1}) f_{j+1}(x_{j+1}) \ldots$$

$$\ldots f_n(x_n)\, d\mu(x_n) \ldots d\mu(x_{j+1})\, d\mu(x_{j-1}) \ldots d\mu(x_1) =$$

$$= \int f_1(x_1)\,d\mu(x_1) \ldots \int f_{j-1}(x_{j-1})\,d\mu(x_{j-1}) \int f_{j+1}(x_{j+1})\,d\mu(x_{j+1}) \ldots$$

$$\ldots \int f_n(x_n)\,d\mu(x_n) \leqq 1 . \qquad \square$$

It follows from the next example that the condition formulated in Statement 2.5 is sufficient but not necessary for the convolution to be bounded by 1 from above. Convolution $f * g$ may be less than 1 even if both integrals $\int f\,d\mu$ and $\int g\,d\mu$ are greater than 1.

Example 2.2. Let $X = R$ be the set of all real numbers, let $\mathscr{B}$ be the Borel σ-algebra on R and let λ be the Lebesgue measure on $(R, \mathscr{B})$. Function f defined on R in the following way

$$\begin{aligned} f(x) &= 0 \quad \text{for} \quad x < 0 \quad \text{or} \quad x \geqq 3 , \\ &= \tfrac{1}{2} \quad \text{for} \quad 0 \leqq x < 1 , \\ &= \tfrac{1}{3} \quad \text{for} \quad 1 \leqq x < 2 , \\ &= \tfrac{2}{3} \quad \text{for} \quad 2 \leqq x < 3 \end{aligned}$$

is measurable on $(R, \mathscr{B}, \lambda)$, and

$$\int f\,d\lambda = \tfrac{3}{2} > 1 .$$

Then the convolution $[f * f](x)$ is a measurable continuous function on $(R, \mathscr{B}, \lambda)$ with the following values

$$\begin{aligned} [f * f](x) &= 0 && \text{for} \quad x \leqq 0 \quad \text{or} \quad x \geqq 6 , \\ &= \tfrac{1}{4}x && \text{for} \quad 0 \leqq x \leqq 1 , \\ &= \tfrac{1}{12}x + \tfrac{1}{6} && \text{for} \quad 1 \leqq x \leqq 2 , \\ &= \tfrac{4}{9}x - \tfrac{5}{9} && \text{for} \quad 2 \leqq x \leqq 3 , \\ &= \tfrac{16}{9} - \tfrac{1}{3}x && \text{for} \quad 3 \leqq x \leqq 4 , \\ &= \tfrac{4}{9} && \text{for} \quad 4 \leqq x \leqq 5 , \\ &= \tfrac{8}{3} - \tfrac{4}{9}x && \text{for} \quad 5 \leqq x \leqq 6 . \end{aligned}$$

It means that

$$\max \{[f * f](x) : x \in R\} = [f * f](3) = \tfrac{7}{9} < 1 .$$

Let us suppose that $(X, \mathscr{X}, \mu)$ and $(X, \mathscr{X}, \nu)$ are σ-finite measure spaces defined in Section 1 and that ν is absolutely continuous with respect to μ (we shall write $\nu \ll \mu$), i.e.

$$\mu(M) = 0 \Rightarrow \nu(M) = 0 \quad \text{for all} \quad M \in \mathscr{X} .$$

Then, according to Radon-Nikodym theorem, there exists a measurable function φ on X such that

$$\nu(M) = \int_M \varphi \, d\mu \quad \text{for all} \quad M \in \mathscr{X} .$$

It follows immediately from Radon-Nikodym theorem that if f and g are measurable functions on X then there exists a function φ on X such that

$$[f * g](x) = \int f(x - y) \, g(y) \, d\nu(y) = \int f(x - y) \, g(y) \, \varphi(y) \, d\mu(y)$$

for all $x \in X$ for which the convolutions exist.

Let us suppose that $(X, \mathscr{X}, \mu)$ and $(Y, \mathscr{Y}, \nu)$ are σ-finite measure spaces defined in Section 1 and such that

$$X \supset Y ,$$

$$\mathscr{Y} = \{M \subset Y : \exists (N \in \mathscr{X}) \, M = N \cap Y\} ,$$

$$\mu(M) = 0 \Rightarrow \nu(Y \cap M) = 0 , \quad M \in \mathscr{X} .$$

Then we may construct a set function $\bar{\nu}$ on $\mathscr{X}$ such that

$$\bar{\nu}(M) = \nu(M \cap Y) \quad \text{for all} \quad M \in \mathscr{X} . \tag{3.1}$$

The mapping $\bar{\nu}$ is a σ-finite Haar measure on $(X, \mathscr{X})$, as follows from (3.1) and from properties of ν. Further, $\bar{\nu} \ll \mu$ and for any integrable measurable function f on X is

$$\int_X f \, d\bar{\nu} = \int_Y f \, d\nu .$$

Consequently, if f and g are functions on X then they are defined also on Y and there exists a measurable function φ on X such that the convolution $f * g$ on Y fulfils the following relation

$$\int_Y f(z - y) \, g(y) \, d\nu(y) = \int_X f(z - y) \, g(y) \, \varphi(y) \, d\mu(y) .$$

4. Fuzzy-Quantities

It was already said above that in this paper we are interested in mathematical models of non-exactly known quantities. It means, we are interested in the quantities the actual values of which are known only approximately. We may describe their possible values by means of fuzzy-sets. It means that if we know that some quantity $\boldsymbol{a}$ takes its values in a set X then we describe those values by a fuzzy-subset f_a of X, where $f_a(x)$, $x \in X$, is the closer to 1 the greater is our expectation that the actual value of $\boldsymbol{a}$ is equal to x.

In the following sections we use the term *fuzzy-quantities* for such non-exactly known quantities with possible values represented by fuzzy-subsets of X.

It follows immediately from the interpretation described above that we may, without any significant loss of generality, suppose that the support-sets of fuzzy-sets representing the values of fuzzy-quantities are always bounded. As $0 \leqq f_a(x) \leqq 1$, it follows from Statement 2.2 that there exist convolutions of those fuzzy-sets if they are measurable and integrable in the measure space $(X, \mathscr{X}, \mu)$.

PART II: SOLUTIONS

In the following sections we use the concepts and results of the first part for solving the problem formulated in introduction. It means that we shall find the fuzzy-sets representing the values of results of addition of some non-exactly known quantities from a group X. In all this part we keep the assumptions about X, $\mathscr{X}$ and μ, which were formulated in Section 1. Moreover, we suppose, in accordance with Section 4, that all functions representing the values of fuzzy-quantities are measurable and integrable on $(X, \mathscr{X}, \mu)$, and that their support sets are bounded, so that their convolutions exist for all $x \in X$.

5. Addition of Fuzzy-Quantities

Let $\boldsymbol{a}$ and $\boldsymbol{b}$ be fuzzy-quantities with values described by fuzzy-subsets f_a and f_b of X, respectively. Then we may represent the possible values of their sum

$$\boldsymbol{c} = \boldsymbol{a} + \boldsymbol{b}$$

by a function $\tilde{f}_c$ (which is not necessarily a fuzzy-set) defined by

$$\tilde{f}_c = f_a * f_b \,. \tag{5.1}$$

Analogously, if

$$\boldsymbol{c} = \boldsymbol{a}_1 + \boldsymbol{a}_2 + \ldots + \boldsymbol{a}_n$$

where $a_1, a_2, \ldots, a_n$ are fuzzy-quantities with values represented by $f_{a_1}, f_{a_2}, \ldots, f_{a_n}$, respectively, then the possible values of c may be represented by a function $\tilde{f}_c$ (which also is not generally a fuzzy-set) defined by

$$\tilde{f}_c = f_{a_1} * f_{a_2} * \ldots * f_{a_n} . \tag{5.2}$$

It follows immediately from Statement 2.4 that

$$\tilde{f}_{(a+b)+c} = \tilde{f}_{a+(b+c)} ,$$

so that the associativity of the group operation is preserved even for the values of fuzzy-quantities. Analogously, it follows from Statement 2.3 and from the remark following after it, that

$$\tilde{f}_{a+b} = \tilde{f}_{b+a}$$

if the group X is commutative.

If at least n-1 among integrals

$$\int f_{a_1}\, d\mu\,, \quad \int f_{a_2}\, d\mu, \ldots, \int f_{a_n}\, d\mu$$

are not greater than 1 then the convolution

$$\tilde{f}_{a_1+a_2+\ldots+a_n} = f_{a_1} * f_{a_2} * \ldots * f_{a_n}$$

is not greater than 1 and, consequently, it is a fuzzy-subset of X as follows from Statement 2.5. The situation becomes to be more complicated if the function $\tilde{f}_{a_1+a_2+\ldots+a_n}$ or especially $\tilde{f}_{a+b}$, is not a fuzzy-subset of X, i.e. if it is greater than 1 for some $x \in X$. As we want to describe the possible values of the sum $a_1 + a_2 + \ldots + a_n$ by means of a fuzzy-set, we have to modify the function $\tilde{f}_{a_1+a_2+\ldots+a_n}$ into a function not greater than 1. It is possible to do so in more ways. We may, for example, divide the function by its supremal value. The simplest, and most lucid, way is to define the fuzzy-subsets

$$\begin{aligned} f_c(x) = f_{a_1+a_2+\ldots+a_n}(x) &= \min\{1; \tilde{f}_{a_1+a_2+\ldots+a_n}(x)\} = \\ &= \min\{1; [f_{a_1} * f_{a_2} * \ldots * f_{a_n}](x)\}, \quad x \in X, \end{aligned} \tag{5.3}$$

and, especially for $c = a + b$,

$$f_c(x) = f_{a+b}(x) = \min\{1; \tilde{f}_c(x)\} = \min\{1; [f_a * f_b](x)\} . \tag{5.4}$$

Formulas (5.3) and (5.4) may be interpreted so that f_c is an intersection of $\tilde{f}_c$ with the maximal fuzzy-subset f_X of X (see Section 0). It means that f_c is an intersection of the set of possible values of the fuzzy-quantity c with the space X of all such achievable values.

It is obvious from (5.3) and (5.4) that always $f_{a+b} = f_{b+a}$ but there may appear certain difficulties connected with the associativity of values of fuzzy-quantities defined by (5.3). Namely, $f_a * f_{b+c}$ may be generally different from $f_{a+b} * f_c$ if f_{b+c} and f_{a+b} were obtained by the intersection (5.4). That discrepancy may be practically eliminated if we well plan and reason out the complete calculation which is to be done, and if we use the intersection operations (5.4) or (5.3) at the very end of the whole calculation of the final fuzzy-quantity. It means that during the whole procedure of finding the possible values of some fuzzy-quantity we realize exactly one operation of intersection with f_X, and it is the last one transforming the final set of values $\tilde{f}_c$ into the form of a fuzzy-subset f_c of X. In this way the fuzzy-subset of X representing the values of the final fuzzy-quantity is minimally deformed and maximally reflects the expected shape of the set of those values. It is necessary to note that also other possible procedures of modification of $\tilde{f}_c$ into the fuzzy-subset f_c of X have the same disadvantage which must be eliminated in analogous way.

The principle concerning the calculation of f_c, intuitively formulated above, will appear in this paper more times. So, it is useful to formulate it in the following more exact condition.

One-Minimum-Condition. Let $a_1, a_2, \ldots, a_n$ be fuzzy-quantities with values represented by fuzzy-sets $f_{a_1}, f_{a_2}, \ldots, f_{a_n}$. Let c be fuzzy-quantity obtained from $a_1, a_2, \ldots, a_n$ by group operations, and let the values of c be represented by f_c. Then f_c is generally defined so that

$$f_c(x) = \min \{1; F[f_{a_1}, f_{a_2}, \ldots, f_{a_n}](x)\},$$

where F is a function on X defined by means of convolutions of functions $f_{a_1}, f_{a_2}, \ldots, f_{a_n}$ without using any minimization operation representing the intersection with f_X. It means that the minimization explicitly written in definition of f_c is the only one applied during the whole process of calculation of f_c from $f_{a_1}, f_{a_2}, \ldots f_{a_n}$.

If this condition is fulfilled then the associativity of fuzzy-quantities summation is guaranteed, and $f_{a+(b+c)} = f_{(a+b)+c}$.

There exists also another way how to avoid the difficulties connected with application of (5.3) and (5.4). We could generalize the definition of fuzzy-sets in such way that the transformation from $\tilde{f}_c$ to f_c will not be necessary. This possibility and its advantages and disadvantages, are briefly discussed in the conclusive Section 8 of this paper.

6. Deterministic Repetitive Addition of Fuzzy-Quantities

The following section is subjected to the mathematical modelling of values of fuzzy-quantities obtained by a few times repeated addition of the same fuzzy-quantity. We suppose, in this section, that the number of repetitions is exactly determined.

It means that we find here a way how to construct the fuzzy-subset of X representing the values of fuzzy-quantity "n-times $\boldsymbol{a}$" where n is a natural number of known value and $\boldsymbol{a}$ is a fuzzy-quantity with values represented by a fuzzy-subset $f_{\boldsymbol{a}}$ or X.

Then we denote by $n\boldsymbol{a}$ the sum

$$n\boldsymbol{a} = \underbrace{\boldsymbol{a} + \boldsymbol{a} + \ldots + \boldsymbol{a}}_{n\text{-times}},$$

which is a fuzzy-quantity with values represented by a fuzzy-subset $f_{n\boldsymbol{a}}$ of X, where

$$f_{na}(x) = \min\{1; \tilde{f}_{na}(x)\}, \quad x \in X, \tag{6.1}$$

and

$$\tilde{f}_{na}(x) = [\underbrace{f_a * f_a * \ldots * f_a}_{n\text{-times}}](x). \tag{6.2}$$

It follows from Statement 2.5 immediately that

$$f_{na} \equiv \tilde{f}_{na} \quad \text{if} \quad \int f_a \, d\mu \leqq 1 .$$

That condition is not necessary, as follows from Example 2.1. Nevertheless, if $\int f_a \, d\mu > 1$ then f_{na} may be generally different from $\tilde{f}_{na}$ and formula (6.1) defines the final fuzzy-subset of X representing the possible values of $n\boldsymbol{a}$. The motivation of choosing (6.1) for that purpose is the same as for (5.3). It is necessary to respect the rule for using the intersection of $\tilde{f}_{na}$ and f_X formulated as One-Minimum-Condition is Section 5. It means that it is advantageous to enumerate all convolutions of fuzzy-sets representing the values of fuzzy-quantities participating in some formula and then to realize the intersection of the final function of values with f_X. For example, if $\boldsymbol{c}$ is a fuzzy-quantity given by the formula

$$\boldsymbol{c} = n\boldsymbol{a} + m\boldsymbol{b} + k(\boldsymbol{a} + \boldsymbol{b})$$

where k, n, m are known natural numbers and $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ are fuzzy-quantities, and if $f_{\boldsymbol{a}}$ and $f_{\boldsymbol{b}}$ are fuzzy-subsets of X representing the values of $\boldsymbol{a}$ and $\boldsymbol{b}$, then it is advantageous to enumerate the function

$$\tilde{f}_c = \tilde{f}_{na} * \tilde{f}_{mb} * (\underbrace{\tilde{f}_{a+b} * \ldots * \tilde{f}_{a+b}}_{k\text{-times}}),$$

and then to construct

$$f_c(x) = \min\{1; \tilde{f}_c(x)\}, \quad x \in X .$$

 This procedure enables us to profit from useful formal properties of convolutions of fuzzy-sets, like their commutativity, if the group X is commutative, associativity and also distributivity as follows from the next statement.

Statement 6.1. Let $\boldsymbol{a}$ and $\boldsymbol{b}$ be fuzzy-quantities with values represented by fuzzy-subsets f_a and f_b of X, and let m and n be natural numbers. Then

$$\tilde{f}_{(n+m)a} = \tilde{f}_{na+ma} \,.$$

If the group X is commutative, then also

$$\tilde{f}_{n(a+b)} = \tilde{f}_{na+nb} \,.$$

Proof.

$$\tilde{f}_{(n+m)a} = \underbrace{f_a * \ldots * f_a}_{(n+m)\text{-}times} = \tilde{f}_{na} * \tilde{f}_{ma} = \tilde{f}_{na+ma}$$

if One-Minimum-Condition is fulfilled. Analogously, if X is commutative then

$$\tilde{f}_{n(a+b)} = \underbrace{\tilde{f}_{a+b} * \ldots * \tilde{f}_{a+b}}_{n\text{-}times} = f_a * f_b * \ldots * f_a * f_b =$$

$$= \underbrace{f_a * \ldots * f_a}_{n\text{-}times} * \underbrace{f_b * \ldots * f_b}_{n\text{-}times} = \tilde{f}_{na} * \tilde{f}_{nb} = \tilde{f}_{na+nb} \,,$$

if One-Minimum-Condition is fulfilled. □

Formulas (5.3) and (6.1) and Statement 6.1 imply that if One-Minimum-Condition is fulfilled then also,

$$f_{n(a+b)} = f_{na+nb} \quad \text{and} \quad f_{(m+n)a} = f_{ma+na} \,.$$

7. Fuzzy-Repetitive Addition of Fuzzy-Quantities

The problem investigated in this section is analogous to the one investigated in the previous Section 6. The difference between them is in the assumption about our knowledge of value of the natural number n. In this section we suppose that we do not know the exact number of repetitions of addition of the fuzzy-quantity with values in X, but that it is represented by a fuzzy-subset of the set of natural numbers. Let us denote by N the set of all natural numbers and let us suppose that $\boldsymbol{a}$ is a fuzzy-quantity with values represented by a fuzzy-subset f_a of X and $\boldsymbol{n}$ is a non-exactly known natural number with possible values represented by fuzzy-subset g_n of N. We shall find the possible values of repetitive addition of $\boldsymbol{a}$ with non-exactly given number of repetitions, it means that we shall try to express the values of fuzzy-quantity "approximately $\boldsymbol{n}$-times $\boldsymbol{a}$" or "a few times $\boldsymbol{a}$".

For any natural number $m \in N$ we may evaluate the function $\tilde{f}_{ma}$ by means of (6.2). Then the properties of the fuzzy-quantity $\boldsymbol{na}$ will be described by fuzzy set $g_{\boldsymbol{n}}$ and by functions $\tilde{f}_{ma}$ for all $m \in N$. It means that we may define for any $m \in N$

$$\tilde{f}^{(\boldsymbol{n})}_{ma} = \tilde{f}_{ma} \cdot g_{\boldsymbol{n}}(m) \tag{7.1}$$

and, analogously to (6.1), we may construct

$$f^{(\boldsymbol{n})}_{ma}(x) = \min\{1; \tilde{f}^{(\boldsymbol{n})}_{ma}(x)\}\,, \quad x \in X\,, \tag{7.2}$$

where the fuzzy-subsets $f^{(\boldsymbol{n})}_{ma}$ of X describe the values of the fuzzy-quantity $\boldsymbol{na}$. It means that the values of $\boldsymbol{na}$ are represented by a class of fuzzy-sets. If it is useful to express those values by means of exactly one fuzzy subset of X then it is the most natural to define a function $f_{\boldsymbol{na}}$ by

$$f_{\boldsymbol{na}}(x) = \sup\{f^{(\boldsymbol{n})}_{ma}(x) : m \in N\} = \min\{1; \sup\{\tilde{f}^{(\boldsymbol{n})}_{ma}(x) : m \in N\}\}\,, \tag{7.3}$$

for $x \in X$. Fuzzy-set $f_{\boldsymbol{na}}$ is a union of fuzzy-sets $f^{(\boldsymbol{n})}_{ma}$ for all $m \in N$, it means that it represents the set of all anyhow available values of the fuzzy-quantity $\boldsymbol{na}$, where the "expectation" of simultaneous appearance of $\boldsymbol{n} = m$ and $m\boldsymbol{a} = x$ i.e. $f^{(\boldsymbol{n})}_{ma}(x)\, g_{\boldsymbol{n}}(m)$ is arbitrarily near to $f_{\boldsymbol{na}}(x)$.

Even in this case it is useful to realize the intersection of obtained convolution s with the maximal fuzzy-subset f_X of X at the very end of all procedure, it means to fulfil the One-Minimum-Condition formulated in Section 5. The reasons for it were discussed in Sections 5 and 6, and they are valid here as well. If we fulfil that condition then the distributivity of the obtained fuzzy-quantities is preserved as follows from the next statement.

Statement 7.1. Let $\boldsymbol{a}$ and $\boldsymbol{b}$ be fuzzy-quantities with values described by fuzzy-subsets $f_{\boldsymbol{a}}$ and $f_{\boldsymbol{b}}$ of X, and let $\boldsymbol{n}$ be a non-exactly known natural number with values described by fuzzy-subset $g_{\boldsymbol{n}}$ of N. Then

$$f_{\boldsymbol{n}(\boldsymbol{a}+\boldsymbol{b})} = f_{\boldsymbol{na}+\boldsymbol{nb}}$$

if One-Minimum-Condition is fulfilled.

Proof. It follows from Statement 6.1 immediately that

$$\tilde{f}^{(\boldsymbol{n})}_{m(\boldsymbol{a}+\boldsymbol{b})} = \tilde{f}_{m(\boldsymbol{a}+\boldsymbol{b})}\, g_{\boldsymbol{n}}(m) = \tilde{f}_{m\boldsymbol{a}+m\boldsymbol{b}}\, g_{\boldsymbol{n}}(m) = \tilde{f}^{(\boldsymbol{n})}_{m\boldsymbol{a}+m\boldsymbol{b}}$$

for all $m \in N$, and (7.3) implies the desired relation. □

8. Some Discussion at the End

The Zadeh's concept of fuzzy-set f was created as a generalization of the classical set characteristic function φ. The generalization is based on the extension of the range of that function from the two-element set $\{0, 1\}$ in the case of φ to the $\langle 0,1\rangle$-interval n the case of f.

Perhaps everyone who studies the theory of fuzzy-sets asks, sooner or later, if the limitation to the $\langle 0, 1\rangle$-interval of values is necessary, and if it corresponds to the real application of fuzzy-sets. In fact, the definition of fuzzy-sets could be modified in the sense that it is any bounded real-valued, in proper sense measurable and, may be, non-negative function on the set X. In this section we shall discusse some arguments for and against such modification of the Zadeh's concept.

The idea of fuzzy-sets was motivated by an intention to prepare a tool for modelling non-exactly known values and non-exact phenomena from the real world by means of mathematical apparatus. Let us suppose, now, that we study n real events $\mathscr{E}_1$, $\mathscr{E}_2, \ldots, \mathscr{E}_n$ which may be represented by fuzzy-subsets $f_1, f_2, \ldots, f_n$ of some basic set X. In the further development of that model we should like to derive from the mathematical properties of functions $f_1, \ldots, f_n$ some non-trivial and more complicated properties of $\mathscr{E}_1, \ldots, \mathscr{E}_n$ which are not directly obvious from the reality. In this moment, in the first step of the whole procedure, we have to construct the functions $f_1, f_2, \ldots, f_n$ so that they fulfil the following conditions:

(a) The relation between $f_1, f_2, \ldots, f_n$, expressed by the values of $f_i(x) - f_j(x)$ or $f_i(x)/f_j(x)$, $i, j = 1, \ldots, n$, $x \in X$, must reflect the respective mutual relation between $\mathscr{E}_1, \mathscr{E}_2, \ldots, \mathscr{E}_n$ in the investigated situation of the world.

(b) $0 \leqq f_i(x) \leqq 1$, $i = 1, 2, \ldots, n$, $x \in X$.

(By the way – this first step of the procedure is also the last one realized in most of fuzzy-sets applications met by the author in the literature; the reason of it, the unmastered mathematical handling with fuzzy-sets as functions, was already discussed in Section 0).

If we think about these two conditions, we see not only their dissimilarity, but also the possibility to complete them by the third, additional, condition

(c) f_i, $i = 1, 2, \ldots, n$ are measurable in proper sense on X, their integrals on X are defined, and

$$\int_X f_i(x)\, d\mu(x) = 1 \quad \text{for all} \quad i = 1, 2, \ldots, n .$$

Then the functions $f_1, \ldots, f_n$ will be the subjectively constructed probability distributions on X with well known properties and with a great deal of exact tools for working with them. Then we may ask, why condition (c) is to be omitted, why we are to leave the deeply investigated and well known field of probability theory and to go to the unknown world of fuzzy-sets. The arguments about "new philosophy" are not worth of discussing.

From the author's point of view, the only important advantage of fuzzy-sets theory is that the omitting of formal condition (c) enables us to concentrate the attention to the essential demand expressed in (a). That concerns the construction of model as well as its further development.

In this sense, condition (b) is also not necessary and, moreover, it takes our attention from the main condition (a). We could omit (b), or to substitute it by some much simpler condition of boundedness of functions f_i. If necessary, it is possible to transform functions fulfilling (a) into the form fulfilling also (b), but then we can transform them, mostly, also into the form fulfilling even (c) and the fuzzy-sets theory looses its sense. So, if we consider (c) not to be necessary for further mathematical elaboration of functions f_i, $i = 1, \ldots, n$, do there exist any essential reasons for preserving (b)? It is the problem which we try to solve here.

The problems which appeared in Sections 5, 6 and 7 of this paper, and which were rather artificially solved by furmulating One-Minimum-Condition in Section 5, illustrate the inconvenience of (b) in fuzzy-quantities theory. It concerns especially the difficulties connected with the correct application of formulas (5.3), (5.4), (6.1), (7.2) and also (7.3). These difficulties would not exist if condition (b) were omitted. Then also the One-Minimum-Condition would be superfluous.

There exists one argument for preserving condition (b), namely, exactly that condition enables us to define the maximal fuzzy-subset $f_X \equiv 1$, and to define for any fuzzy-set f its complement f' as $f'(x) = 1 - f(x)$, $x \in X$. The possibility to define their complements helps to illustrate the analogy between fuzzy-stes and the classical sets. This analogy is, generally, strong, but not in the case of complement. The set theoretical properties of complements of fuzzy-sets are very different from the properties of the classical ones. There exist, for example, fuzzy-sets which are subsets of their own complements (choose $f \equiv \frac{1}{3}$; then $f' \equiv \frac{2}{3} > f(x)$ for all $x \in X$). It means that the possibility to define complements of fuzzy-sets has rather esthetical than mathematical value. The notion of the maximal fuzzy-subset f_X of X is useful especially for the definition of complement, and it looses most of its importance without this utilization.

All reasons formulated briefly and rather simplified above support, according to author's opinion, the suggestion to modify the fuzzy-sets definition. It would be advantageous to define fuzzy-sets as real-valued, bounded, non-negative and measurable functions on the given set X with some σ-algebra $\mathscr{X}$. This modification can simplify the construction of fuzzy-models of the real events, as its simplicity enables us to concentrate all our attention to the adequateness of the model in the sense of (a). Moreover, it simplifies the computation of values of fuzzy-quantities as follows from this work, and it probably simplifies also other practical applications of fuzzy-sets which will be surely done in future.

(Received June 22, 1976.)

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